

AN OPERATOR EXTENSION OF THE PARALLELOGRAM LAW AND RELATED NORM INEQUALITIES

MOHAMMAD SAL MOSLEHIAN

Abstract. We establish a general operator parallelogram law concerning a characterization of inner product spaces, get an operator extension of Bohr's inequality and present several norm inequalities. More precisely, let \mathfrak{A} be a C^* -algebra, T be a locally compact Hausdorff space equipped with a Radon measure μ and let $(A_t)_{t \in T}$ be a continuous field of operators in \mathfrak{A} such that the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. If $\alpha : T \times T \rightarrow \mathbb{C}$ is a measurable function such that $\overline{\alpha(t,s)}\alpha(s,t) = 1$ for all $t, s \in T$, then we show that

$$\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)A_s|^2 d\mu(t)d\mu(s) + \int_T \int_T |\alpha(t,s)B_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s) \\ = 2 \int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s) - 2 \left| \int_T (A_t - B_t) d\mu(t) \right|^2.$$

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