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GENERAL CLOSED 4–POINT QUADRATURE FORMULAE OF EULER TYPE

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Abstract. General closed 4-point quadrature formulae where the integral over $[0,1]$ is approximated with the values of the function in points: $0, x, 1-x$ and 1 are derived. As special cases, Simpson's formula, Simpson's 3/8 formula and the Gauss 2-point formula are recaptured. Next, general corrected closed 4-point quadrature formulae are derived, and the Lobatto 4-point, corrected Simpson's, corrected Simpson's 3/8 formula and the corrected Gauss 2-point formula are obtained as special cases. We call "corrected" such quadrature formulae where the integral is approximated not only with the values of the integrand at certain points but with the values of its first derivative at the end points of the interval as well.

1. Introduction

The aim of this paper is to derive closed 4-point quadrature formulae using the extended Euler identities. First, we derive general closed quadrature formulae where the integral over $[0,1]$ is approximated by values of the function at points: 0, x, $1-x$ and 1. As special cases, Simpson's formula, Simpson's 3/8 formula and the Gauss 2-point formula are recaptured.

Next, general corrected closed 4-point quadrature formulae are derived. We call "corrected" such quadrature formulae where the integral is approximated not only with the values of the integrand at certain points but with the values of its first derivative at the end points of the interval as well. Corrected formulae have a degree of exactness higher than the adjoint original formulae (see [13], [11]). The Lobatto 4-point, corrected Simpson's, corrected Simpson's 3/8 formula and the corrected Gauss 2-point formula are obtained as special cases.

The main tool used are the extended Euler formulae, obtained in [3]: for $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$, for some $n \geqslant 1$, for every $x \in [a,b]$ we have

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = f(x) - T_n(x) + \frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)\mathrm{d}f^{(n-1)}(t) \tag{1.1}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t &= f(x) - T_{n-1}(x) \\ &\quad + \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t)\end{aligned} \tag{1.2}$$

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where $T_0(x)=0$ and for $1\leqslant m\leqslant n$

$$T_m(x)=\sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right].$$

where $B_k(t)$ is the k-th Bernoulli polynomial and $B_k^*(t)=B_k(t-\lfloor t\rfloor)$, $t\in\mathbf{R}$. Recall that Bernoulli numbers are defined by $B_k=B_k(0)$, $k\geqslant 0$. For further details on Bernoulli polynomials see [1] and [10].

2. On General Closed 4-Point Quadrature Formulae

Let $x\in(0,1/2]$ and $f:[0,1]\to\mathbf{R}$ be such that $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geqslant 0$. Put $x\equiv 0,\ x,\ 1-x$ and 1 in (1.2), multiply by $1/2-A(x),\ A(x),\ A(x),\ 1/2-A(x)$, respectively and add. The following formula is obtained:

$$\int_0^1 f(t)dt-(1/2-A(x))[f(0)+f(1)]-A(x)[f(x)+f(1-x)]+\tilde{T}_{2n}(x)$$
$$=\frac{1}{(2n+2)!}\int_0^1\tilde{F}_{2n+2}(x,t)df^{(2n+1)}(t),\qquad(2.1)$$

where, for $t\in\mathbf{R}$,

$$\begin{aligned}
&\tilde{T}_{2n}(x)=\sum_{k=2}^{2n}\frac{1}{k!}\,\tilde{G}_k(x,0)\,[f^{(k-1)}(1)-f^{(k-1)}(0)],\\
&\tilde{G}_k(x,t)=[1-2A(x)]B_k^*(1-t)+A(x)\,[B_k^*(x-t)+B_k^*(1-x-t)]\,,\ k\geqslant 1\qquad(2.2)\\
&\tilde{F}_k(x,t)=\tilde{G}_k(x,t)-\tilde{G}_k(x,0),\ \ k\geqslant 2.
\end{aligned}$$

It is easy to verify that

$$\tilde{G}_k(x,1-t)=(-1)^k\tilde{G}_k(x,t)\quad\text{for}\quad 0\leqslant t\leqslant 1$$

and

$$\frac{\partial\tilde{G}_k(x,t)}{\partial t}=-k\tilde{G}_{k-1}(x,t).$$

Further, notice that $\tilde{G}_{2k-1}(x,0)=0$ for $k\geqslant 2$ and this is not affected with any choice of the coefficient $A(x)$. On the other hand, in general, $\tilde{G}_{2k}(x,0)\neq 0$. To obtain the closed 4-point quadrature formulae with the highest possible degree of exactness, it is clear from (2.1) that we have to impose the following condition: $\tilde{G}_2(x,0)=0$. This condition produces:

$$A_2(x):=A(x)=-\frac{B_2}{2(B_2(x)-B_2)}=\frac{1}{12x(1-x)}.\qquad(2.3)$$

This is the coefficient we will work with in this section. Formula (2.1) now becomes:

$$\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}(x)=\frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,t)df^{(2n+1)}(t)\quad(2.4)$$

where

$$Q(0,x,1-x,1)=\frac{1}{12x(1-x)}\left[-6B_2(x)f(0)+f(x)+f(1-x)-6B_2(x)f(1)\right],$$

$$T_{2n}(x)=\sum_{k=2}^{n}\frac{1}{(2k)!}\,G_{2k}(x,0)\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)],$$

$$G_k(x,t)=\frac{1}{12x(1-x)}\left[B_k^*(x-t)-12B_2(x)\cdot B_k^*(1-t)+B_k^*(1-x-t)\right],$$

$$F_k(x,t)=G_k(x,t)-G_k(x,0),\;\; k\geqslant 2\,.$$

Changing the assumptions on function f, we can obtain two more identities where the remainder of the quadrature formula on the left-hand side of (2.4) is expressed differently. Namely, assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geqslant 1$, from (1.1) we get:

$$\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}(x)=\frac{1}{(2n)!}\int_0^1 G_{2n}(x,t)df^{(2n-1)}(t),\qquad (2.5)$$

and assuming $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n\geqslant 0$, from (1.1) (or (1.2)) we get:

$$\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}(x)=\frac{1}{(2n+1)!}\int_0^1 G_{2n+1}(x,t)df^{(2n)}(t).\quad (2.6)$$

If we assumed $\tilde{G}_{2k}(x,0)=0$ for some $k\geqslant 2$, it would increase the exactness but the quadrature formula thus produced would include values of up to $(2k-3)$-th order derivatives at end points of the interval. In cases when those values are easy to calculate, this is not an obstacle, especially when $f^{(k)}(1)=f^{(k)}(0)$ for $k\geqslant 1$. This will be the topic of the next section.

The following lemma is the key step for obtaining sharp estimates of error for this type of quadrature formulae.

LEMMA 1. *For* $x\in(0,\frac{1}{2}-\frac{\sqrt{3}}{6}]\cup[\frac{1}{3},\frac{1}{2}]$ *and* $k\geqslant 1$, $G_{2k+1}(x,t)$ *has no zeros in variable* t *in the interval* $(0,1/2)$. *The sign of this function is determined by:*

$$(-1)^k G_{2k+1}(x,t)>0\quad \text{for } x\in(0,1/2-\sqrt{3}/6],$$
$$(-1)^{k+1} G_{2k+1}(x,t)>0\quad \text{for } x\in[1/3,1/2].$$

Proof. Observe $G_3(x,t)$. For $0\leqslant t\leqslant x$, it takes the form:

$$G_3(x,t)=-t^2\left(t+\frac{3B_2(x)}{2x(1-x)}\right).$$

Its only zero, except 0, is $t_1=\frac{3B_2(x)}{2x(x-1)}$. It is easy to see that $0<t_1\leqslant x$ iff $x\in(\frac{1}{2}-\frac{\sqrt{3}}{6},1-\frac{\sqrt{2}}{2}]$. Further, for $x\leqslant t\leqslant 1/2$, function $G_3(x,t)$ takes the form:

$$G_3(x,t)=-t^3+\frac{3t^2}{2}-\frac{t}{2(1-x)}+\frac{x}{4(1-x)}.$$

Here it has 3 zeros:

$$t = \frac{1}{2}, \quad t_2 = \frac{1}{2} - \frac{\sqrt{3x^2-4x+1}}{2(1-x)}, \quad t_3 = \frac{1}{2} + \frac{\sqrt{3x^2-4x+1}}{2(1-x)}.$$

It only needs to be checked if t_2 is a zero for $x \in (0,1/3)$ since $t_2, t_3 \in \mathbf{R}$ iff $x \in (0,1/3]$ and it is obvious that $t_3 \geqslant 1/2$. That $t_2 < 1/2$ is trivial and it is easy to see that $t_2 \geqslant x$ iff $x \in [1-\frac{\sqrt{2}}{2}, \frac{1}{3})$. Therefore, our statement is valid for $k=1$. Assuming the opposite, the statement for $k \geqslant 2$ follows by induction.

It is elementary to determine the sign of $G_3(x,t)$ since we know its form. From there we can conclude whether $G_5(x,t)$ is convex or concave on $(0,1/2)$ and as it has no zeros there, that is enough to determine its sign. By this procedure we can determine the sign of $G_{2k+1}(x,t)$ for $k \geqslant 3$ which completes the proof.

REMARK 1. From Lemma 1 it follows immediately that for $k \geqslant 1$ and $x \in (0, 1/2 - \sqrt{3}/6]$, function $(-1)^{k+1} F_{2k+2}(x,t)$ is strictly increasing in variable t on $(0,1/2)$ and strictly decreasing on $(1/2,1)$. Since $F_{2k+2}(x,0) = F_{2k+2}(x,1) = 0$, it has constant sign on $(0,1)$ and obtains its maximum at $t = 1/2$. Analogous statement, but with the opposite sign, is valid in the case when $x \in [1/3, 1/2]$.

THEOREM 1. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(2n+2)}$ is continuous and has constant sign on $[0,1]$ for some $n \geqslant 1$ and let $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$. Then there exists a point $\theta \in [0,1]$ such that*

$$R_{2n+2}(x,f) = \frac{\theta}{(2n+2)!} \cdot F_{2n+2}\left(x, \frac{1}{2}\right) \cdot \left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right] \tag{2.7}$$

where $R_{2n+2}(x,f)$ denotes the right-hand side of (2.4) and

$$\begin{aligned} F_{2n+2}(x,1/2) = {} & \frac{1}{6x(1-x)} \left[B_{2n+2}(1/2-x) - B_{2n+2}(x) + \left(2-2^{-2n-1}\right) B_{2n+2}\right] \\ & - \left(2-2^{-2n-1}\right) B_{2n+2}. \end{aligned} \tag{2.8}$$

Proof. Let $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}]$ and suppose $f^{(2n+2)}(t) \geqslant 0$ on $[0,1]$. Then we have

$$0 \leqslant \int_0^1 (-1)^{n+1} F_{2n+2}(x,t) f^{(2n+2)}(t) \mathrm{d}t \leqslant (-1)^{n+1} F_{2n+2}(x,1/2) \cdot \int_0^1 f^{(2n+2)}(t) \mathrm{d}t,$$

which means there exists $\theta \in [0,1]$ such that

$$(2n+2)! \cdot R_{2n+2}(x,f) = \theta \cdot F_{2n+2}(x,1/2) \left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right].$$

When $x \in [\frac{1}{3}, \frac{1}{2}]$ or $f^{(2n+2)}(t) \leqslant 0$ the statement follows similarly.

THEOREM 2. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geqslant 1$ and $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$, then there exists a point $\xi \in [0,1]$ such that*

$$R_{2n+2}(x,f) = -\frac{G_{2n+2}(x,0)}{(2n+2)!} \cdot f^{(2n+2)}(\xi). \tag{2.9}$$

where $R_{2n+2}(x,f)$ denotes the right-hand side of (2.4) and

$$G_{2n+2}(x,0) = \frac{1}{6x(1-x)}\left[B_{2n+2}(x) - B_{2n+2}\right] + B_{2n+2}. \tag{2.10}$$

Proof. From Remark 1 we know that function $F_{2n+2}(x,t)$ has constant sign under these conditions, so the claim follows from the mean value theorem for integrals.

When we apply (2.9) to the remainder in formula (2.4) for $n = 1$, we obtain:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) = \frac{1}{720}(5x^2 - 5x + 1) \cdot f^{(4)}(\xi). \tag{2.11}$$

For $x = 1/3$, this formula becomes the classical Simpson's 3/8 formula, for $x = 1/2$ it becomes the well-known Simpson's formula, and finally for $x = 1/2 - \sqrt{3}/6 \Leftrightarrow A_2(x) = 1/2$ it becomes the classical Gauss 2-point formula (stated on $[0,1]$). These three formulae were studied and generalized using a similar technique as in this paper in [4], [5] and [6], respectively. Of course, all related results from those papers follow as special cases of our results.

REMARK 2. Although only $x \in (0,1/2]$ were taken into consideration here, results for $x = 0$ can easily be obtained by considering the limit process when x tends to 0. Namely,

$$\lim_{x\to 0} Q(0,x,1-x,1) = \frac{1}{2}[f(0)+f(1)] - \frac{1}{12}[f'(1) - f'(0)]$$
$$\lim_{x\to 0} G_k(x,t) = B_k^*(1-t)$$

Consequently, from (2.11) it follows:

$$\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + \frac{1}{12}[f'(1) - f'(0)] = \frac{1}{720} f^{(4)}(\xi). \tag{2.12}$$

THEOREM 3. *Let $p,q \in \mathbf{R}$ be such that $1 \leqslant p,\ q \leqslant \infty$ and $1/p + 1/q = 1$. If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \geqslant 1$, then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - Q(0,x,1-x,1) + T_{2n}(x) \right| \leqslant K(2n,q) \cdot \|f^{(2n)}\|_p. \tag{2.13}$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \geqslant 0$, then we have

$$\left| \int_0^1 f(t)\mathrm{d}t - Q(0,x,1-x,1) + T_{2n}(x) \right| \leqslant K(2n+1,q) \cdot \|f^{(2n+1)}\|_p. \tag{2.14}$$

If $f^{(2n+2)} \in L_p[0,1]$ for some $n \geqslant 0$, then we have

$$\left| \int_0^1 f(t)\mathrm{d}t - Q(0,x,1-x,1) + T_{2n}(x) \right| \leqslant K^*(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{2.15}$$

where

$$K(m,q) = \frac{1}{m!}\left[\int_0^1 |G_m(x,t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}} \quad \textit{and} \quad K^*(m,q) = \frac{1}{m!}\left[\int_0^1 |F_m(x,t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Proof. Inequalities (2.13), (2.14) and (2.15) follow immediately after applying Hölder's inequality to the remainders in formulae (2.5), (2.6) and (2.4). To prove inequalities are sharp, put

$$\begin{aligned}
&f^{(m)}(t) = \operatorname{sgn} G_m(x,t) \cdot |G_m(x,t)|^{1/(p-1)} \quad \text{for} \quad 1 < p < \infty \quad \text{and} \\
&f^{(m)}(t) = \operatorname{sgn} G_m(x,t) \quad \text{for} \quad p = \infty \quad \text{in} \quad (2.13) \quad \text{and} \quad (2.14), \\
&f^{(m)}(t) = \operatorname{sgn} F_m(x,t) \cdot |F_m(x,t)|^{1/(p-1)} \quad \text{for} \quad 1 < p < \infty \quad \text{and} \\
&f^{(m)}(t) = \operatorname{sgn} F_m(x,t) \quad \text{for} \quad p = \infty \quad \text{in} \quad (2.15).
\end{aligned}$$

The proof that these inequalities are the best possible for $p = 1$ is the same as in the proof of Theorem 2 in [12].

For $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $n \geqslant 1$, using Lemma 1 and Remark 1, we can calculate the following constants as special cases of the previous Theorem:

$$\begin{aligned}
K^*(2n+2,1) &= \frac{1}{(2n+2)!}\left|G_{2n+2}(x,0)\right|, \\
K^*(2n+2,\infty) &= \frac{1}{2}\,K(2n+1,1) = \frac{1}{(2n+2)!}\left|F_{2n+2}\left(x,\frac{1}{2}\right)\right|,
\end{aligned}$$

where $G_{2n+2}(x,0)$ is as in (2.10) and $F_{2n+2}(x,1/2)$ is as in (2.8). In view of this, let us consider inequalities (2.14) and (2.15) for $n = 1$ and $p = \infty$:

$$\begin{aligned}
\left| \int_0^1 f(t)\mathrm{d}t - Q(0,x,1-x,1) \right| &\leqslant \frac{1}{576}\left|\frac{16x^2-15x+3}{1-x}\right| \cdot \|f'''\|_\infty \\
\left| \int_0^1 f(t)\mathrm{d}t - Q(0,x,1-x,1) \right| &\leqslant \frac{1}{720}\left|5x^2-5x+1\right| \cdot \|f^{(4)}\|_\infty
\end{aligned}$$

In order to find which admissible x gives the least estimate of error, we have to minimize the functions on the right-hand side. It is easy to see that both those functions are decreasing on $(0, \frac{1}{2} - \frac{\sqrt{3}}{6}]$ and increasing on $[\frac{1}{3}, \frac{1}{2}]$ and that they reach their minimal values at $x = 1/3$. In fact, the same is valid in the case when $n = 1$ and $p = 1$, since $K^*(4,\infty) = \frac{1}{2}\,K(3,1)$.

Therefore, the node that gives the least estimate of error in these three cases is $x = 1/3$, i.e. the optimal closed 4-point quadrature formula is Simpson's 3/8 formula.

3. On General Corrected Closed 4-Point Quadrature Formulae

In this section, we consider closed 4-point quadrature formulae with a degree of exactness higher than the formulae obtained in the previous section but such that involve values of the first derivative of the integrand at both ends of the interval. Such quadrature formulae are sometimes called ”corrected” (cf. [13], [11]).

Let us observe formula (2.1) again. If, instead of the condition $\tilde{G}_2(x,0)=0$, we impose condition $\tilde{G}_4(x,0)=0$, thus leaving the values of the first derivative in the quadrature formula and removing the values of the third, we will get corrected closed 4-point quadrature formulae. This new condition produces the following coefficient:

$$A_4(x) := A(x) = -\frac{B_4}{2(B_4(x)-B_4)} = \frac{1}{60x^2(1-x)^2}. \tag{3.1}$$

Now, assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 1$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^C(x) = \frac{1}{(2n)!}\int_0^1 G_{2n}^C(x,t)df^{(2n-1)}(t); \tag{3.2}$$

assuming $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 0$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^C(x) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^C(x,t)df^{(2n)}(t); \tag{3.3}$$

and finally, assuming $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 0$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^C(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^C(x,t)df^{(2n+1)}(t), \tag{3.4}$$

where

$$\begin{aligned}
Q_C(0,x,1-x,1) &= \frac{1}{60x^2(1-x)^2}[30B_4(x)f(0)+f(x)+f(1-x)+30B_4(x)f(1)],\\
G_k^C(x,t) &= [1-2A_4(x)]B_k^*(1-t)+A_4(x)\left[B_k^*(x-t)+B_k^*(1-x-t)\right],\ \ k\geqslant 1\\
F_k^C(x,t) &= G_k^C(x,t)-G_k^C(x,0),\ \ k\geqslant 2\\
T_{2n}^C(x) &= \sum_{k=2}^{2n}\frac{1}{k!}\,G_k^C(x,0)\,[f^{(k-1)}(1)-f^{(k-1)}(0)]\\
&= \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)]+\sum_{k=3}^{n}\frac{G_{2k}^C(x,0)}{(2k)!}\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)]
\end{aligned}$$

Now we proceed to the key lemma.

LEMMA 2. *For $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $k \geqslant 2$, $G_{2k+1}^C(x,t)$ has no zeros in variable t in the interval $(0, 1/2)$. The sign of the function is determined by:*

$$(-1)^k G_{2k+1}^C(x,t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{5}/10],$$
$$(-1)^{k+1} G_{2k+1}^C(x,t) > 0 \quad \text{for } x \in [1/3, 1/2].$$

Proof. We start from $G_5^C(x,t)$ and claim that for $x \in (1/2 - \sqrt{5}/10, 1/3)$, $G_5^C(x,t)$ has at least one zero in variable t in $(0, 1/2)$. To prove this, first notice that $G_5^C(x,0) = \frac{\partial G_5^C}{\partial t}(x,0) = \frac{\partial^2 G_5^C}{\partial t^2}(x,0) = G_5^C(x,1/2) = 0$ and that $x \in (1/2 - \sqrt{5}/10, 1/3)$ is equivalent to $\frac{\partial^3 G_5^C}{\partial t^3}(x,0) < 0$ and $\frac{\partial G_5^C}{\partial t}(x, \frac{1}{2}) < 0$. From $\frac{\partial^3 G_5^C}{\partial t^3}(x,0) < 0$ we conclude $\frac{\partial^3 G_5^C}{\partial t^3}(x,t) < 0$ in some neighborhood of $t = 0$. Therefore, $\frac{\partial^2 G_5^C}{\partial t^2}(x,t)$ is decreasing in some neighborhood of $t = 0$ and since $\frac{\partial^2 G_5^C}{\partial t^2}(x,0) = 0$, it follows that there we have $\frac{\partial^2 G_5^C}{\partial t^2}(x,t) < 0$. Further, $\frac{\partial G_5^C}{\partial t}(x,t)$ is then also decreasing and since $\frac{\partial G_5^C}{\partial t}(x,0) = 0$, we conclude $\frac{\partial G_5^C}{\partial t}(x,t) < 0$ in some neighborhood of $t = 0$. Finally, from here we see that $G_5^C(x,t)$ is decreasing and since $G_5^C(x,0) = 0$ we have $G_5^C(x,t) < 0$ in some neighborhood of 0. On the other hand, from $\frac{\partial G_5^C}{\partial t}(x, \frac{1}{2}) < 0$ we conclude that $\frac{\partial G_5^C}{\partial t}(x,t) < 0$ in some neighborhood of $t = 1/2$. Then $G_5^C(x,t)$ is decreasing and since $G_5^C(x,1/2) = 0$ we see that $G_5^C(x,t) > 0$ in that neighborhood. Now it is clear that when $\frac{\partial^3 G_5^C}{\partial t^3}(x,0) < 0$ and $\frac{\partial G_5^C}{\partial t}(x, \frac{1}{2}) < 0$, i.e. when $x \in (1/2 - \sqrt{5}/10, 1/3)$, $G_5^C(x,t)$ has at least one zero on $(0, 1/2)$.

It is left to prove that for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$, $G_5^C(x,t)$ has constant sign. This will be done by showing that $G_5^C(x,t)$ is decreasing in the variable x and after checking its behavior at the end points, our statement will follow. First assume $0 < t \leqslant x \leqslant 1/2$. Then

$$\frac{\partial G_5^C}{\partial x}(x,t) = t^3 \cdot \frac{1-2x}{6x^3(1-x)^3}\, [t - 2x(1-x)].$$

Since $t \leqslant x \leqslant 2x(1-x)$, it follows that $\frac{\partial G_5^C}{\partial x}(x,t) < 0$ on this interval. When $0 < x \leqslant t < 1/2$, we have

$$\frac{\partial G_5^C}{\partial x}(x,t) = \frac{(1-2t)}{6(1-x)^3}\, [x - 2t(1-t)].$$

Similarly as before, now $x \leqslant t \leqslant 2t(1-t)$. Therefore $G_5^C(x,t)$ is decreasing in x. To complete our proof, we need to consider the sign of $G_5^C\left(\frac{5-\sqrt{5}}{10}, t\right)$ and $G_5^C\left(\frac{1}{3}, t\right)$. Assume $0 < t \leqslant x \leqslant 1/2$. Then $G_5^C(x,t) = \frac{-t^3}{12x^2(1-x)^2} \cdot g(x,t)$ where $g(x,t) = 12t^2(1-x)^2 \cdot x^2 + t(-30x^4 + 60x^3 - 30x^2 + 1) + 4x(5x^3 - 10x^2 + 6x - 1)$. Now, it is trivial to see that $G_5^C\left(\frac{5-\sqrt{5}}{10}, t\right) > 0$ and that $G_5^C\left(\frac{1}{3}, t\right) < 0$. Similarly, when $0 < x \leqslant t \leqslant 1/2$, we have $G_5^C(x,t) = \frac{1-2t}{12(1-x)^2} \cdot h(x,t)$ where $h(x,t) = 6t^4(1-x)^2 - 12t^3(1-x)^3 + t^2(4x^2 -$

$8x+6)+2t\cdot x(x-2)+x^2$ and again $G_5^C\left(\frac{5-\sqrt{5}}{10},t\right)>0$ and $G_5^C\left(\frac{1}{3},t\right)<0$. Therefore, since $G_5^C(x,t)$ is decreasing in x, it follows that $G_5^C(x,t)>0$ for $x\in(0,\frac{1}{2}-\frac{\sqrt{5}}{10}]$ and that $G_5^C(x,t)<0$ for $x\in[\frac{1}{3},\frac{1}{2}]$.

Thus, the assertion is true for $k=2$. For $k\geqslant 3$ it follows by induction. As for the sign of functions $G_{2k+1}^C(x,t)$, the proof is analogous to the same part of the proof of Lemma 1.

THEOREM 4. *Let $f:[0,1]\to\mathbf{R}$ be such that $f^{(2n+2)}$ is continuous and has constant sign on $[0,1]$ for some $n\geqslant 2$ and let $x\in(0,\frac{1}{2}-\frac{\sqrt{5}}{10}]\cup[\frac{1}{3},\frac{1}{2}]$. Then there exists a point $\theta\in[0,1]$ such that*

$$R_{2n+2}^C(x,f)=\frac{\theta}{(2n+2)!}\cdot F_{2n+2}^C\left(x,\frac{1}{2}\right)\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right] \tag{3.5}$$

where $R_{2n+2}^C(x,f)$ denotes the right-hand side of (3.4) *and*

$$\begin{aligned}F_{2n+2}^C(x,1/2)=&\frac{1}{30x^2(1-x)^2}\left[B_{2n+2}(1/2-x)-B_{2n+2}(x)+\left(2-2^{-2n-1}\right)B_{2n+2}\right]\\&-\left(2-2^{-2n-1}\right)B_{2n+2}.\end{aligned} \tag{3.6}$$

Proof. Analogous to the proof of Theorem 1.

THEOREM 5. *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geqslant 2$ and $x\in(0,\frac{1}{2}-\frac{\sqrt{5}}{10}]\cup[\frac{1}{3},\frac{1}{2}]$, then there exists a point $\xi\in[0,1]$ such that*

$$R_{2n+2}^C(x,f)=-\frac{G_{2n+2}^C(x,0)}{(2n+2)!}\cdot f^{(2n+2)}(\xi). \tag{3.7}$$

where $R_{2n+2}^C(x,f)$ denotes the right-hand side of (3.4) *and*

$$G_{2n+2}^C(x,0)=\frac{1}{30x^2(1-x)^2}\left[B_{2n+2}(x)-B_{2n+2}\right]+B_{2n+2}. \tag{3.8}$$

Proof. Analogous to the proof of Theorem 2.

When we apply (3.7) to the remainder in formula (3.4) for $n=2$, we obtain:

$$\begin{aligned}\int_0^1 f(t)dt-Q_C(0,x,1-x,1)+\frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)]&\\=-\frac{14x^2-14x+3}{302400}\cdot f^{(6)}(\xi).&\end{aligned} \tag{3.9}$$

The above formula produces corrected Simpson's 3/8, corrected Simpson's and the corrected Gauss 2-point formula (the case when $A_4(x)=1/2$) as special cases for appropriate x. Those three formulae were studied and generalized in [7], [8] and [6], respectively. All results from those papers follow as special cases of results from this section.

For $x = 1/2 - \sqrt{5}/10$, formula (3.9) produces the Lobatto 4-point formula which will be considered in more detail in a separate subsection.

REMARK 3. Similarly as in Remark 2, one might wonder if similar results can be obtained for $x = 0$. By considering the limit process we get:

$$\lim_{x\to 0}\left(Q_C(0,x,1-x,1) - \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)]\right)$$
$$= \frac{1}{2}[f(0)+f(1)] - \frac{1}{10}[f'(1)-f'(0)] + \frac{1}{120}[f''(0)+f''(1)]$$
$$\lim_{x\to 0} G_k(x,t) = B_k^*(1-t) + \frac{k(k-1)}{60}B_{k-2}^*(1-t)$$

Consequently, from (3.9) it follows:

$$\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + \frac{1}{10}[f'(1)-f'(0)] - \frac{1}{120}[f''(0)+f''(1)] = -\frac{1}{100800}f^{(6)}(\xi). \tag{3.10}$$

Note that quadrature formulae (2.12) and (3.10) were derived in [2], by integrating the two-point Taylor interpolation formula.

REMARK 4. For the sake of contrast, let us give the $n = 2$ case of (2.4), applying (2.9) for the remainder:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) - \frac{5x^2-5x+1}{720}[f'''(1)-f'''(0)]$$
$$= \frac{14x^4-28x^3+7x^2+7x-2}{60480}\cdot f^{(6)}(\xi).$$

The obtained quadrature formula has the same degree of exactness as formula (3.9), only it makes use of the third derivative values instead of the first.

THEOREM 6. *Let $p,q \in \mathbf{R}$ be such that $1 \leqslant p,\ q \leqslant \infty$ and $1/p + 1/q = 1$. If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \geqslant 1$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - Q_C(0,x,1-x,1) + T_{2n}^C(x)\right| \leqslant K_C(2n,q)\cdot\|f^{(2n)}\|_p. \tag{3.11}$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \geqslant 0$, then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - Q_C(0,x,1-x,1) + T_{2n}^C(x)\right| \leqslant K_C(2n+1,q)\cdot\|f^{(2n+1)}\|_p \tag{3.12}$$

and if $f^{(2n+2)} \in L_p[0,1]$ for some $n \geqslant 0$, then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - Q_C(0,x,1-x,1) + T_{2n}^C(x)\right| \leqslant K_C^*(2n+2,q)\cdot\|f^{(2n+2)}\|_p, \tag{3.13}$$

where

$$K_C(m,q) = \frac{1}{m!}\left[\int_0^1 \left|G_m^C(x,t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}} \quad \textit{and} \quad K_C^*(m,q) = \frac{1}{m!}\left[\int_0^1 \left|F_m^C(x,t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Analogous to the proof of Theorem 3.

Similarly as in the previous section, for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $n \geqslant 2$, we can calculate the following constants as special cases of the previous Theorem:

$$K_C^*(2n+2,1) = \frac{1}{(2n+2)!}\left|G_{2n+2}^C(x,0)\right|,$$

$$K_C^*(2n+2,\infty) = \frac{1}{2}\, K_C(2n+1,1) = \frac{1}{(2n+2)!}\left|F_{2n+2}^C\left(x,\frac{1}{2}\right)\right|,$$

where $G_{2n+2}^C(x,0)$ and $F_{2n+2}^C(x,1/2)$ are as in (3.8) and (3.6), respectively.

We now seek for the optimal corrected closed 4-point quadrature formula for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$, $n = 2$ and $p = \infty$. Theorem 6 gives:

$$\left|\int_0^1 f(t)\mathrm{d}t - Q_C(0,x,1-x,1) + \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)]\right| \leqslant \frac{|32x^3-55x^2+30x-5|}{115200(1-x)^2}\cdot\|f^{(5)}\|_\infty$$

$$\left|\int_0^1 f(t)\mathrm{d}t - Q_C(0,x,1-x,1) + \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)]\right| \leqslant \frac{|14x^2-14x+3|}{302400}\cdot\|f^{(6)}\|_\infty$$

It is not hard to see that functions on the right-hand sides of both of these inequalities are decreasing on $(0, \frac{1}{2} - \frac{\sqrt{5}}{10}]$ and increasing on $[\frac{1}{3}, \frac{1}{2}]$ and they reach their minimum at $x = 1/3$. The same goes for the case when $n = 2$ and $p = 1$.

Thus, once again, we conclude that the node which gives the best estimation of error in these three cases is $x = 1/3$, i.e. the optimal corrected closed 4-point quadrature formula is corrected Simpson's 3/8 formula.

3.1. Lobatto 4-Point Quadrature Formulae

The Lobatto formulae are quadrature formulae which involve both ends of the interval as nodes and choose the other nodes to obtain maximal exactness. They are especially useful for functions which vanish at both ends of the interval of integration. In such cases, a higher effective degree of exactness is attained than that afforded even by the formulae of Gaussian type. Namely, while in this case the use of r nodes in the Lobatto formulae leads to a degree of exactness $2r+1$, same number of nodes used in the Gauss formulae leads to a degree of exactness of "only" $2r-1$. For further details on the Lobatto formulae see [9] or [10].

In the simplest nontrivial case, the Lobatto 3-point formula reduces to the classical Simpson's rule. As for the Lobatto 4-point formula, its nodes and coefficients are in fact the unique solution of the system:

$$\tilde{G}_2(x,0) = \tilde{G}_4(x,0) = 0 \;\Leftrightarrow\; A_2(x) = A_4(x).$$

This system is consistent with wanting to obtain from (2.1) the quadrature formula with the highest possible degree of exactness, i.e. a formula which has a degree of exactness higher than the quadrature formulae studied in Section 2 and equal to that of the corrected quadrature formulae studied in Section 3, not being corrected itself at the same time. This is of course a well-known fact and our method illustrates it nicely.

It is customary to study the Lobatto formulae on the interval $[-1,1]$, in order to make use of the symmetry of the nodes and coefficients, so by a simple linear transformation we will transform the results to this interval. Formulae (3.2), (3.3) and (3.4) now become:

$$\int_{-1}^{1} f(t)dt - Q_L + T_{2n}^L = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^L(t)\,df^{(2n-1)}(t), \tag{3.14}$$

$$\int_{-1}^{1} f(t)dt - Q_L + T_{2n}^L = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^L(t)\,df^{(2n)}(t), \tag{3.15}$$

$$\int_{-1}^{1} f(t)dt - Q_L + T_{2n}^L = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^L(t)\,df^{(2n+1)}(t), \tag{3.16}$$

where

$$Q_L = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right],$$

$$G_k^L(t) = \frac{1}{3}B_k^*\left(\frac{1}{2}-\frac{t}{2}\right) + \frac{5}{6}\left[B_k^*\left(\frac{\sqrt{5}}{10}-\frac{t}{2}\right) + B_k^*\left(-\frac{\sqrt{5}}{10}-\frac{t}{2}\right)\right], \quad k \geqslant 1$$

$$F_k^L(t) = G_k^L(t) - G_k^L(-1), \quad k \geqslant 2$$

$$T_{2n}^L = \sum_{k=3}^{n} \frac{2^{2k-1}}{(2k)!}\, G_{2k}^L(-1)\,[f^{(2k-1)}(1) - f^{(2k-1)}(-1)]$$

Theorem 5 now becomes:

COROLLARY 1. *If $f:[-1,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[-1,1]$ for some $n \geqslant 2$, then there exists a point $\xi \in [-1,1]$ such that*

$$R_{2n+2}^L(f) = -\frac{2^{2n+2}}{(2n+2)!}\, G_{2n+2}^L(-1)\cdot f^{(2n+2)}(\xi) \tag{3.17}$$

where

$$G_{2n+2}^L(-1) = \frac{1}{3}\left[B_{2n+2} + 5B_{2n+2}\left(\frac{1}{2}-\frac{\sqrt{5}}{10}\right)\right].$$

Applying (3.17) for $n = 2$ to the remainder in (3.16) we get the Lobatto 4-point formula:

$$\int_{-1}^{1} f(t)dt - Q_L = -\frac{2}{23625} f^{(6)}(\xi)$$

Using Hölder's inequality we can easily obtain an analogue of Theorem 6. As immediate consequences for $p = 1$ and $p = \infty$ the following estimations are obtained:

$$\left| \int_{-1}^{1} f(t)dt - Q_L \right| \leqslant C(m,q) \cdot \|f^{(m)}\|_p, \ \ m = 1, \dots, 6$$

where

$$C(1,1) \approx 3.76866 \cdot 10^{-1}, \qquad C(1,\infty) = \left| G_1^L \left(\frac{1}{\sqrt{5}} \right) \right| = \frac{1}{\sqrt{5}} \approx 4.47214 \cdot 10^{-1},$$
$$C(2,1) \approx 4.17772 \cdot 10^{-2}, \qquad C(2,\infty) = G_2^L \left(\frac{1}{\sqrt{5}} \right) \approx 6.06553 \cdot 10^{-2},$$
$$C(3,1) \approx 6.4048 \cdot 10^{-3}, \qquad C(3,\infty) \approx 7.35788 \cdot 10^{-3}$$
$$C(4,1) \approx 1.13265 \cdot 10^{-3}, \qquad C(4,\infty) = G_4^L(0)/3 \approx 1.46629 \cdot 10^{-3},$$
$$C(5,1) = \frac{4}{45} |F_6^L(0)| \approx 2.48452 \cdot 10^{-4}, \qquad C(5,\infty) \approx 2.83162 \cdot 10^{-4},$$
$$C(6,1) = \frac{4}{45} |G_6^L(-1)| \approx 8.46561 \cdot 10^{-5},$$
$$C(6,\infty) = \frac{2}{45} |F_6^L(0)| = \frac{\sqrt{5}}{18000} \approx 1.24226 \cdot 10^{-4}.$$

REMARK 5. Similar estimations can be obtained for $m \geqslant 7$. However, the values of derivatives, starting from the 5th, at the end points of the interval are then also included in the quadrature formula. As we've mentioned before, in cases when those values are easy to calculate, this is not an obstacle and if $f^{(k)}(1) = f^{(k)}(-1)$ for $k \geqslant 5$, we get a formula with an even higher degree of exactness.

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