

## ON STOLARSKY AND RELATED MEANS

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*Abstract.* We give a simple proof of the Stolarsky means inequality as well as some related inequalities for similar means of Stolarsky type.

### 1. Introduction and Preliminaries

Let us consider the following means

$$\begin{aligned}
 E(x, y; r, s) &= \left( \frac{s(y^r - x^r)}{r(y^s - x^s)} \right)^{\frac{1}{r-s}} \\
 E(x, y; r, 0) &= \left( \frac{y^r - x^r}{r(\ln y - \ln x)} \right)^{\frac{1}{r}} \\
 E(x, y; r, r) &= e^{-\frac{1}{r}} \left( \frac{x^{x^r}}{y^{y^r}} \right)^{\frac{1}{x^r - y^r}}, \\
 E(x, y; 0, 0) &= \sqrt{xy},
 \end{aligned}$$

where  $0 < x < y < \infty$ ,  $r$  and  $s$  are any real numbers but 0.

These means, known in literature, are called Stolarsky means. Namely Stolarsky in [1] (see also [2, p.120]) introduced these means. Stolarsky proved that the function  $E(x, y; r, s)$  is increasing in both  $r$  and  $s$  i.e. for  $r \leq u$  and  $s \leq v$ , we have

$$E(x, y; r, s) \leq E(x, y; u, v). \tag{1}$$

In this paper, first we shall give a simple proof of inequality (1). Further we shall introduce two new classes of means of Stolarsky type.

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### 2. A Simple Proof of Stolarsky Means Inequality

Note that  $E(x, y; r, s)$  is continuous in  $r$  and  $s$ , so it is enough to prove (1) in the case where  $r, s, u, v \neq 0$ ,  $r \neq s$  and  $u \neq v$ .

We consider the following function

$$f(x) = p^2 \varphi_r(x) + 2pq\varphi_t(x) + q^2 \varphi_s(x) \quad \text{where } t = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R},$$

and

$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0; \\ \ln x, & r = 0. \end{cases}$$

Now

$$\begin{aligned} f'(x) &= p^2 x^{r-1} + 2pqx^{t-1} + q^2 x^{s-1} \\ &= \left( px^{(r-1)/2} + qx^{(s-1)/2} \right)^2 \geq 0. \end{aligned}$$

This implies  $f$  is monotonically increasing. So for  $x \neq y$

$$\frac{f(x) - f(y)}{x - y} \geq 0,$$

i.e.

$$p^2 \frac{\varphi_r(x) - \varphi_r(y)}{x - y} + 2pq \frac{\varphi_t(x) - \varphi_t(y)}{x - y} + q^2 \frac{\varphi_s(x) - \varphi_s(y)}{x - y} \geq 0.$$

Let

$$\phi(r) = \frac{\varphi_r(x) - \varphi_r(y)}{x - y},$$

then

$$p^2 \phi(r) + 2pq\phi(t) + q^2 \phi(s) \geq 0$$

i.e.

$$\phi^2(t) \leq \phi(r) \cdot \phi(s) \quad \text{where } t = \frac{r+s}{2}.$$

This implies  $\phi$  is log-convex in Jensen sense.

We observe  $\lim_{r \rightarrow 0} \phi(r) = \phi(0)$ , hence  $\phi$  is continuous on  $\mathbb{R}$  and therefore is log-convex function.

We need following lemma (see [2], p. 4).

**LEMMA 2.1.** *Let  $f$  be log-convex function and if  $x_1 \leq y_1$ ,  $x_2 \leq y_2$ ,  $x_1 \neq x_2$ ,  $y_1 \neq y_2$ , then the following inequality is valid:*

$$\left( \frac{f(x_2)}{f(x_1)} \right)^{\frac{1}{x_2 - x_1}} \leq \left( \frac{f(y_2)}{f(y_1)} \right)^{\frac{1}{y_2 - y_1}}. \tag{2}$$

Applying Lemma 2.1 for  $f = \phi$ ,  $x_1 = r$ ,  $y_1 = u$ ,  $x_2 = s$ ,  $y_2 = v$  we get an inequality

$$\left( \frac{r(y^s - x^s)}{s(y^r - x^r)} \right)^{1/(s-r)} \leq \left( \frac{u(y^v - x^v)}{v(y^u - x^u)} \right)^{1/(v-u)}.$$

Since  $E(x, y; r, s)$  is continuous in  $r$  and  $s$ , we have (1).

**3. New Class of Means of Stolarsky Type:  $E_1(r, t)$**

Let us begin with the following lemma, which is a special case of Theorem 6 in [3].

**THEOREM 3.1.** *Let  $\phi, \psi \in C^2([x, y])$ . Then for some  $\xi \in [x, y]$ ,*

$$\frac{\frac{1}{y-x} \int_x^y \phi(u) du - \phi\left(\frac{x+y}{2}\right)}{\frac{1}{y-x} \int_x^y \psi(u) du - \psi\left(\frac{x+y}{2}\right)} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{3}$$

**REMARK 3.2.** Theorem 3.1 enables us to define various types of means, because if  $\phi''/\psi''$  has inverse, from (3) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\frac{1}{y-x} \int_x^y \phi(u) du - \phi\left(\frac{x+y}{2}\right)}{\frac{1}{y-x} \int_x^y \psi(u) du - \psi\left(\frac{x+y}{2}\right)}\right). \tag{4}$$

Specially, if we take  $\phi(x) = x^{r-1}$ ,  $\psi(x) = x^{t-1}$ , and using continuous extension, we have the following expression

$$E_1(x, y; r, t) = \left(\frac{(t-1)(t-2) \frac{x^t - y^t}{t(x-y)} - \left(\frac{x+y}{2}\right)^{t-1}}{(r-1)(r-2) \frac{x^r - y^r}{r(x-y)} - \left(\frac{x+y}{2}\right)^{r-1}}\right)^{\frac{1}{r-t}}. \tag{5}$$

Of course, we suppose  $0 < x < y < \infty$ ,  $r, t \neq 0, 1, 2$ ,  $r \neq t$ . However, we have to deduce continuous extensions of (5) in order to cover all choices of  $r$  and  $t$ . For that purpose we consider the following function

$$\phi(t) = \begin{cases} \frac{1}{(t-1)(t-2)} \left[ \frac{x^t - y^t}{t(x-y)} - \left(\frac{x+y}{2}\right)^{t-1} \right], & t \neq 0, 1, 2; \\ \frac{\ln x - \ln y}{2(x-y)} - \frac{1}{x+y}, & t = 0; \\ \ln\left(\frac{x+y}{2}\right) + \frac{x(1-\ln x) - y(1-\ln y)}{x-y}, & t = 1; \\ \frac{x^2 \ln x - y^2 \ln y}{2(x-y)} - \left(\frac{x+y}{2}\right) \ln\left(\frac{x+y}{2}\right) - \frac{x+y}{4}, & t = 2. \end{cases} \tag{6}$$

It is easy to see that  $\phi$  is a continuous function. Now

$$E_1(x, y; r, t) = \left(\frac{\phi(r)}{\phi(t)}\right)^{\frac{1}{r-t}}$$

and all continuous extensions of (5) are now obvious but the case  $r = t$  :

$$E_1(x, y; r, r) = \exp\left(\lim_{t \rightarrow r} \left(\frac{\phi(r)}{\phi(t)} - 1\right) \frac{1}{r-t}\right) = \exp\left(\frac{\phi'(r)}{\phi(r)}\right);$$

$$E_1(x, y; r, r) = \exp\left(\frac{\frac{x^r \ln x - y^r \ln y}{r(x-y)} - \frac{3r^2 - 6r + 2}{r^2(r-1)(r-2)} \frac{x^r - y^r}{x-y} + \frac{2r-3}{(r-1)(r-2)} \left(\frac{x+y}{2}\right)^{r-1} - \left(\frac{x+y}{2}\right)^{r-1} \ln\left(\frac{x+y}{2}\right)}{\frac{x^r - y^r}{r(x-y)} - \left(\frac{x+y}{2}\right)^{r-1}}\right)^{\frac{1}{r}},$$

for  $r \neq 0, 1, 2$ ;

$$E_1(x, y; 0, 0) = \exp\left(\frac{\frac{\ln^2 x - \ln^2 y}{x-y} - \frac{3}{2} \frac{\ln x - \ln y}{x-y} - \frac{6}{x+y} - \frac{2}{x+y} \ln\left(\frac{x+y}{2}\right)}{\frac{\ln x - \ln y}{x-y} - \frac{2}{x+y}}\right);$$

$$E_1(x, y; 1, 1) = \exp\left(\frac{\frac{x \ln^2 x - y \ln^2 y}{4(x-y)} - \ln\left(\frac{x+y}{2}\right) - \frac{1}{2} \ln^2\left(\frac{x+y}{2}\right)}{\frac{x(x+y \ln x - 1) - y(\ln y - 1)}{x-y} - \ln\left(\frac{x+y}{2}\right)}\right);$$

$$E_1(x, y; 2, 2) = \exp\left(\frac{\frac{x^2 \ln^2 x - y^2 \ln^2 y}{4(x-y)} + \frac{x+y}{2} \ln\left(\frac{x+y}{2}\right) - \frac{x+y}{4} \ln^2\left(\frac{x+y}{2}\right) - \frac{3}{4} \frac{x^2 \ln x - y^2 \ln y}{x-y} + \frac{3}{8}(x+y)}{\frac{x^2 \ln x - y^2 \ln y}{2(x-y)} - \frac{x+y}{2} \ln\left(\frac{x+y}{2}\right) - \frac{x+y}{4}}\right).$$

We have proved, in fact

**THEOREM 3.3.**  $E_1(x, y; r, t)$ , defined above, is mean of the numbers  $x, y$ .

Monotonicity property of new mean is the object of the following theorem.

**THEOREM 3.4.** Let  $r \leq u, t \leq v$ . Then

$$E_1(x, y; r, t) \leq E_1(x, y; u, v) \tag{7}$$

za sve  $0 < x < y < \infty$ .

We will need following lemma.

**LEMMA 3.5.** Function  $\phi$  defined by (6) is a log-convex function.

*Proof.* Let us consider the following function

$$\varphi_r(x) = \begin{cases} \frac{x^r}{r(r-1)}, & r \neq 0, 1; \\ -\ln x, & r = 0; \\ x \ln x, & r = 1. \end{cases} \tag{8}$$

Then  $\varphi_r$  is a convex function because  $\varphi_r''(x) = x^{r-2} \geq 0, x > 0$ .

Next we consider the following function

$$f(x) = p^2 \varphi_{r-1}(x) + 2pq \varphi_{z-1}(x) + q^2 \varphi_{s-1}(x) \quad \text{where } z = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R}.$$

This function is convex because

$$f''(x) = (px^{\frac{r-3}{2}} + qx^{\frac{s-3}{2}})^2 \geq 0.$$

We now define the function

$$\phi(r) = \frac{1}{y-x} \int_x^y \varphi_{r-1}(u) du - \varphi_{r-1}\left(\frac{x+y}{2}\right) \tag{9}$$

i.e.

$$\phi(t) = \begin{cases} \frac{1}{(t-1)(t-2)} \left[ \frac{x^t - y^t}{t(x-y)} - \left(\frac{x+y}{2}\right)^{t-1} \right], & t \neq 0, 1, 2; \\ \frac{\ln x - \ln y}{2(x-y)} - \frac{1}{x+y}, & t = 0; \\ \ln\left(\frac{x+y}{2}\right) + \frac{x(1-\ln x) - y(1-\ln y)}{x-y}, & t = 1; \\ \frac{x^2 \ln x - y^2 \ln y}{2(x-y)} - \left(\frac{x+y}{2}\right) \ln\left(\frac{x+y}{2}\right) - \frac{x+y}{4}, & t = 2. \end{cases}$$

As we noticed before this function is continuous on  $\mathbb{R}$ .

We now apply Hadamard inequality on the function  $f$ , and we conclude

$$p^2 \phi(r) + 2pq\phi(z) + q^2 \phi(s) \geq 0, \quad \text{where } z = \frac{r+s}{2} \text{ and } p, q \in \mathbb{R},$$

which means

$$\phi^2(z) \leq \phi(r) \cdot \phi(s) \quad \text{where } z = \frac{r+s}{2}.$$

So  $\phi$  is log-convex in Jensen sense and because  $\phi$  is continuous, it is a log-convex function.  $\square$

*Proof of Theorem 3.3.* We now apply inequality Lemma 2.1 for  $f = \phi$ ,  $x_1 = r$ ,  $y_1 = u$ ,  $x_2 = t$ ,  $y_2 = v$  to deduce that

$$\left(\frac{\phi(r)}{\phi(t)}\right)^{\frac{1}{r-t}} \leq \left(\frac{\phi(u)}{\phi(v)}\right)^{\frac{1}{u-v}}.$$

By continuous extensions of  $E_1(x, y; r, t)$ , given above, we have for  $r \leq u$ ,  $t \leq v$ ,

$$E_1(x, y; r, t) \leq E_1(x, y; u, v). \quad \square$$

### 4. Generalized $E_1$ means

We will use Remark 3.2 in the case  $\phi(x) = x^{p-1}$ ,  $\psi(x) = x^{q-1}$ , and then we take substitutions  $p \rightarrow \frac{r}{s}$ ,  $q \rightarrow \frac{t}{s}$ ,  $x \rightarrow x^s$ ,  $y \rightarrow y^s$ . So we define new mean for numbers  $0 < x < y < \infty$ , by

$$E_1(x, y; r, t; s) = \left( \frac{(t-s)(t-2s)}{(r-s)(r-2s)} \frac{\frac{s(y^r - x^r)}{r(y^s - x^s)} - \left(\frac{x^s + y^s}{2}\right)^{\frac{r-s}{s}}}{\frac{s(y^t - x^t)}{t(y^s - x^s)} - \left(\frac{x^s + y^s}{2}\right)^{\frac{t-s}{s}}} \right)^{\frac{1}{r-t}}, \quad (10)$$

$r \neq t$ ,  $r, t \neq 0, s, 2s$ .

Again, we have to cover all continuous extensions of (10). For that purpose we consider, for  $s \neq 0$ ,

$$\psi_s(t) = \begin{cases} \left[ \frac{1}{(t-s)(t-2s)} \left[ \frac{s(x^t - y^t)}{t(x^s - y^s)} - (x^s + y^s) \frac{t-s}{s} \right] \right], & t \neq 0, s, 2s; \\ \frac{\ln x - \ln y}{2s(x^s - y^s)} - \frac{1}{s^2(x^s + y^s)}, & t = 0; \\ \frac{1}{s^2} \left[ \ln \left( \frac{x^s + y^s}{2} \right) + \frac{x^s(1 - \ln x^s) - y^s(1 - \ln y^s)}{x^s - y^s} \right], & t = s; \\ \frac{1}{s^2} \left[ \frac{s(x^{2s} \ln x - y^{2s} \ln y)}{2(x^s - y^s)} - (x^s + y^s) \ln \left( \frac{x^s + y^s}{2} \right) - \frac{x^s + y^s}{4} \right], & t = 2s \end{cases} \tag{11}$$

and in the case  $s = 0$  we consider

$$\psi_0(t) = \begin{cases} \frac{1}{t^2} \left[ \frac{t(x^t - y^t)}{\ln x - \ln y} - \sqrt{x^t y^t} \right], & t \neq 0; \\ \frac{1}{24} (\ln x - \ln y)^2, & t = 0. \end{cases} \tag{12}$$

It is easy to see that  $s \mapsto \psi_s(t)$  and  $t \mapsto \psi_s(t)$  are continuous functions and that

$$E_1(x, y; r, t; s) = \left( \frac{\psi_s(r)}{\psi_s(t)} \right)^{\frac{1}{r-t}}.$$

Now all continuous extensions can be deduced easily but cases  $r = t$  :

$$E_1(r, r; s) = \exp \left( \frac{\frac{s(x^r \ln x - y^r \ln y)}{r(x^s - y^s)} - \frac{s(3r^2 - 6sr + 2s^2)}{r^2(r-s)(r-2s)} \frac{x^r - y^r}{x^s - y^s} \left( \frac{x^s + y^s}{2} \right)^{\frac{r-s}{s}} \left[ \frac{1}{s} \ln \left( \frac{x^s + y^s}{2} \right) - \frac{2r-3s}{(r-s)(r-2s)} \right]}{\frac{s(y^r - x^r)}{r(y^s - x^s)} - \left( \frac{x+y}{2} \right)^{\frac{r-s}{s}}} \right),$$

$rs(r-s)(r-2s) \neq 0$ ;

$$E_1(s, s; s) = \exp \left( \frac{\frac{x \ln^2 x - y \ln^2 y}{4(x^s - y^s)} - \frac{1}{s^2} \ln \left( \frac{x^s + y^s}{2} \right) - \frac{1}{2s^2} \ln^2 \left( \frac{x^s + y^s}{2} \right)}{\frac{x^s \ln x - y^s \ln y}{x^s - y^s} - \frac{1}{s} \ln \left( \frac{x^s + y^s}{2} \right) - \frac{1}{s}} \right), s \neq 0;$$

$$E_1(2s, 2s; s) = \exp \left( \frac{\frac{x^{2s} \ln^2 x - y^{2s} \ln^2 y}{4(x^{2s} - y^{2s})} + \frac{x^s + y^s}{2s^2} \ln \left( \frac{x^s + y^s}{2} \right) - \frac{3}{4s} \frac{x^{2s} \ln x - y^{2s} \ln y}{x^s - y^s} - \frac{x^s + y^s}{4s^2} \ln^2 \left( \frac{x^s + y^s}{2} \right) + \frac{3}{8s^2} (x^s + y^s)}{\frac{x^{2s} \ln x - y^{2s} \ln y}{2(x^s - y^s)} - \frac{x^s + y^s}{2s} \ln \left( \frac{x^s + y^s}{2} \right) - \frac{x^s + y^s}{4s}} \right),$$

$s \neq 0$ ;

$$E_1(0, 0; s) = \exp \left( \frac{\frac{s(\ln^2 x - \ln^2 y)}{x^s - y^s} + \frac{3}{2} \frac{\ln x - \ln y}{x^s - y^s} - \frac{6}{s(x^s + y^s)} - \frac{2}{s(x^s + y^s)} \ln \left( \frac{x^s + y^s}{2} \right)}{\frac{s(\ln x - \ln y)}{x^s - y^s} - \frac{2}{x^s + y^s}} \right), s \neq 0;$$

$$E_1(r, r; 0) = \exp \left( \frac{\frac{r(x^r \ln x - y^r \ln y) - 3(x^r - y^r)}{r^2(\ln x - \ln y)} + \frac{\sqrt{x^r y^r}}{2} \left( \frac{4}{r} - \ln(xy) \right)}{\frac{r(x^r - y^r)}{\ln x - \ln y} - \sqrt{x^r y^r}} \right), r \neq 0;$$

$$E_1(0, 0; 0) = \sqrt{xy}.$$

We will end this section showing monotonicity of generalized  $E_1$  means.

**THEOREM 4.1.** *Let  $r \leq u, t \leq v$ . Then*

$$E_1(x, y; r, t; s) \leq E_1(x, y; u, v; s), \tag{13}$$

for all  $s \in \mathbb{R}, 0 < x < y < \infty$ .

*Proof.* Let us consider the case  $s \neq 0$  first. In this case we use Lemma 3.5 and log-convex function  $\phi$  defined there. From Lemma 2.1 we have

$$\left(\frac{\phi(t)}{\phi(r)}\right)^{\frac{1}{t-r}} \leq \left(\frac{\phi(v)}{\phi(u)}\right)^{\frac{1}{v-u}} \tag{14}$$

for  $r, t, u, v \in \mathbb{R}, t \leq v, r \leq v, t \neq r, v \neq u$ .

For  $s > 0$  we will make replacements  $x \rightarrow x^s, y \rightarrow y^s, t \rightarrow t/s$  in (14) and we get

$$\phi_s(t) = \begin{cases} \frac{s^2}{(t-s)(t-2s)} \left[ \frac{s(x^t - y^t)}{t(x^s - y^s)} - \left(\frac{x^s + y^s}{2}\right)^{\frac{t-s}{s}} \right], & t \neq 0, s, 2s; \\ \frac{s(\ln x - \ln y)}{2(x^s - y^s)} - \frac{1}{x^s + y^s}, & t = 0; \\ \ln\left(\frac{x^s + y^s}{2}\right) + \frac{x^s(1 - \ln x^s) - y^s(1 - \ln y^s)}{x^s - y^s}, & t = s; \\ \frac{s(x^{2s} \ln x - y^{2s} \ln y)}{2(x^s - y^s)} - \left(\frac{x^s + y^s}{2}\right) \ln\left(\frac{x^s + y^s}{2}\right) - \frac{x^s + y^s}{4}, & t = 2s. \end{cases} \tag{15}$$

Now, in (14) we replace  $\phi \rightarrow \phi_s, x \rightarrow x^s, y \rightarrow y^s, r \rightarrow r/s, t \rightarrow t/s, u \rightarrow u/s, v \rightarrow v/s$  so we get

$$\left(\frac{\phi_s(t)}{\phi_s(r)}\right)^{\frac{s}{t-r}} \leq \left(\frac{\phi_s(v)}{\phi_s(u)}\right)^{\frac{s}{v-u}}. \tag{16}$$

By raising both sides in (16) to the power  $1/s$  we have required result.

For  $s < 0$  we make the same substitutions  $\phi \rightarrow \phi_s, x \rightarrow x^s, y \rightarrow y^s, r \rightarrow r/s, t \rightarrow t/s, u \rightarrow u/s, v \rightarrow v/s$ , and noting  $v/s \leq t/s, u/s \leq r/s$ , we have

$$\left(\frac{\phi_s(v)}{\phi_s(u)}\right)^{\frac{s}{v-u}} \leq \left(\frac{\phi_s(t)}{\phi_s(r)}\right)^{\frac{s}{t-r}}. \tag{17}$$

By raising both sides in (17) to the power  $1/s$  we have required result again. Hence we proved (13) for  $s \neq 0$ . Using continuity of the function  $s \mapsto E_1(x, y; r, t; s)$  and taking limit in (13) when  $s \rightarrow 0$  we conclude that

$$E_1(x, y; r, t; 0) \leq E_1(x, y; u, v; 0). \quad \square$$

### 5. New Class of Means of Stolarsky Type: $E_2(r, t)$

Next theorem enables us to define new means, similar to  $E_1$  type means. This is just a special case of Theorem 7 in [3].

**THEOREM 5.1.** *Let  $\phi, \psi \in C^2([x, y])$ . Then for some  $\xi \in [x, y]$ ,*

$$\frac{\frac{\phi(x) + \phi(y)}{2} - \frac{1}{y-x} \int_x^y \phi(u) du}{\frac{\psi(x) + \psi(y)}{2} - \frac{1}{y-x} \int_x^y \psi(u) du} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{18}$$

**REMARK 5.2.** If  $\phi''/\psi''$  has inverse, from (18) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\frac{\phi(x) + \phi(y)}{2} - \frac{1}{y-x} \int_x^y \phi(u) du}{\frac{\psi(x) + \psi(y)}{2} - \frac{1}{y-x} \int_x^y \psi(u) du}\right) \tag{19}$$

Specially, if we take  $\phi(x) = x^{r-1}$ ,  $\psi(x) = x^{t-1}$ , and using continuous extensions, we can define new means.

Suppose  $0 < x < y < \infty$  are given. Let us define the following expressions

$$E_2(x, y; r; t) = \left( \frac{(t-1)(t-2)}{(r-1)(r-2)} \frac{\frac{x^{t-1}+y^{t-1}}{2} - \frac{x^r-y^r}{r(x-y)}}{\frac{x^{t-1}+y^{t-1}}{2} - \frac{x^r-y^r}{r(x-y)}} \right)^{\frac{1}{r-t}}, \tag{20}$$

$r \neq t, r, t \neq 0, 1, 2$ .

Again, to deduce all continuous extensions of (20) we consider the function

$$\lambda(t) = \begin{cases} \frac{1}{(t-1)(t-2)} \left[ \frac{x^{t-1}+y^{t-1}}{2} - \frac{x^t-y^t}{t(x-y)} \right], & t \neq 0, 1, 2; \\ \frac{x+y}{4xy} - \frac{\ln x - \ln y}{x-y}, & t = 0; \\ \frac{x(\ln x - 1) - y(\ln y - 1)}{x-y} - \frac{\ln x + \ln y}{2}, & t = 1; \\ \frac{x \ln x + y \ln y}{2} + \frac{x+y}{2} - \frac{x^2 \ln x - y^2 \ln y}{2(x-y)}, & t = 2. \end{cases} \tag{21}$$

It is easy to see that  $\lambda$  is a continuous function. Now

$$E_2(x, y; r; t) = \left( \frac{\lambda(r)}{\lambda(t)} \right)^{\frac{1}{r-t}}$$

and all continuous extensions of (20) are now obvious but the case  $r = t$  :

$$E_2(x, y; r; r) = \exp \left( \lim_{t \rightarrow r} \left( \frac{\lambda(r)}{\lambda(t)} - 1 \right) \frac{1}{r-t} \right) = \exp \left( \frac{\lambda'(r)}{\lambda(r)} \right);$$

$$E_2(x, y; t; t) = \exp \left( \frac{\frac{x^{t-1} \ln x + y^{t-1} \ln y}{2} + \frac{3t^2 - 6t + 2}{t^2(t-1)(t-2)} \frac{x^t - y^t}{x-y} - \frac{2t-3}{(t-1)(t-2)} \frac{x^{t-1} + y^{t-1}}{2} - \frac{x^t \ln x - y^t \ln y}{t(x-y)}}{\frac{x^{t-1} + y^{t-1}}{2} - \frac{x^t - y^t}{t(x-y)}} \right),$$

$t \neq 0, 1, 2;$

$$E_2(x, y; 0; 0) = \exp \left( \frac{\frac{2y \ln x + 2x \ln y + 3x + 3y}{4xy} - \frac{2(\ln^2 x - \ln^2 y) + 3(\ln x - \ln y)}{2(x-y)}}{\frac{x+y}{2xy} - \frac{\ln x - \ln y}{x-y}} \right);$$

$$E_2(x, y; 1; 1) = \exp \left( \frac{\frac{x \ln^2 x - y \ln^2 y}{2(x-y)} - \frac{\ln^2 x + \ln^2 y}{4} - \frac{\ln x + \ln y}{2}}{\frac{x(\ln x - 1) - y(\ln y - 1)}{x-y} - \frac{\ln x + \ln y}{2}} \right);$$

$$E_2(x, y; 2; 2) = \exp \left( \frac{\frac{x \ln^2 x + y \ln^2 y}{4} - \frac{x \ln x + y \ln y}{2} - \frac{x^2 \ln^2 x - y^2 \ln^2 y}{4(x-y)}}{\frac{x \ln x + y \ln y}{2} + \frac{x+y}{4} - \frac{x^2 \ln x - y^2 \ln y}{2(x-y)}} \right).$$

Combining Theorem 5.1 and Remark 5.2 we have

**THEOREM 5.3.**  $E_2(x, y; r; t)$ , defined above, is mean of the numbers  $x, y$ .

Next step is monotonicity laws of new means.

THEOREM 5.4. *Let  $r \leq u, t \leq v$ . Then*

$$E_2(x, y; r, t) \leq E_2(x, y; u, v) \tag{22}$$

for all  $0 < x < y < \infty$ .

This theorem has proof quite similar to Theorem 4.1 using the following lemma instead of Lemma 3.5.

LEMMA 5.5. *Function  $\lambda$  defined by (21) is  $\phi$  a log-convex function.*

*Proof.* We will use function  $\phi_r$  defined by (8) and further we define

$$\lambda(t) = \frac{\phi_{t-1}(x) + \phi_{t-1}(y)}{2} - \frac{1}{y-x} \int_x^y \phi_{t-1}(u) du. \tag{23}$$

Writing down each case,  $t = 0, 1, 2$ , we see that this is precisely the function defined by (21) and this function is continuous. The rest part of proof is using same arguments as in Lemma 3.5.  $\square$

### 6. Generalized $E_2$ means

Now we will use Remark 5.2 in case  $\phi(x) = x^{p-1}, \psi(x) = x^{q-1}$ , and then we take substitutions  $p \rightarrow \frac{r}{s}, q \rightarrow \frac{t}{s}, x \rightarrow x^s, y \rightarrow y^s$ . So we define new mean for numbers  $x, y; 0 < x < y < \infty$

$$E_2(x, y; r, t; s) = \left( \frac{(t-s)(t-2s)}{(r-s)(r-2s)} \frac{\frac{x^{r-s} + y^{r-s}}{2} - \frac{s(y^r - x^r)}{r(y^s - x^s)}}{\frac{x^{t-s} + y^{t-s}}{2} - \frac{s(y^t - x^t)}{t(y^s - x^s)}} \right)^{\frac{1}{r-t}}, \tag{24}$$

$r \neq t, r, t \neq 0, s, 2s$ .

In order to cover all continuous extensions of (24) we consider, for  $s \neq 0$

$$\lambda_s(t) = \begin{cases} \frac{1}{(t-s)(t-2s)} \left[ \frac{x^{t-s} + y^{t-s}}{2} - \frac{s(y^t - x^t)}{t(y^s - x^s)} \right], & t \neq 0, s, 2s; \\ \frac{x^s + y^s}{4s^2 x^s y^s} - \frac{\ln x - \ln y}{2s(x^s - y^s)}, & t = 0; \\ \frac{x^s \ln x - y^s \ln y}{s(x^s - y^s)} - \frac{\ln x + \ln y}{2s} - \frac{1}{s^2}, & t = s; \\ \frac{x^s \ln x + y^s \ln y}{2s} + \frac{x^s + y^s}{4s^2} - \frac{x^{2s} \ln x - y^{2s} \ln y}{2s(x^s - y^s)}, & t = 2s \end{cases} \tag{25}$$

and in case  $s = 0$  we consider

$$\lambda_0(t) = \begin{cases} \frac{1}{t^2} \left[ \frac{x^t + y^t}{2} - \frac{y^t - x^t}{t(\ln y - \ln x)} \right], & t \neq 0; \\ \frac{1}{12} (\ln x - \ln y)^2, & t = 0. \end{cases} \tag{26}$$

It is easy to see that  $s \mapsto \lambda_s(t)$  and  $t \mapsto \lambda_s(t)$  are continuous functions and that

$$E_2(x, y; r, t; s) = \left( \frac{\lambda_s(r)}{\lambda_s(t)} \right)^{\frac{1}{r-t}}.$$

Now all continuous extensions can be deduced easily but cases  $r = t$  :

$$E_2(t, t; s) = \exp \left( \frac{\frac{s(3t^2 - 6st + 2s^2)}{t^2(t-s)(t-2s)} \frac{x^t - y^t}{x^s - y^s} - \frac{s(x^t \ln x - y^t \ln y)}{t(x^s - y^s)} - \frac{2(t-3s)}{(t-s)(t-2s)} \frac{x^{t-s} + y^{t-s}}{2} + \frac{x^{t-s} \ln x + y^{t-s} \ln y}{2}}{\frac{x^{t-s} + y^{t-s}}{2} - \frac{s(y^t - x^t)}{t(y^s - x^s)}} \right),$$

$ts(t-s)(t-2s) \neq 0;$

$$E_2(s, s; s) = \exp \left( \frac{\frac{\ln x + \ln y}{2s} + \frac{\ln^2 x + \ln^2 y}{4} - \frac{x^s \ln^2 x - y^s \ln^2 y}{2(x^s - y^s)}}{\frac{\ln x + \ln y}{2} - \frac{x^s \ln x - y^s \ln y}{x^s - y^s} + \frac{1}{s}} \right), s \neq 0;$$

$$E_2(2s, 2s; s) = \exp \left( \frac{\frac{x^s \ln^2 x + y^s \ln^2 y}{4} - \frac{x^s \ln x + y^s \ln y}{2s} - \frac{x^{2s} \ln^2 x - y^{2s} \ln^2 y}{4(x^s - y^s)}}{\frac{x^s \ln x + y^s \ln y}{2} + \frac{x^s + y^s}{4s} - \frac{x^{2s} \ln x - y^{2s} \ln y}{2(x^s - y^s)}} \right), s \neq 0;$$

$$E_2(0, 0; s) = \exp \left( \frac{\frac{3(x^s + y^s)}{4s x^s y^s} + \frac{x^s \ln y + y^s \ln x}{2x^s y^s} - \frac{2s(\ln^2 x - \ln^2 y) + 3(\ln x - \ln y)}{2(x^s - y^s)}}{\frac{x^s + y^s}{2x^s y^s} - \frac{s(\ln x - \ln y)}{x^s - y^s}} \right), s \neq 0;$$

$$E_2(t, t; 0) = \exp \left( \frac{\frac{x^t \ln x + y^t \ln y}{2} + \frac{3(x^t - y^t) - t(x^t \ln x - y^t \ln y)}{t^2(\ln x - \ln y)} - \frac{x^t + y^t}{t}}{\frac{x^t + y^t}{2} - \frac{y^t - x^t}{t(\ln y - \ln x)}} \right), t \neq 0;$$

$$E_2(0, 0; 0) = \sqrt{xy}.$$

At the end of paper we have monotonicity for generalized  $E_2$  means.

Technique of proof of the next theorem is the same as in Theorem 4.1 so we omit it here.

**THEOREM 6.1.** *Let  $r \leq u, t \leq v$ . Then*

$$E_2(x, y; r, t; s) \leq E_2(x, y; u, v; s), \tag{27}$$

for all  $s \in \mathbb{R}, 0 < x < y < \infty$ .

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