Mathematical
Inequalities
& Applications
Volume 14, Number 3 (2011), 543–553

ON THE WEIGHTED HARDY TYPE INEQUALITY IN A FIXED DOMAIN FOR FUNCTIONS VANISHING ON THE PART OF THE BOUNDARY

YU. O. KOROLEVA

(*Communicated by L.-E. Persson*)

Abstract. We derive and discuss a new two-dimensional weighted Hardy-type inequality in a rectangle for the class of functions from the Sobolev space H^1 vanishing on small alternating pieces of the boundary.

1. Introduction

Inequalities of Hardy-type are very important for many applications. These inequalities are important tools e.g. for deriving some estimates for operator norms, for proving some embedding theorems, for estimating eigenvalues, etc.

The following basic one-dimensional Hardy-type inequality is well known:

$$\int_0^a \left(\frac{u}{t}\right)^p dt \leqslant \left(\frac{p}{p-1}\right)^p \int_0^a (u')^p\, dt, \tag{1.1}$$

where $u \in L_p(0,a),\ u' \in L_p(0,a),\ p > 1,\ u(0) = 0$.

This inequality could be generalized to the multidimensional weighted form:

$$\left(\int_{\mathbb{R}^n} V(x)|u(x)|^q\, dx\right)^{\frac{1}{q}} \leqslant \mathscr{C} \left(\int_{\mathbb{R}^n} W(x)|\nabla u(x)|^p\, dx\right)^{\frac{1}{p}}, \tag{1.2}$$

where $n \in \mathbb{Z}_+,\ u(x) \in C_0^\infty(\mathbb{R}^n),\ V(x) \geqslant 0, W(x) \geqslant 0, p,q \geqslant 1$, and the constant $\mathscr{C}$ depends only on $V(x)$ and $W(x)$. There are several results concerning weighted Hardy-type inequality (see e.g. the books [10], [11] and [14] and the references given there).

For the case $n = 1,\ 1 < p \leqslant q < \infty$ we have the following necessary and sufficient condition for the validity of (1.2):

$$A_M := \sup_{x>0} \left(\int_x^\infty V(t)\, dt\right)^{\frac{1}{q}} \left(\int_0^x W^{1-p^*}(t)\, dt\right)^{\frac{1}{p^*}} < \infty, \tag{1.3}$$

where $p^* = \frac{p}{p-1}$.

Let us mention also the following result of V. Maz'ya (see [12, Corollary of Theorem 1.4.1.2, Theorem 1.4.2.2]):

Mathematics subject classification (2010): 39A10, 39A11, 39A70, 39B62, 41A44, 45A05.
Keywords and phrases: Inequalities, partial differential equations, functional analysis, spectral theory, homogenization theory, Hardy-type inequalities.

© ELEMENT, Zagreb
Paper MIA-14-47

THEOREM 1.1. *Let $1 < p < q < \infty$, $p < n$ or $1 = p \leqslant q < \infty$. Then the Hardy inequality (1.2) with $W(x) \equiv 1$ holds for every $u \in C_0^\infty(\mathbb{R}^n)$ with a finite constant $C > 0$ if and only if*

$$B := \sup_{x \in \mathbb{R}^n} \sup_{R>0} R^{1-\frac{n}{p}} \left(\int\limits_{B_{R(x)}} V(y)\,dy \right)^{\frac{1}{q}} < \infty,$$

where $B_{R(x)}$ is a ball of the radius R centered at the point x.

The weighted Hardy-type inequality can be generalized to domains in $\mathbb{R}^n$. It was first done by J. Nečas in [13]. He proved that if Ω is a bounded domain with Lipschitz boundary, $1 < p < \infty$, $\alpha < p-1$, then for $u \in C_0^\infty(\Omega)$ the inequality

$$\int\limits_{\Omega} |u(x)|^p \rho^{-p+\alpha}(x)\,dx \leqslant C \int\limits_{\Omega} |\nabla u(x)|^p \rho^{\alpha}(x)\,dx \tag{1.4}$$

holds, where $\rho(x) = dist(x, \partial\Omega)$. After that this inequality was generalized by A. Kufner in [9] to domains with the Hölder boundary and later by A. Wannebo (see [16]) to domains with the generalized Hölder condition. All results related to (1.4) in the case $\alpha = 0$ was described in [7].

The aim of this paper is to prove a Hardy-type inequality (1.4) with $p = 2$ for functions from H^1, vanishing on small alternating pieces of the boundary of the domain. It is assumed for the simplicity that Ω is a rectangle in $\mathbb{R}^2$.

Such a result is completely new in the theory of Hardy-type inequalities and it gives us possibility to apply the tools of homogenization theory to obtain the asymptotics of the best constant in the Hardy-type inequality.

An analogous result was obtained earlier in [1] and in [2] for Friedrichs inequality which can be regarded as a special case of weighted Hardy-type inequalities when we assume that the weight functions equals to 1.

The paper is organized as follows: in Section 2 we give some necessary definitions and formulate auxiliary lemmas. The main results are presented and proved in Section 3 and Section 4 is reserved for some concluding remarks.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be the rectangle $[0,a] \times [0,b]$:

$$\Omega = \{(x_1, x_2) : 0 \leqslant x_1 \leqslant a, 0 \leqslant x_2 \leqslant b\}.$$

Assume that $\varepsilon > 0$ is a small positive parameter, $\varepsilon = \frac{b}{N}$, $N \gg 1$, $N \in \mathbb{N}$, and $\delta = const, 0 < \delta < 1$. Moreover, let

$$\Gamma := \{(x_1, x_2) \in \partial\Omega : \ x_1 = 0\}.$$

We suppose that Γ is represented in the form:

$$\Gamma = \overline{\Gamma_\varepsilon} \cup \gamma_\varepsilon, \ \Gamma_\varepsilon = \bigcup_i (\Gamma_i^\varepsilon), \ \gamma_\varepsilon = \bigcup_i (\gamma_i^\varepsilon), \ \Gamma_i^\varepsilon \cap \gamma_i^\varepsilon = \varnothing,$$

$$mes\,\Gamma_i^\varepsilon = \varepsilon\delta, \quad mes\,(\Gamma_i^\varepsilon \cup \gamma_i^\varepsilon) = \varepsilon,$$

where Γ_i^ε and γ_i^ε are alternating (see Figure 2).

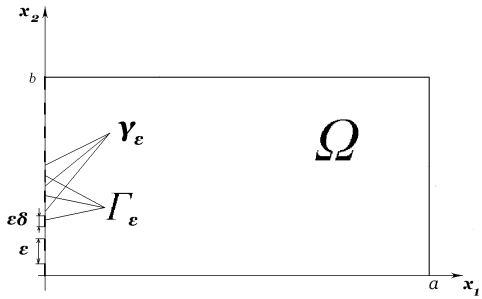

Figure 1: *The domain* Ω.

Denote by

$$\Pi_1^i := \{(x_1,x_2) \in \Omega : 0 < x_1 < a, x_2 \in \Gamma_i^\varepsilon\},$$
$$\Pi_2^i := \{(x_1,x_2) \in \Omega : 0 < x_1 < a, x_2 \in \gamma_i^\varepsilon\}.$$
$$\Pi_1 = \bigcup_i \Pi_1^i,\ \Pi_2 = \bigcup_i \Pi_2^i.$$

Fix a parameter $\theta > 0$. Define the set $\Omega^\theta := \{x = (x_1,x_2) \in \Omega :\ x_1 > \theta\}$. The sets $\Pi_1^{i,\theta}, \Pi_2^{i,\theta}, \Pi_1^\theta$ and Π_2^θ are defined analogously. Moreover, we use the notation

$$B(x,r) := \{(y_1,y_2) \in \mathbb{R}^2 : (y_1-x_1)^2 + (y_2-x_2)^2 \leqslant r^2\},$$

and the average value of the function u over $B(\cdot,r) \in \mathbb{R}^2$ is defined as

$$u_B := \frac{1}{\pi r^2} \int_B u(x)\,dx.$$

Let u be a locally integrable function on $\mathbb{R}^2$. The maximal functions $M(u)$ and $M_R(u)$ of u are defined by

$$M(u)(x) := \sup_{r>0} u_B,\ M_R(u)(x) := \sup_{0<r\leqslant R} u_B.$$

Let us define the Sobolev space

$$H^1(\Omega,\Gamma_\varepsilon) = \{u_\varepsilon \in H^1(\Omega) :\ u_\varepsilon|_{\Gamma_\varepsilon} = 0\}.$$

Analogously,

$$C^\infty(\Omega,\Gamma_\varepsilon) = \{u_\varepsilon \in C^\infty(\Omega) : u_\varepsilon = 0 \text{ in a neighborhood of } \Gamma_\varepsilon = 0\}.$$

Let $x \in \Omega$, $\rho(x) = \mathrm{dist}(x, \Gamma)$. Define the following functions:

$$r_1(x) = \rho(x),\ r_2(x) = \mathrm{dist}(x, \Gamma_\varepsilon) := \inf_{y \in \Gamma_\varepsilon} \mathrm{dist}(\mathrm{x}, \mathrm{y}).$$

According to the geometrical construction of the domain,
$r_1(x) < r_2(x) < \rho(x) + (1-\delta)\frac{\varepsilon}{2}$ (see Figure 2).

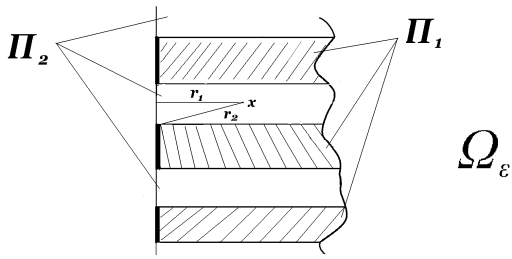

Figure 2: *Periodical structure of the domain.*

We need to derive the following auxiliary Lemma of independent interest:

LEMMA 2.1. *Let $u_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon)$. Then the Friedrichs type inequality*

$$\int_{\Pi_2} u_\varepsilon^2\, dx \leqslant K(a, \varepsilon, \delta) \int_{\Omega} |\nabla u_\varepsilon|^2\, dx, \tag{2.1}$$

holds with $K(a, \varepsilon, \delta) = 2\left(a^2 \frac{1-\delta}{\delta} + \varepsilon^2 (1-\delta)^2\right)$.

Proof. Fix the point $(x_1, x_2) \in \Pi_1^i$. By using the Newton-Leibnitz formula, we have

$$u_\varepsilon(x_1, x_2) = u_\varepsilon(x_1, x_2) - u(0, x_2) = \int_0^{x_1} \frac{\partial u_\varepsilon}{\partial x_1}\, dx_1 \leqslant \int_0^a \frac{\partial u_\varepsilon}{\partial x_1}\, dx_1.$$

Hence,

$$u_\varepsilon^2(x_1, x_2) \leqslant \left(\int_0^a \frac{\partial u_\varepsilon}{\partial x_1}\, dx_1\right)^2 \leqslant a \int_0^a |\nabla u_\varepsilon|^2\, dx_1.$$

Then, by integrating the last inequality with respect to x_2 and after that with respect to x_1 over Π_1^i, we obtain that

$$\int_{\Pi_1^i} u_\varepsilon^2\, dx \leqslant a^2 \int_{\Pi_1^i} |\nabla u_\varepsilon|^2\, dx. \tag{2.2}$$

Now choose the point $(x_1,\widetilde{x}_2) \in \Pi_2^i$ such that $\widetilde{x}_2 = \frac{1-\delta}{\delta}x_2 + \varepsilon\delta$. This means that $(x_1,\widetilde{x}_2) \in \Pi_2^i$ if and only if $(x_1,x_2) \in \Pi_1^i$. By again using the Newton-Leibnitz formula, we fined that

$$u_\varepsilon(x_1,\widetilde{x}_2) = u(x_1,x_2) + \int\limits_{x_2}^{\widetilde{x}_2} \frac{\partial u_\varepsilon}{\partial x_2}\, d\widetilde{x}_2.$$

Consequently,

$$u_\varepsilon^2(x_1,\widetilde{x}_2) \leqslant 2u^2(x_1,x_2) + 2(\widetilde{x}_2 - x_2) \int\limits_{x_2}^{\widetilde{x}_2} |\nabla u_\varepsilon|^2\, d\widetilde{x}_2.$$

Integrating the last inequality over Π_2^i and substituting the integral on the right-hand side by the greater integral, we get that

$$\int\limits_{\Pi_2^i} u_\varepsilon^2\, dx_1\, d\widetilde{x}_2 \leqslant 2\frac{1-\delta}{\delta} \int\limits_{\Pi_1^i} u_\varepsilon^2\, dx_1\, dx_2 + 2\varepsilon^2(1-\delta)^2 \int\limits_{\Pi_2^i} |\nabla u_\varepsilon|^2\, dx_1\, d\widetilde{x}_2.$$

Finally, by applying the estimate (2.2) to the first integral on the right-hand side and substituting both integrals by the greater integral, we obtain that

$$\int\limits_{\Pi_2^i} u_\varepsilon^2\, dx \leqslant 2\left(a^2\frac{1-\delta}{\delta} + \varepsilon^2(1-\delta)^2\right) \int\limits_{\Pi_1^i \cup \Pi_2^i} |\nabla u_\varepsilon|^2\, dx. \tag{2.3}$$

By summarizing up the inequalities (2.2) and (2.3) with respect to i, we obtain the desired estimate:

$$\begin{aligned}\int\limits_{\Pi_2} u_\varepsilon^2\, dx &= \int\limits_{\bigcup\limits_i \Pi_2^i} u_\varepsilon^2\, dx \leqslant 2\left(a^2\frac{1-\delta}{\delta} + \varepsilon^2(1-\delta)^2\right) \int\limits_{\bigcup\limits_i (\Pi_1^i \cup \Pi_2^i)} |\nabla u_\varepsilon|^2\, dx \\ &= 2\left(a^2\frac{1-\delta}{\delta} + \varepsilon^2(1-\delta)^2\right) \int\limits_{\Omega} |\nabla u_\varepsilon|^2\, dx. \quad \square\end{aligned}$$

We also need the following well-known Lemmas:

LEMMA 2.2. *Let $u \in W_1^1(B)$. Then*

$$|u(x) - u_B| \leqslant 2\int\limits_B \frac{|\nabla u(y)|}{|x-y|}\, dy.$$

For the proof see in [3, Lemma 7.16].

The following important inequality was derived in [4]:

LEMMA 2.3. *Let* $B = B(x,r) \subset \Omega$, $u \in C^\infty(\Omega)$, $\Gamma \subset \partial\Omega$. *Then*

$$\inf_{y\in\Gamma\cap B} \int_B \frac{|\nabla u(z)|}{|y-z|}\,dz \leqslant \int_B \frac{|\nabla u(z)|}{|x-z|}\,dz.$$

LEMMA 2.4. *If* $0 < \alpha < 2$ *and* $r > 0$, *then there exists a constant* c_2, $c_2 \leqslant \left(\frac{1}{2}\right)^{\alpha-2}$, *such that for each* $x \in \mathbb{R}^2$,

$$\int_{B(x,r)} \frac{|u(y)|}{|x-y|^{2-\alpha}}\,dy \leqslant c_2 r^\alpha M(u)(x).$$

For the proof we refer to [15, Lemma 2.8.3]. We will use this result for the case $\alpha = 1$. Here, as usual, $M(u)$ stands for the Hardy-Littlewood maximal operator. Moreover, the following important theorem (the Hardy-Littlewood theorem on Maximal Operator) will be used:

THEOREM 2.1. *If* $u \in L_2(\mathbb{R}^2)$, *then there exist a constant* $c_3 > 0$ *such that*

$$\|M(u)\|_2 \leqslant c_3 \|u\|_2.$$

For the proof see e.g. in [15, Theorem 2.8.2].

3. The main results

Consider the function

$$\rho_\varepsilon(x) = \begin{cases} \rho(x), & \text{if } x \in \Pi_1, \\ \rho(x) + (1-\delta)\frac{\varepsilon}{2}, & \text{if } x \in \Pi_2. \end{cases} \tag{3.1}$$

Our first main result is the following pointwise inequality:

THEOREM 3.1. *Let* $u_\varepsilon \in C^\infty(\Omega, \Gamma_\varepsilon)$. *Then there exist a constant* C, $C \leqslant 4$, *such that the pointwise inequality*

$$|u_\varepsilon(x)| \leqslant C\rho_\varepsilon(x) M_{\rho_\varepsilon(x)} |\nabla u_\varepsilon| \chi_{B(\overline{x},\rho_\varepsilon(x))}(x) \tag{3.2}$$

holds, where $\overline{x} \in \Gamma$ *is satisfying that* $|x-\overline{x}| = \rho(x)$.

Proof. Choose the point $x \in \Omega$ and denote by $B := B(\overline{x}, \rho_\varepsilon(x))$, where $\rho_\varepsilon(x)$ is defined in (3.1).

Then $B \cap \Gamma_\varepsilon \neq \emptyset$ for each $x \in \Omega$. Extend the function u_ε in $\mathbb{R}^2 \setminus \overline{\Omega}$ by reflecting it across the boundary. By applying Lemma 2.2 to the extended u_ε, we have for any $y \in B \cap \Gamma_\varepsilon$:

$$\begin{aligned} |u_\varepsilon(x)| &= |u_\varepsilon(x) - u_\varepsilon(y)| \leqslant |u_\varepsilon(x) - u_{\varepsilon B}| + |u_\varepsilon(y) - u_{\varepsilon B}| \\ &\leqslant 2\left(\int_B \frac{|\nabla u_\varepsilon(z)|}{|x-z|}\,dz + \int_B \frac{|\nabla u_\varepsilon(z)|}{|y-z|}\,dz \right). \end{aligned} \tag{3.3}$$

Hence, by applying Lemma 2.3 to (3.3), we obtain that

$$|u_\varepsilon(x)| \leqslant 2 \int_B \frac{|\nabla u_\varepsilon(z)|}{|x-z|}\, dz. \tag{3.4}$$

Finally, taking into account Lemma 2.4 and (3.1), we have that

$$\begin{aligned} |u_\varepsilon(x)| &\leqslant 2c_2\, \rho_\varepsilon(x)\, M_{\rho_\varepsilon(x)}(|\nabla u_\varepsilon|)(x) \\ &= \begin{cases} 4\rho(x)\, M_{\rho(x)}(|\nabla u_\varepsilon|), & x \in \Pi_1 \cap B(\overline{x}, \rho(x)), \\ 4\Big(\rho(x) + \frac{(1-\delta)}{2}\varepsilon\Big) M_{\rho(x)+\frac{(1-\delta)}{2}\varepsilon}(|\nabla u_\varepsilon|), & x \in \Pi_2 \cap B\Big(\overline{x}, \rho(x) + \frac{(1-\delta)}{2}\varepsilon\Big). \end{cases} \end{aligned} \tag{3.5}$$

The proof is complete. □

The next two theorems generalize some well-known classical Hardy-type for $p = 2$ inequalities to a much more wide class of functions.

THEOREM 3.2. *Let $\rho_\varepsilon(x)$ be the function defined in* (3.1) *and $0 \leqslant \alpha < \alpha_0$. Then the estimate*

$$\int_\Omega \rho_\varepsilon^{-2+\alpha}(x) u_\varepsilon^2(x)\, dx \leqslant C_1 \int_\Omega \rho_\varepsilon^{\alpha}(x) |\nabla u_\varepsilon(x)|^2\, dx \tag{3.6}$$

holds for each fixed ε for all functions $u_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon)$, where the constant C_1 does not depend on u_ε and on ε.

Proof. Fix $u_\varepsilon \in C^\infty(\Omega, \Gamma_\varepsilon)$ extended in $\mathbb{R}^n \setminus \overline{\Omega}$. According to (3.2) the inequality

$$\frac{|u_\varepsilon(x)|}{\rho_\varepsilon(x)} \leqslant 4 M_{\rho_\varepsilon(x)} \left(|\nabla u_\varepsilon| \chi_{B(\overline{x}, \rho_\varepsilon(x))} \right)(x)$$

holds for all $x \in \Omega$. Then we have that

$$\int_\Omega \frac{|u_\varepsilon(x)|^2}{\rho_\varepsilon^2(x)}\, dx \leqslant 16 \int_\Omega M^2_{\rho_\varepsilon(x)} \left(|\nabla u_\varepsilon| \chi_{B(\overline{x}, \rho_\varepsilon(x))}(x) \right) dx.$$

The statement in Theorem 3.1 implies that

$$\int_\Omega M^2_{\rho_\varepsilon(x)} \left(|\nabla u_\varepsilon| \chi_{B(\overline{x}, \rho_\varepsilon(x))} \right)(x)\, dx \leqslant c_3 \int_\Omega |\nabla u_\varepsilon(x)|^2\, dx.$$

Thus, it yields that

$$\int_\Omega \left(\frac{|u_\varepsilon(x)|}{\rho_\varepsilon(x)} \right)^2 dx \leqslant \mathscr{C}_1 \int_\Omega |\nabla u_\varepsilon(x)|^2\, dx,$$

where $\mathscr{C}_1 = 16c_3$.

Hence, the inequality (3.6) holds with $\alpha = 0$. The next step is to prove (3.6) for $\alpha > 0$. Choose $\sigma > 0$ and put $v_\varepsilon = |u_\varepsilon|\rho_\varepsilon^\sigma$. It is not difficult to derive that

$$\begin{aligned} |\nabla v_\varepsilon|^2 &= \left(\frac{\partial u_\varepsilon}{\partial x_1}\rho_\varepsilon^\sigma + \sigma\rho^{\sigma-1}\frac{\partial \rho_\varepsilon}{\partial x_1}u_\varepsilon\right)^2 + \rho_\varepsilon^{2\sigma}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 \\ &\leqslant 2\rho_\varepsilon^{2\sigma}|\nabla u_\varepsilon|^2 + 2\sigma^2\rho_\varepsilon^{2\sigma-2}u_\varepsilon^2. \end{aligned} \tag{3.7}$$

By now applying (3.6) with $\alpha = 0$ to v_ε, we obtain that

$$\int_\Omega \rho_\varepsilon^{-2+2\sigma}u_\varepsilon^2\,dx \leqslant \mathscr{C}_1\left(\int_\Omega \rho_\varepsilon^{2\sigma}|\nabla u_\varepsilon|^2\,dx + \sigma^2\int_\Omega \rho_\varepsilon^{2(\sigma-1)}|u_\varepsilon|^2\,dx\right).$$

If $1 - \mathscr{C}_1\sigma^2 > 0$, then we have that

$$\int_\Omega \rho_\varepsilon^{-2+2\sigma}u_\varepsilon^2\,dx \leqslant \frac{2\mathscr{C}_1}{1-2\mathscr{C}_1\sigma^2}\int_\Omega \rho_\varepsilon^{2\sigma}|\nabla u_\varepsilon|^2\,dx.$$

Substituting σ by $\frac{\alpha}{2}$ and denoting C_1 by $\frac{2\mathscr{C}_1}{1-2\mathscr{C}_1\sigma^2}$, we prove inequality (3.6). Finally, by approximating the functions from $H^1(\Omega,\Gamma_\varepsilon)$ by smooth functions belonging to $C^\infty(\Omega,\Gamma_\varepsilon)$, we can complete the proof. $\square$

Our final main result reads:

THEOREM 3.3. *Let* $\rho(x) = dist(x,\Gamma)$,

$$K := K(a,\varepsilon,\delta,\theta) = 4 + \frac{2}{\theta^2}\left(a^2\frac{1-\delta}{\delta} + \varepsilon^2(1-\delta)^2\right)$$

and $0 \leqslant \alpha < \alpha_0 := \sqrt{\frac{2}{K}}$. *Then the estimate*

$$\int_{\Omega^\theta} \rho^{-2+\alpha}(x)u_\varepsilon 2(x)\,dx \leqslant C(a,\varepsilon,\delta,\theta,\alpha)\int_\Omega \rho^\alpha(x)|\nabla u_\varepsilon(x)|^2\,dx \tag{3.8}$$

holds for each fixed $\theta > 0$ *for all functions* $u_\varepsilon \in H^1(\Omega,\Gamma_\varepsilon)$, *where the constant*

$$C(a,\varepsilon,\delta,\theta,\alpha) = \frac{4K}{2-K\alpha^2}.$$

Proof. First we note that

$$\int_{\Pi_1}\left(\frac{u_\varepsilon}{\rho}\right)^2 dx \leqslant 4\int_{\Pi_1}|\nabla u_\varepsilon|^2\,dx \leqslant 4\int_\Omega |\nabla u_\varepsilon|^2\,dx. \tag{3.9}$$

Indeed, if $u_\varepsilon(0) = 0$, then, according to the classical one-dimensional inequality (1.1), we have that

$$\int_0^a {u'_\varepsilon}^2\,dx_1 \geqslant \frac{1}{4}\int_0^a\left(\frac{u_\varepsilon}{x_1}\right)^2 dx_1.$$

Using this fact, we obtain that

$$\int_{\Pi_1} |\nabla u_\varepsilon|^2\,dx \geqslant \int_{\Gamma_\varepsilon}\int_0^a \left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 dx \geqslant \frac{1}{4}\int_{\Gamma_\varepsilon}\int_0^a \left(\frac{u_\varepsilon}{x_1}\right)^2 dx$$
$$= \frac{1}{4}\int_{\Pi_1} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx.$$

In particular, the estimate (3.9) holds in Π_1^θ. The next step is to prove that

$$\int_{\Pi_2^\theta} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx \leqslant \frac{K(a,\varepsilon,\delta)}{\theta^2}\int_\Omega |\nabla u_\varepsilon|^2\,dx. \tag{3.10}$$

Using the Friedrichs inequality (2.1) and taking into account the fact that $\rho(x) > \theta$ when $x \in \Pi_2^\theta$, we obtain the following estimate:

$$\int_{\Pi_2^\theta} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx \leqslant \frac{1}{\theta^2}\int_{\Pi_2} u_\varepsilon^2\,dx \leqslant \frac{K(a,\varepsilon,\delta)}{\theta^2}\int_\Omega |\nabla u_\varepsilon|^2\,dx.$$

Now, summarizing the inequalities (3.9) and (3.10), we derive the desired estimate:

$$\int_{\Omega^\theta} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx = \int_{\Pi_1^\theta} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx + \int_{\Pi_2^\theta} \left(\frac{u_\varepsilon}{\rho}\right)^2 dx$$
$$\leqslant \left(4 + \frac{K(a,\varepsilon,\delta)}{\theta^2}\right)\int_\Omega |\nabla u_\varepsilon|^2\,dx.$$

We have derived the inequality (3.8) for the case $\alpha = 0$. The proof of (3.8) for the case $\alpha > 0$ is identically to the proof of the second part of Theorem 3.2, so we omit the details. □

4. Concluding remarks

REMARK 4.1. The condition $\delta = 1$ in the definition (3.1) corresponds to the case $\Gamma_\varepsilon = \Gamma$.

REMARK 4.2. If $\alpha = 0$, then (3.6) takes the form

$$\int_\Omega \left(\frac{u_\varepsilon(x)}{\rho_\varepsilon(x)}\right)^2 dx \leqslant C_1\int_\Omega |\nabla u_\varepsilon(x)|^2\,dx, \tag{4.1}$$

while (3.8) becomes

$$\int_{\Omega^\theta} \left(\frac{u_\varepsilon(x)}{\rho(x)}\right)^2 dx \leqslant C(a,\varepsilon,\delta,\theta,\alpha)\int_\Omega |\nabla u_\varepsilon(x)|^2\,dx.$$

We conjecture that these Hardy-type inequalities holds also when $p = 2$ is replaced by any $p > 1$ but then another type of proof must be found.

REMARK 4.3. In this paper we have succeeded to prove a weighted Hardy-type inequality in a fixed domain for functions vanishing on a part of the boundary. We can see several open questions equipped with this result. For instance, one interesting problem is to try to find a weighted Hardy-type inequality for perforated domains in the case when the size of perforation depends on the small parameter.

Acknowledgments

It is a pleasure for me to thank Professor Lars-Erik Persson and the referee for various remarks and suggestions, which have improved the final version of the paper. I also thank Professor Gregory A. Chechkin for interesting discussions and help concerning the problem studied in this paper.

The work was partially supported by RFBR (project 09-01-00353) and by the grant of the President of Russian Federation for supporting "Leading Scientific Schools" and by Luleå University of Technology (Sweden).

REFERENCES

[1] G. A. CHECHKIN, YU. O. KOROLEVA AND L. -E. PERSSON, *On the precise asymptotics of the constant in the Friedrich's inequality for functions, vanishing on the part of the boundary with microinhomogeneous structure*, J. Inequal. Appl., **2007** (2007), Article ID 34138, 13 pages.

[2] G. A. CHECHKIN, YU. O. KOROLEVA, A. MEIDELL AND L. -E. PERSSON, *On the Friedrichs inequality in a domain perforated nonperiodically along the boundary. Homogenization procedure. Asymptotics in parabolic problems*, Russ. J. Math. Phys., **16**, 1 (2009), 1–16.

[3] D. GILBARG AND N. TRUDINGER, *Elliptic Partial Differenrial Equations of Second Order,* Springer-Verlag, 1983.

[4] P. HAJLASZ, *Pointwise Hardy inequalities*, Proc. Amer. Math. Soc., **127** (1999), 417–423.

[5] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Mathematical Library (Reprint of the 1952 edition ed.), Cambridge: Cambridge University Press, 1988.
Math. Anal., **109** (1934), 463–487, 685–713.

[6] I. S. KAC AND M. G. KREIN, *Criteria for discreteness of the spectrum of a singular string*, Izv. Vyss. Uchebn. Zaved. Mat., **2** (1958), 136–153.[in Russian]

[7] J. KINNUNEN AND R. KORTE, *Characterizations for Hardy's Inequality*, Around the Research of Vladimir Maz'ya, International Mathematical Series, **11** (2010), 239–254.

[8] V. KOKILASHVILI, A. MESKHI AND L.-E. PERSSON, *Weighted Norm Inequalities For Integral Transforms With Product Kernals*, Nova Science Publishers Inc (United States), 2009.

[9] A. KUFNER, *Weighted Sobolev Spaces*, John Wiley and Sons, 1985.

[10] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy Inequality. About its History and some Related Results*, Vydavetelsky Servis Publishing House, Pilsen, 2007.

[11] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey-London-Singapore-Hong Kong, 2003.

[12] V. G. MAZ'JA, *Sobolev spaces*, Translated from the Russian by T.O. Shaposhnikova, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.

[13] J. NEČAS, *Sur une méthode pour résourde les équations aux dérivées partielle du type elliptique, voisine de la variationelle*, Ann. Scuola Norm. Sup. Pisa, **16** (1962), 305–362.
[14] B. OPIC AND A. KUFNER, *Hardy-Type Inequalities*, Longman, Harlow, 1990.
[15] W. ZIEMER, *Weakly Differentiable Functions,* Graduate Texts in Mathematics 120, Springer-Verlag, 1989.
[16] A. WANNEBO, *Hardy Inequalities and Imbedding in Domains Generalizing $C^{0,\lambda}$ Domains*, Proc. Amer. Math. Soc., **122** (1994), 1181–1190.
Reprint of the sixth (1980) edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995.

(Received March 28, 2010)

Yu. O. Koroleva
Department of Differential Equations
Faculty of Mechanics and Mathematics
Moscow Lomonosov State University
Moscow 119991
Russia
&
Department of Mathematics
Luleå University of Technology
SE-971 87 Luleå, Sweden
e-mail: korolevajula@mail.ru