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CHARACTERIZATION OF UNITARY OPERATORS BY ELEMENTARY OPERATORS AND UNITARILY INVARIANT NORMS

CRISTIAN CONDE

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Abstract. In this work we characterize unitary operators via inequalities of elementary operators with unitarily invariant norms.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space, and let $(B(\mathscr{H}), \|.\|)$ the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the usual norm. We denote by $Gl(\mathscr{H})$ the group of invertible elements of $B(\mathscr{H})$, $U(\mathscr{H})$ the unitary operators and $Gl_s(\mathscr{H})$ the set of all invertible and selfadjoint operators.

A linear operator $R : B(\mathscr{H}) \to B(\mathscr{H})$ defined by $R(X) = \sum_{i=1}^n A_i X B_i$, where $A_i, B_i \in B(\mathscr{H})$, with $1 \leqslant i \leqslant n$, is called an elementary operator on $B(\mathscr{H})$; and we denote by $R = R_{\overline{A},\overline{B}}$, where $\overline{A} = (A_1, \ldots, A_n)$ and $\overline{B} = (B_1, \ldots, B_n)$. This class of operators includes many important operators of $B(\mathscr{H})$ such as the inner derivation $\delta_A(X) = AX - XA$, the multiplication operator $M_{A,B}(X) = AXB$, the symmetrized two-sided multiplication $U_{A,B}(X) = AXB + BXA$ and the operator $V_{A,B} = AXB - BXA$. We denote by Φ_S the operator $U_{S,S^{-1}}$.

In [8], Nakamoto proved that a bounded linear operator A on $\mathscr{H}$ is normal if and only if $\|\delta_A(X)\|_2 = \|\delta_{A^*}(X)\|_2$ for all $X \in B_2(\mathscr{H})$ (Hilbert-Schmidt class). In [9], Seedik characterizes the operators S for which the Corach-Porta-Recht inequality ([4], [1]) holds, more precisely he proved that an invertible operator S is a non zero complex multiple of some selfadjoint operator if and only if $\|\Phi_S(X)\| \geqslant 2\|X\|$ for all $X \in B(\mathscr{H})$.

On the other hand, in [7], the authors ask whether the same characterization obtained on [9] is true for other unitarily invariant norms. They proved that S is necessarily a normal operator if $2\|X\|_{\mathscr{I}} \leqslant \|\Phi_S(X)\|_{\mathscr{I}}$ for all $X \in B(\mathscr{H})$, with rank one (Corollary 2.2). Furthermore, in this work Magajna et al. obtained that if $\mathscr{I}$ a norm ideal and we denote by $\overline{A} = (tS, \frac{1}{t}S^{-1})$ and $\overline{B} = (S^{-1}, S)$ for $t > 0$, then

$$\gamma S \in Gl_s(\mathscr{H}), \gamma \in \mathbb{C} - \{0\} \text{ if and only if } \inf_{t>0} \|R_{\overline{A},\overline{B}}(X)\|_{\mathscr{I}} \geqslant 2\|X\|_{\mathscr{I}}$$

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for all $X \in \mathscr{I}$ of rank 1.

In a recent work [11], Seedik obtains some characterizations of some subclasses of normal operators in $B(\mathscr{H})$ by inequalities or equalities (associated with elementary operators). Motivated by these results, in [3] we characterized the normal invertible operators of $B(\mathscr{H})$ via unitarily invariant norms and elementary operators.

The purpose of this work is to find the set for a given norm ideal $\mathscr{I}$

$$E_{\mathscr{I}} = \{S \in Gl(\mathscr{H}) : \|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}} \text{ for every } X \in \mathscr{I}\}.$$

By a result obtained in [3] this subset is contained in $N(\mathscr{H}) = \{T \in B(\mathscr{H}) : TT^* = T^*T\}$.

2. Preliminaries

We recall that $\mathscr{I}$ is a norm ideal of $B(\mathscr{H})$ if $\mathscr{I}$ is a two-sided ideal of $B(\mathscr{H})$ and a Banach space with respect to the norm $\|.\|_{\mathscr{I}}$ satisfying:

1. $\|XTY\|_{\mathscr{I}} \leqslant \|X\|\|T\|_{\mathscr{I}}\|Y\|$ for $T \in \mathscr{I}$ and $X, Y \in B(\mathscr{H})$,
2. $\|X\|_{\mathscr{I}} = \|X\|$ if X is of rank one.

In particular, condition 1. implies that the norm is unitarily invariant, $\|UXV^*\|_{\mathscr{I}} = \|X\|_{\mathscr{I}}$ for $X \in \mathscr{I}$ and any $U, V \in U(\mathscr{H})$. The most known examples of norm ideals of $B(\mathscr{H})$ are the so called p-Schatten class with $p \geqslant 1$ defined by

$$B_p(\mathscr{H}) = \{X \in B_0(\mathscr{H}) : \{s_j(X)\} \in l^p\},$$

where $\{s_j(X)\}$ denotes the sequence of singular values of X, rearranged such that $s_1(X) \geqslant s_2(X) \geqslant \cdots$ with multiplicies counted, with norm given by $\|X\|_p^p = \sum s_j(X)^p$ and $B_0(\mathscr{H})$ is the ideal of compact operators. When $p = \infty$, the norm $\|.\|_\infty$ coincides with the usual norm $\|X\| = s_1(X)$. For a complete account of the theory of unitarily invariant norms the reader is referred to [5].

For sake of completness, we recall three well-known statements that we will use in the following section. Given a norm ideal $\mathscr{I}$ and a linear operator $P : \mathscr{I} \to \mathscr{I}$ we denote by

$$\|P\|_{B(\mathscr{I})} = \sup\{\|P(X)\|_{\mathscr{I}} : \|X\|_{\mathscr{I}} = 1\}.$$

THEOREM 1. ([10], Theorem 2.1.) *Let $S \in B(\mathscr{H})$ be an invertible and selfadjoint operator and $\mathscr{I}$ a norm ideal. Then we have the following inequality:*

$$\|\Phi_S\|_{B(\mathscr{I})} \geqslant \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}. \tag{1}$$

LEMMA 1. *Let $S \in Gl(\mathscr{H})$. Then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$, for some unitary operator V.*

THEOREM 2. ([3], Theorem 2.1.) *Let $S \in Gl(\mathscr{H})$ and $\mathscr{I}$ a norm ideal. Then the following conditions are equivalent:*

1. *S is normal,*
2. $\|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} = \|M_{S^*,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,
3. $\|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} \geqslant 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,
4. $\|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} \geqslant 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$ *of rank one.*

REMARK 1. Other characterization of normal invertible operators is given in the following statement.

PROPOSITION 1. *Let $S \in Gl(\mathscr{H})$ and $\mathscr{I}$ a norm ideal. Then the following conditions are equivalent:*

1. *S is normal,*
2. $\|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} \leqslant \|M_{S^*,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,
3. $\|M_{S,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S}(X)\|_{\mathscr{I}} \leqslant \|M_{S^*,S^{-1}}(X)\|_{\mathscr{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$ *of rank one.*

Proof. The implication *1* $\Rightarrow$ *2* follows inmediately from Theorem 2 and *2* $\Rightarrow$ *3* is trivial. *3* $\Rightarrow$ *1* We consider rank one operators $X = x \otimes y \in \mathscr{I}$ defined by $(x \otimes y)z = \langle z, y\rangle x$ for $x, y, z \in \mathscr{H}$, then it follows that the following inequality holds

$$\|M_{S,S^{-1}}(x \otimes y)\|_{\mathscr{I}} + \|M_{S^{-1},S}(x \otimes y)\|_{\mathscr{I}} \leqslant \|M_{S^*,S^{-1}}(x \otimes y)\|_{\mathscr{I}} + \|M_{S^{-1},S^*}(x \otimes y)\|_{\mathscr{I}}$$

or equivalently,

$$\|S(x \otimes y)S^{-1}\|_{\mathscr{I}} + \|S^{-1}(x \otimes y)S\|_{\mathscr{I}} \leqslant \|S^*(x \otimes y)S^{-1}\|_{\mathscr{I}} + \|S^{-1}(x \otimes y)S^*\|_{\mathscr{I}}.$$

It is easy to see that $A(u \otimes v)B = Au \otimes B^*v$ and $\|u \otimes v\|_{\mathscr{I}} = \|u \otimes v\| = \|u\|\|v\|$, for all $u, v \in \mathscr{H}$ and $A, B \in B(\mathscr{H})$. Then

$$\|S(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S^*(y)\| \leqslant \|S^*(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S(y)\|. \tag{2}$$

Assume that S is not a normal operator. Consequently there exist a vector $x \in \mathscr{H}$, $\|x\| = 1$ such that $\|Sx\| > \|S^*x\|$ (or $\|Sx\| < \|S^*x\|$). It follows, from (2), that for all $y \in \mathscr{H}$ with $\|y\| = 1$, $\|Sy\| > \|S^*y\|$ (or $\|Sy\| < \|S^*y\|$), so we have from (2) that

$$\begin{aligned} 0 < (\|Sx\| - \|S^*x\|) &\leqslant (\|Sy\| - \|S^*y\|)\|(S^{-1})(x)\|\|(S^*)^{-1}(y)\|^{-1} \\ &\leqslant (\|Sy\| - \|S^*y\|)\|S^{-1}\|\|S^*\|. \end{aligned}$$

Hence, for all $y \in \mathscr{H}$ with $\|y\| = 1$

$$\|Sx\| + \|S^{-1}\|\|S\|\|S^*y\| \leqslant \|S^*x\| + \|S^{-1}\|\|S\|\|Sy\|$$

Thus $\|Sx\| + \|S^{-1}\|\|S\|\|S^*\| \leqslant \|S^*x\| + \|S^{-1}\|\|S\|\|S\|$. It follows that $\|Sx\| \leqslant \|S^*x\|$, which it is a contradiction. Therefore S is a normal operator. $\square$

In [6], Kittaneh obtained the generalization of the Corach-Porta-Recht inequality in any norm ideal $\mathscr{I}$. More precisely, for Hilbert-space operators T,R,X with T,R invertible operators and a unitarily invariant norm $\mathscr{I}$, the inequality

$$2\|X\|_{\mathscr{I}} \leqslant \|R^*XT^{-1} + R^{-1}XT^*\|_{\mathscr{I}}, \tag{3}$$

holds for all $X \in \mathscr{I}$.

In this work, we consider the polar decomposition of $S \in Gl(\mathscr{H})$ given by $S = U|S|$ with $|S| = (S^*S)^{1/2}$ positive and $U \in U(\mathscr{H})$.

3. Main results

PROPOSITION 2. *If $S \in Gl_s(\mathscr{H})$ and $\|\Phi_S(X)\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ for all $X \in \mathscr{I}$, then $S = \|S\|V$ with $V \in U(\mathscr{H})$.*

Proof. Since $\|\Phi_S(X)\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ for all $X \in \mathscr{I}$, it follows that $\|\Phi_S\|_{B(\mathscr{I})} \leqslant 2$. By (1) we have that $2 \geqslant \|\Phi_S\|_{B(\mathscr{I})} \geqslant \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} \geqslant 2$.

From this we derive that $\|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} = 2$, so it follows immediately that $\|S\|\|S^{-1}\| = 1$. So from Lemma 2 it turns out that $S = \|S\|V$ with V an unitary operator and this finishes the proof. □

Now, we obtain a generalization of [11], Th. 8.

COROLLARY 1. *Let $\mathscr{I}$ a norm ideal, then*

$$U_s(\mathscr{H}) = \{S \in Gl_s(\mathscr{H}) : \|S\| = 1 \text{ and } \|\Phi_S(X)\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\}.$$

where $U_s(\mathscr{H})$ denotes the unitary selfadjoint operators in $B(\mathscr{H})$.

Proof. If $S \in U_s(\mathscr{H})$ then $S \in Gl_s(H), \|S\| = 1$ and for any $X \in \mathscr{I}$ we have

$$\|\Phi_S(X)\|_{\mathscr{I}} = \|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}},$$

by the unitary invariance of the norm. Thus, the result follows immediately from Proposition 2. □

In the previous statement, if we omit the hypothesis $\|S\| = 1$ we obtain a characterization of $\mathbb{R}^*U_s(\mathscr{H})$, with $\mathbb{R}^* = \mathbb{R} - \{0\}$.

If $S = U|S| \in U(\mathscr{H})$ then necessarily $|S| \in U_s(\mathscr{H})$ and in consequence $|S|$ is characterized by the previous corollary. In the following result we prove that a condition which holds for the modules of S is a sufficient condition for determinate if S is an unitary operator in $B(\mathscr{H})$.

THEOREM 3. *Let $\mathscr{I}$ a norm ideal then*

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1, S = U|S| \text{ and } \|\Phi_{|S|}(X)\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\}.$$

Proof. By the hypothesis, $\|\Phi_{|S|}\|_{B(\mathscr{I})} \leqslant 2$. By (1) we have the following lower bound for the operator $\Phi_{|S|}$,

$$\|\Phi_{|S|}\|_{B(\mathscr{I})} \geqslant \| \, |S| \, \| \, \| \, |S|^{-1} \, \| + \frac{1}{\| \, |S| \, \| \, \| \, |S|^{-1} \, \|} \geqslant 2.$$

From this inequality and the condition obtained above, we get that

$$\|\Phi_{|S|}\|_{B(\mathscr{I})} = \| \, |S| \, \| \, \| \, |S|^{-1} \, \| + \frac{1}{\| \, |S| \, \| \, \| \, |S|^{-1} \, \|} = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} = 2.$$

In other words, $\|S\|\|S^{-1}\| = 1$. Then the result follows immediately from the Lemma 1.

On the other hand, if $S = U|S| \in U(\mathscr{H})$ then for any $X \in \mathscr{I}$ we may write $X = U^*Y$ with $Y \in \mathscr{I}$ and since $\|S\| = 1$ we have that

$$\begin{aligned}\|\Phi_{|S|}(X)\|_{\mathscr{I}} &= \||S|U^*Y|S|^{-1} + |S|^{-1}U^*Y|S|\|_{\mathscr{I}} = \|S^*YS^{-1} + S^{-1}YS^*\|_{\mathscr{I}} \leqslant 2\|Y\|_{\mathscr{I}} \\ &= 2\|X\|_{\mathscr{I}}.\end{aligned}$$

This concludes the proof. □

Since the norm of the operators $S^*YS^{-1} + S^{-1}YS^*, SXS^{-1} + (S^*)^{-1}XS^*$ and $\Phi_{|S|}$ are related via unitaries operators, more precisely for all $X \in \mathscr{I}$

$$\|\Phi_{|S|}(X)\|_{\mathscr{I}} = \||S|X|S|^{-1} + |S|^{-1}X|S|\|_{\mathscr{I}} = \|S^*(UX)S^{-1} + S^{-1}(UX)S^*\|_{\mathscr{I}}$$

and

$$\|\Phi_{|S|}(X)\|_{\mathscr{I}} = \|U^*(SXS^{-1} + (S^*)^{-1}XS^*)U\|_{\mathscr{I}} = \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathscr{I}}$$

where $S = U|S|$, we obtain the following characterizations of $U(\mathscr{H})$.

THEOREM 4. *Let $\mathscr{I}$ a norm ideal then*

$$\begin{aligned}U(\mathscr{H}) &= \{S \in Gl(\mathscr{H}) : \|S\| = 1, \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1, \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1, \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1, \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}\}.\end{aligned}$$

REMARK 2. The operator which characterizes the unitary operators of $B(\mathscr{H})$ can be written as follows

$$SXS^{-1} + (S^*)^{-1}XS^* = SXS^{-1} + (S^{-1})^*XS^* = SXS^{-1} + (SX^*S^{-1})^*$$

in particular if X is a selfadjoint operator then

$$SXS^{-1} + (S^*)^{-1}XS^* = 2Re(SXS^{-1}),$$

where $Re(T) = \frac{1}{2}(T + T^*)$.

A natural question is if with the selfadjoint operators of $\mathscr{I}$ we can describe all $U(\mathscr{H})$, more precisely

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_{\mathscr{I}} \leqslant \|X\|_{\mathscr{I}} \text{ for all } X \in \mathscr{I}, X = X^*\}?$$

We give a particular example when $\mathscr{I}$ is the 2-Schatten ideal. Every $X \in B(\mathscr{H})$ can be written as $X = Re(X) + iIm(X)$, where $Re(X)$ (defined above), $Im(X)$ are selfadjoint operators and $Im(X) = \frac{1}{2i}(X - X^*)$. This decomposition is called the Cartesian decomposition of X.

In [2], Bhatia and Kittaneh prove sharp inequalities comparing the norm $\|X\|_p$ with $(\|Re(X)\|_p^2 + \|Im(X)\|_p^2)$. More precisely, for $p = 2$ we get

$$\|Re(X)\|_2^2 + \|Im(X)\|_2^2 = \|X\|_2^2.$$

THEOREM 5.

$$\begin{aligned} U(\mathscr{H}) &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_2 \leqslant \|X\|_2 \text{ for all } X \in B_2^s(\mathscr{H})\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_2 = \|X\|_2 \text{ for all } X \in B_2^s(\mathscr{H})\}. \end{aligned}$$

where $B_2^s(\mathscr{H})$ denotes the selfadjoint operators in $B_2(\mathscr{H})$

Proof. Let $Z = Re(Z) + iIm(Z) \in B_2(\mathscr{H})$ then

$$\begin{aligned} \|SZS^{-1} + (S^*)^{-1}ZS^*\|_2^2 &= \|S(Re(Z) + iIm(Z))S^{-1} + (S^*)^{-1}(Re(Z) + iIm(Z))S^*\|_2^2 \\ &= \|2(Re(SRe(Z)S^{-1}) + iRe(SIm(Z)S^{-1}))\|_2^2 \\ &= 4\|Re(SRe(Z)S^{-1}) + iRe(SIm(Z)S^{-1})\|_2^2 \\ &= 4(\|Re(SRe(Z)S^{-1})\|_2^2 + \|Re(SIm(Z)S^{-1})\|_2^2) \\ &\leqslant 4(\|Re(Z)\|_2^2 + \|Im(Z)\|_2^2) = 4\|Z\|_2^2. \qquad \square \end{aligned}$$

We observe that if $S = \lambda V$ with $V \in U(\mathscr{H})$ and $\lambda \in \mathbb{R}^*$, then for every $X \in \mathscr{I}$

$$\|SXS^{-1}\|_{\mathscr{I}} + \|S^{-1}XS\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}. \tag{4}$$

Now motivated by the conclusion of Proposition 2 and the equality (4), we characterize the real multiples of some unitary operator.

THEOREM 6. *Let $S \in Gl(\mathscr{H})$ and $\mathscr{I}$ a norm ideal. Then the following conditions are equivalent:*

1. *$S = \lambda V$ with $\lambda \in \mathbb{R}^*$ and $V \in U(\mathscr{H})$,*
2. *$\|SXS^{-1}\|_{\mathscr{I}} + \|S^{-1}XS\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ for every $X \in \mathscr{I}$,*
3. *$\|S^*XS^{-1}\|_{\mathscr{I}} + \|S^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ for every $X \in \mathscr{I}$,*
4. *$\|S^*XS^{-1}\|_{\mathscr{I}} + \|S^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ for every $X \in \mathscr{I}$,*

5. $\|S^*XS^{-1}+S^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

6. $\|S^*XS^{-1}+S^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

7. $\|S^*XS^{-1}+S^{-1}XS^*\|_{\mathscr{I}} = 2$ *for every* $X \in \mathscr{I}$, $\|X\|_{\mathscr{I}} = 1$.

8. $\|SXS^{-1}\|_{\mathscr{I}} + \|(S^*)^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

9. $\|SXS^{-1}\|_{\mathscr{I}} + \|(S^*)^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

10. $\|SXS^{-1}+(S^*)^{-1}XS^*\|_{\mathscr{I}} \leqslant 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

11. $\|SXS^{-1}+(S^*)^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,

12. $\|SXS^{-1}+(S^*)^{-1}XS^*\|_{\mathscr{I}} = 2$ *for every* $X \in \mathscr{I}$, $\|X\|_{\mathscr{I}} = 1$.

Proof. The implications *1.* $\Rightarrow$ *2.*, *3.* $\Rightarrow$ *4.*, *4.* $\Rightarrow$ *5.* and *6.* $\Rightarrow$ *7.* are trivial.

2. $\Rightarrow$ *3.* This implication is a consequence of the unitary invariance of the norm and the fact that S is a normal operator (see Th. 2).

5. $\Rightarrow$ *6.* Let $X \in \mathscr{I}$ then

$$2\|X\|_{\mathscr{I}} \geqslant \quad \|S^*XS^{-1}+S^{-1}XS^*\|_{\mathscr{I}} = \|\Phi_{|S|}(U^*X)\|_{\mathscr{I}} \geqslant 2\|U^*X\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}},$$

in the last inequality we use (3).

7. $\Rightarrow$ *1.* By the hypothesis and the inequality (3) we get that $\|\Phi_{|S|}\|_{B(\mathscr{I})} = 2$. In other words, $\|S\|\|S^{-1}\| = 1$. Then the result follows immediately.

We actually showed that the first seven conditions are equivalent. With a similar argument (using (3)) we obtain that the conditions *2*, *8*, *9*, *10*, *11* and *12* are also equivalent and this concludes the proof $\square$

REMARK 3.

1. This theorem is a generalization of [11], Th. 6.

2. If $\mathscr{I} = \mathscr{I}_\phi$ is a norm ideal associated with a ϕ regular symmetric norming function (we refer to [5] for details on norm ideals generated by a symmetric norming function), that is

$$\lim_{n\to\infty} \phi(\xi_{n+1}, \xi_{n+2}, ..) = 0, \tag{5}$$

or the equivalent condition $\mathscr{I}_\phi^{(0)} = \mathscr{I}_\phi$ where $\mathscr{I}_\phi^{(0)}$ denotes the closure of th ideal of finite rank operators, $B_{0,0}(\mathscr{H})$, with respect to the norm $\|.\|_{\mathscr{I}}$, then in the previous results we can relax the hypothesis for all $X \in B_{0,0}(\mathscr{H})$. For example, the ideal $B_p(\mathscr{H})$ with $1 \leqslant p \leqslant \infty$ satisfies the condition (5).

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Cristian Conde
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento
J. M. Gutierrez 1150 (B1613GSX) Los Polvorines.
and
Instituto Argentino de Matemática "Alberto P. Calderón"
Saavedra 15 3ş piso
(C1083ACA) Buenos Aires
Argentina
e-mail: cconde@ungs.edu.ar