

HÖLDER MEAN INEQUALITIES FOR THE GENERALIZED GRÖTZSCH RING AND HERSCH-PFLUGER DISTORTION FUNCTIONS

SONG-LIANG QIU, YE-FANG QIU, MIAO-KUN WANG AND YU-MING CHU

(Communicated by J. Pečarić)

Abstract. In this paper, we study the monotonicity properties of the generalized elliptic integrals and establish two Hölder mean inequalities for the generalized Grötzsch ring function $\mu_a(r)$ and the generalized Hersch-Pfluger distortion function $\varphi_K^a(r)$.

1. Introduction

For real numbers a , b and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad |x| < 1. \quad (1.1)$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) denotes the shifted factorial function

$$(a, n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n = 1, 2, \dots$. For a survey of these functions, see [6].

Recently, the Gaussian hypergeometric function $F(a, b; c; x)$ has attracted the attention of many mathematicians. In particular, many important properties and remarkable inequalities can be found in the literature [5, 9, 12, 15, 17-19].

For $r \in (0, 1)$, $r' = \sqrt{1-r^2}$ and $a \in (0, 1)$, the generalized elliptic integrals [3] are defined by

$$\begin{cases} \mathcal{H}_a = \mathcal{H}_a(r) = \pi F(a, 1-a; 1; r^2)/2, \\ \mathcal{H}_a' = \mathcal{H}_a'(r) = \mathcal{H}_a(r'), \\ \mathcal{H}_a(0) = \pi/2, \quad \mathcal{H}_a(1) = \infty \end{cases} \quad (1.2)$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \pi F(a-1, 1-a; 1; r^2)/2, \\ \mathcal{E}_a' = \mathcal{E}_a'(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \pi/2, \quad \mathcal{E}_a(1) = \sin(\pi a)/[2(1-a)]. \end{cases} \quad (1.3)$$

Mathematics subject classification (2010): 30C62, 33E05.

Keywords and phrases: Generalized elliptic integrals; generalized Grötzsch ring function; generalized Hersch-Pfluger distortion function; Hölder mean.

This research is supported by the Natural Science Foundation of China (No. 11071069), Natural Science Foundation of Zhejiang Province (Nos. D7080080, Y6100170) and Innovation Team Foundation of the Department of Education of Zhejiang Province (No. T200924).

In particular, if $a = 1/2$, then the functions $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ reduce to $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively, which are the well-known complete elliptic integrals of the first and second kinds, respectively [1, 2, 7, 10, 13]. By symmetry of (1.2), unless stated otherwise, we assume that $a \in (0, 1/2]$. For $a \in (0, 1/2]$ and $r \in (0, 1)$ define

$$m_a(r) = \frac{2}{\pi \sin \pi a} r'^2 \mathcal{K}_a(r) \mathcal{K}'_a(r) \tag{1.4}$$

and

$$\mu_a(r) = \frac{\pi}{2 \sin \pi a} \frac{\mathcal{K}'_a(r)}{\mathcal{K}_a(r)}. \tag{1.5}$$

Moreover, if $a = 1/2$, then $m_{1/2}(r) = m(r)$ and $\mu_{1/2}(r) = \mu(r)$, where $\mu(r)$ is the modulus of the plane Grötzsch ring $\mathbb{B}^2 \setminus [0, r]$ and \mathbb{B}^2 is the unit disk.

It is well known that the Ramanujan’s generalized modular equation with signature $1/a$ and degree p is

$$\mu_a(s) = p\mu_a(r), \tag{1.6}$$

and the solution of equation (1.6) is given by

$$s = \varphi_K^a(r) = \mu_a^{-1}(\mu_a(r)/K). \tag{1.7}$$

In the special case of $a = 1/2$, the solution $\varphi_K^a(r)$ of equation (1.6) reduce to the Hersch-Pfluger distortion function $\varphi_K(r)$, which has many important applications in the theory of plane quasiconformal mappings [4, 8, 11, 16, 20]. As usual, we call $\varphi_K^a(r)$ the generalized Hersch-Pfluger distortion function.

For $\lambda \in \mathbb{R}$, the Hölder mean $M_\lambda(x, y)$ of order λ of two positive numbers x and y is defined by

$$M_\lambda(x, y) = \begin{cases} (\frac{x^\lambda + y^\lambda}{2})^{1/\lambda}, & \lambda \neq 0, \\ \sqrt{xy}, & \lambda = 0. \end{cases} \tag{1.8}$$

The main properties of the Hölder mean are given in [14].

In [7, Theorems 5.12 and 10.28], the authors presented the the following Hölder mean inequalities

$$M_0(\mu(x), \mu(y)) \leq \mu(M_0(x, y)) \tag{1.9}$$

and

$$M_0(\varphi_K(x), \varphi_K(y)) \leq \varphi_K(M_0(x, y)) \tag{1.10}$$

for all $x, y \in (0, 1)$ and $K > 1$.

The purpose of this paper is to generalize the inequalities (1.9) and (1.10) to the generalized Grötzsch ring function $\mu_a(r)$ and the generalized Hersch-Pfluger distortion function $\varphi_K^a(r)$ for any $a \in (0, 1/2]$.

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$, $0 < a \leq 1/2$ and $s = \varphi_K^a(r)$, the following derivative formulas were presented in [3, Theorem 4.1].

$$\begin{aligned} \frac{d\mathcal{K}_a}{dr} &= 2(1-a)\frac{\mathcal{E}_a - r'^2\mathcal{K}_a}{r'^2}, & \frac{d\mathcal{E}_a}{dr} &= 2(1-a)\frac{\mathcal{E}_a - \mathcal{K}_a}{r}, \\ \frac{dm_a(r)}{dr} &= \frac{2}{\pi r \sin \pi a} \left[\frac{\sin \pi a}{2} - 4(1-a)\mathcal{K}_a\mathcal{E}_a' + 2(1-2a)r^2\mathcal{K}_a\mathcal{K}_a' \right], \\ \frac{d\mu_a(r)}{dr} &= -\frac{\pi^2}{4r'^2\mathcal{K}_a^2}, \\ \frac{d\varphi_K^a(r)}{dr} &= \frac{ss'\mathcal{K}_a(s)\mathcal{K}_a'(s)}{r'^2\mathcal{K}_a(r)\mathcal{K}_a'(r)}. \end{aligned}$$

LEMMA 2.1. (See [3, Lemmas 5.2 and 5.4])

- (1) $(\mathcal{E}_a - r'^2\mathcal{K}_a)/r^2$ is strictly increasing from $(0, 1)$ onto $((\pi a)/2, [\sin(\pi a)]/[2(1-a)])$;
- (2) $(\mathcal{E}_a - r'^2\mathcal{K}_a)/(r^2\mathcal{K}_a)$ is strictly decreasing from $(0, 1)$ onto $(0, a)$;
- (3) $r'^2\mathcal{K}_a$ is strictly decreasing from $(0, 1)$ onto $(0, \pi/2)$.

LEMMA 2.2. Let λ be a real number. Then the function

$$h(r) = \frac{\mu_a(r)^{\lambda-1}}{r^\lambda r'^2 \mathcal{K}_a(r)^2}$$

is strictly increasing on $(0, 1)$ if and only if $\lambda \leq 0$.

Proof. By logarithmic differentiation,

$$\begin{aligned} \frac{h'(r)}{h(r)} &= (\lambda - 1)\frac{1}{\mu_a(r)}\left(-\frac{\pi^2}{4rr'^2\mathcal{K}_a^2}\right) - \frac{\lambda}{r} + \frac{2r}{r'^2} - \frac{4(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{r'^2\mathcal{K}_a} \\ &= \frac{\pi \sin \pi a + 2r'^2\mathcal{K}_a\mathcal{K}_a'}{2rr'^2\mathcal{K}_a\mathcal{K}_a'} [h_1(r) - \lambda], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} h_1(r) &= \frac{\pi \sin \pi a + 4r^2\mathcal{K}_a\mathcal{K}_a' - 8(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)\mathcal{K}_a'}{\pi \sin \pi a + 2r'^2\mathcal{K}_a\mathcal{K}_a'} \\ &= \frac{\pi \sin \pi a + 4r^2\mathcal{K}_a'\mathcal{K}_a[1 - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/(r^2\mathcal{K}_a)]}{\pi \sin \pi a + 2r'^2\mathcal{K}_a\mathcal{K}_a'}. \end{aligned}$$

From Lemma 2.1 we clearly see that $h_1(r) > 0$ and strictly increasing from $(0, 1)$ onto $(0, +\infty)$. Therefore, Lemma 2.2 follows from (2.1). \square

LEMMA 2.3. *Let $0 < r < 1$. Then the function $g(r) = 4(1-a)\mathcal{E}_a - 2(1-2a)r^2\mathcal{K}_a$ is strictly decreasing from $(0, 1)$ onto $(2[\sin(\pi a)], \pi)$.*

Proof. Making use of series expansion, we have

$$\begin{aligned}
 g(r) &= \frac{\pi}{2} \left[4(1-a) \sum_{n=0}^{\infty} \frac{(a-1, n)(1-a, n)}{(n!)^2} r^{2n} - 2(1-2a) \sum_{n=0}^{\infty} \frac{(a, n)(1-a, n)}{(n!)^2} r^{2n} \right. \\
 &\quad \left. + 2(1-2a) \sum_{n=0}^{\infty} \frac{(a, n)(1-a, n)}{(n!)^2} r^{2n+2} \right] \\
 &= \pi \left[1 - \sum_{n=0}^{\infty} A_n \frac{(a, n)(1-a, n)}{[(n+1)!]^2} r^{2n+2} \right], \tag{2.2}
 \end{aligned}$$

where

$$A_n = (2a^2 + 2a + 1)n + (3a^2 - 3a + 1) > 0 \tag{2.3}$$

for $a \in (0, 1/2]$ and $n = 0, 1, 2, \dots$. Therefore, Lemma 2.3 follows from (2.2) and (2.3) together with (1.3) and Lemma 2.1(3). \square

LEMMA 2.4. *For $0 < r < 1$, we have*

$$\begin{aligned}
 &8(1-a)^2\mathcal{E}_a\mathcal{E}'_a - 4(1-a)[2(1-a) - (1-2a)r^2]\mathcal{K}_a\mathcal{E}'_a \\
 &+ [4(4a^2 - 5a + 2) - 4(1-2a)^2r^2]r^2\mathcal{K}_a\mathcal{K}'_a \\
 &- 4(1-2a)(1-a)r^2\mathcal{K}'_a\mathcal{E}_a > 0. \tag{2.4}
 \end{aligned}$$

Proof. Let

$$f(r) = \mathcal{K}_a[4(1-a)\mathcal{E}'_a - 2(1-2a)r^2\mathcal{K}'_a]. \tag{2.5}$$

From Lemma 2.3 we know that $f(r) > 0$ and strictly increasing on $(0, 1)$. By differentiation we get

$$f'(r) = \frac{f_1(r)}{rr^2}, \tag{2.6}$$

where

$$\begin{aligned}
 f_1(r) &= 8(1-a)^2\mathcal{E}_a\mathcal{E}'_a - 4(1-a)[2(1-a) - (1-2a)r^2]\mathcal{K}_a\mathcal{E}'_a \\
 &+ [4(4a^2 - 5a + 2) - 4(1-2a)^2r^2]r^2\mathcal{K}_a\mathcal{K}'_a \\
 &- 4(1-2a)(1-a)r^2\mathcal{K}'_a\mathcal{E}_a. \tag{2.7}
 \end{aligned}$$

Hence, Lemma 2.4 follows from (2.5)-(2.7) and the monotonicity of $f(r)$. \square

LEMMA 2.5. *Let $0 < r < 1$, and let $f(r)$ and $f_1(r)$ be defined as in Lemma 2.4, and $H(r) = f_1(r)/\{r^2[f(r) - (\pi \sin(\pi a))/2]\}$. Then the range of $H(r)$ is $(0, +\infty)$.*

Proof. From Lemma 2.3 we clearly see that $f(r) - [\pi \sin(\pi a)]/2$ is strictly increasing from $(0, 1)$ onto $([\pi \sin(\pi a)]/2, +\infty)$. Then $f_1(r) > 0$ leads to the conclusion that $H(r) > 0$ for $r \in (0, 1)$.

Noting that $f_1(r)$ can be rewritten as

$$\begin{aligned}
 f_1(r) = & 8(1-a)^2 \mathcal{E}_a \mathcal{E}'_a - 4(1-a)r^2 \mathcal{K}_a \left(\frac{\mathcal{E}'_a - r^2 \mathcal{K}'_a}{r^2} \right) \\
 & - 4(1-2a)(1-a)r^2 \mathcal{K}_a \mathcal{E}'_a + 4(1-2a)^2 r^2 r'^2 \mathcal{K}_a \mathcal{K}'_a \\
 & - 4(1-2a)(1-a)r^2 \mathcal{K}'_a \mathcal{E}_a.
 \end{aligned} \tag{2.8}$$

Equations (2.7) and (2.8) together with Lemma 2.1 imply that

$$\lim_{r \rightarrow 0} f_1(r) = 0 \tag{2.9}$$

and

$$\lim_{r \rightarrow 1} f_1(r) = \pi \sin \pi a. \tag{2.10}$$

Moreover,

$$\lim_{r \rightarrow 0} \frac{1}{r'^2 \{f(r) - [\pi \sin(\pi a)]/2\}} = \frac{2}{\pi \sin \pi a} \tag{2.11}$$

and

$$\lim_{r \rightarrow 1} \frac{1}{r'^2 \{f(r) - [\pi \sin(\pi a)]/2\}} = +\infty. \tag{2.12}$$

Therefore, Lemma 2.5 follows from (2.9)-(2.12) and $H(r) > 0$. \square

LEMMA 2.6. *If λ is a real number and $f(r)$ is defined as in Lemma 2.4, then the function $G(r) = \lambda r'^2 \mathcal{K}_a(r) \mathcal{K}'_a(r) - f(r)$ is strictly decreasing on $(0, 1)$ if and only if $\lambda \geq 0$, and $G(r)$ is not monotone on $(0, 1)$ for $\lambda < 0$.*

Proof. By differentiation one has

$$\begin{aligned}
 G'(r) = & -\frac{\lambda}{r} \left\{ \mathcal{K}_a [4(1-a)\mathcal{E}'_a - 2(1-2a)r^2 \mathcal{K}'_a] - \frac{\pi \sin \pi a}{2} \right\} \\
 & - 8(1-a)^2 \left[\frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r r'^2} \mathcal{E}'_a - r \mathcal{K}_a \frac{\mathcal{E}'_a - \mathcal{K}'_a}{r'^2} \right] \\
 & + 4(1-2a)r \mathcal{K}_a \mathcal{K}'_a + 4(1-a)(1-2a)r \mathcal{K}'_a \frac{\mathcal{E}_a - r'^2 \mathcal{K}_a}{r'^2} \\
 & - 4(1-a)(1-2a)r \mathcal{K}_a \frac{\mathcal{E}'_a - r^2 \mathcal{K}'_a}{r'^2} \\
 = & \frac{f(r) - [\pi \sin(\pi a)]/2}{r} [-\lambda - H(r)],
 \end{aligned} \tag{2.13}$$

where $H(r)$ is defined as in Lemma 2.5.

It follows from (2.13) and Lemma 2.5 that $G'(r) < 0$ for all $r \in (0, 1)$ if and only if $\lambda \geq 0$. \square

LEMMA 2.7. Let $0 < r < 1$, $m_a(r) = 2r^2 \mathcal{H}_a \mathcal{H}'_a / [\pi \sin(\pi a)]$ and $s = \varphi_K^a(r)$. If $K > 1$, then the function $F(r) = [s^\lambda m_a(s)]/[r^\lambda m_a(r)]$ is strictly decreasing on $(0, 1)$ if and only if $\lambda \geq 0$; If $0 < K < 1$, then $F(r)$ is strictly increasing on $(0, 1)$ if and only if $\lambda \geq 0$; $F(r)$ is not monotone on $(0, 1)$ for any $\lambda < 0$.

Proof. By logarithmic differentiation we have

$$\begin{aligned} \frac{F'(r)}{F(r)} &= \frac{\lambda}{s} \frac{ss'^2 \mathcal{H}_a(s) \mathcal{H}'_a(s)}{rr'^2 \mathcal{H}_a(r) \mathcal{H}'_a(r)} + \frac{1}{m_a(s)} \frac{ss'^2 \mathcal{H}_a(s) \mathcal{H}'_a(s)}{rr'^2 \mathcal{H}_a(r) \mathcal{H}'_a(r)} \\ &\quad \times \frac{2\{[\sin(\pi a)]/2 - 4(1-a)\mathcal{H}_a(s)\mathcal{E}'_a(s) + 2(1-2a)s^2 \mathcal{H}_a(s) \mathcal{H}'_a(s)\}}{\pi s \sin(\pi a)} \\ &\quad - \frac{\lambda}{r} \frac{2\{[\sin(\pi a)]/2 - 4(1-a)\mathcal{H}_a(r)\mathcal{E}'_a(r) + 2(1-2a)r^2 \mathcal{H}_a(r) \mathcal{H}'_a(r)\}}{\pi r m_a(r) \sin(\pi a)} \\ &= \frac{G(s) - G(r)}{rr'^2 \mathcal{H}_a(r) \mathcal{H}'_a(r)}, \end{aligned} \tag{2.14}$$

where $G(r)$ is defined as in Lemma 2.6.

It is well known that $s > r (< r)$ for $K > 1 (< 1)$. Therefore, (2.14) and Lemma 2.6 imply that $F'(r) < 0 (> 0)$ for $K > 1 (< 1)$ and all $r \in (0, 1)$ if and only if $\lambda \geq 0$. \square

3. Main Results

THEOREM 3.1. Let λ be a real number. Then the inequality

$$M_\lambda(\mu_a(x), \mu_a(y)) \leq \mu_a(M_\lambda(x, y)) \tag{3.1}$$

holds for all $x, y \in (0, 1)$ if and only if $\lambda \leq 0$, with equality if and only if $x = y$ for all $\lambda \leq 0$.

Proof. If $\lambda = 0$, then inequality (3.1) follows from [3, Corollary 5.7]. Next, we prove that inequality (3.1) holds for $\lambda < 0$. Without loss of generality, we assume that $x \leq y$. Define

$$J(x, y) = \mu_a(M_\lambda(x, y))^\lambda - \frac{\mu_a(x)^\lambda + \mu_a(y)^\lambda}{2}, \quad \lambda \neq 0. \tag{3.2}$$

Let $t = M_\lambda(x, y)$, then $\partial t / \partial x = (x/t)^{\lambda-1} / 2$. If $x < y$, then we clearly see that $t > x$. By differentiation, we have

$$\begin{aligned} \frac{\partial J}{\partial x} &= \frac{\lambda}{2} \mu_a(t)^{\lambda-1} \left(-\frac{\pi^2}{4t^2 \mathcal{H}_a(t)^2} \right) \left(\frac{x}{t} \right)^{\lambda-1} - \frac{\lambda}{2} \mu_a(x)^{\lambda-1} \left(-\frac{\pi^2}{4xx'^2 \mathcal{H}_a(x)^2} \right) \\ &= -\frac{\pi^2 \lambda x^{\lambda-1}}{8} \left[\frac{\mu_a(t)^{\lambda-1}}{t^\lambda t'^2 \mathcal{H}_a(t)^2} - \frac{\mu_a(x)^{\lambda-1}}{x^\lambda x'^2 \mathcal{H}_a(x)^2} \right]. \end{aligned} \tag{3.3}$$

From Lemma 2.2 and (3.3) we know that $\partial J/\partial x > 0$ if and only if $\lambda < 0$. Therefore, $J(x, y)$ is strictly increasing with respect to x and

$$J(x, y) \leq J(y, y) = 0. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$\mu_a(M_\lambda(x, y))^\lambda \leq \frac{\mu_a(x)^\lambda + \mu_a(y)^\lambda}{2} \tag{3.5}$$

for all $x, y \in (0, 1)$ if and only if $\lambda < 0$.

Therefore, inequality (3.1) follows from inequality (3.5), and we clearly see that inequality (3.1) holds equality if and only if $x = y$. \square

THEOREM 3.2. *For $\lambda \in \mathbb{R}$ we have the following statements:*

(1) *If $K > 1$, then inequality*

$$M_\lambda(\varphi_K^a(x), \varphi_K^a(y)) \leq \varphi_K^a(M_\lambda(x, y)) \tag{3.6}$$

holds for all $x, y \in (0, 1)$ if and only if $\lambda \geq 0$.

(2) *If $0 < K < 1$, then inequality*

$$M_\lambda(\varphi_K^a(x), \varphi_K^a(y)) \geq \varphi_K^a(M_\lambda(x, y)) \tag{3.7}$$

holds for all $x, y \in (0, 1)$ if and only if $\lambda \geq 0$. Inequality (3.6) or (3.7) becomes equality if and only if $x = y$.

Proof. If $\lambda = 0$, then inequalities (3.6) and (3.7) follows from [3, Theorem 1.14]. Next, we only prove that inequality (3.6) for $\lambda > 0$ and $K > 1$, since the case of $0 < K < 1$ is completely similar. Without loss of generality, we assume that $x \leq y$. Define

$$I(x, y) = \varphi_K^a(M_\lambda(x, y))^\lambda - \frac{\varphi_K^a(x)^\lambda + \varphi_K^a(y)^\lambda}{2}, \quad \lambda \neq 0. \tag{3.8}$$

Let $t = M_\lambda(x, y)$, $u = \varphi_K^a(t)$ and $v = \varphi_K^a(x)$. If $x < y$, then $t > x$ and $u > v$. By differentiation, we have

$$\begin{aligned} \frac{\partial I}{\partial x} &= \frac{\lambda}{2} \frac{u^\lambda u'^2 \mathcal{H}_a(u) \mathcal{H}'_a(u)}{t t'^2 \mathcal{H}_a(t) \mathcal{H}'_a(t)} \left(\frac{x}{t}\right)^{\lambda-1} - \frac{\lambda}{2} \frac{v^\lambda v'^2 \mathcal{H}_a(v) \mathcal{H}'_a(v)}{x x'^2 \mathcal{H}_a(x) \mathcal{H}'_a(x)} \\ &= \frac{\lambda}{2} x^{\lambda-1} \left(\frac{u^\lambda u'^2 \mathcal{H}_a(u) \mathcal{H}'_a(u)}{t^\lambda t'^2 \mathcal{H}_a(t) \mathcal{H}'_a(t)} - \frac{v^\lambda v'^2 \mathcal{H}_a(v) \mathcal{H}'_a(v)}{x^\lambda x'^2 \mathcal{H}_a(x) \mathcal{H}'_a(x)} \right) \\ &= \frac{\lambda}{2} x^{\lambda-1} \left(\frac{u^\lambda m_a(u)}{t^\lambda m_a(t)} - \frac{v^\lambda m_a(v)}{x^\lambda m_a(x)} \right). \end{aligned} \tag{3.9}$$

From Lemma 2.7 and (3.9) we know that $\partial I/\partial x < 0$ if and only if $\lambda > 0$. Therefore, $I(x, y)$ is strictly increasing with respect to x and

$$I(x, y) \geq I(y, y) = 0. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\varphi_K^\alpha (M_\lambda(x, y))^\lambda \geq \frac{\varphi_K^\alpha(x)^\lambda + \varphi_K^\alpha(y)^\lambda}{2}, \quad (3.11)$$

for all $x, y \in (0, 1)$ if and only if $\lambda > 0$.

Therefore, inequality (3.6) follows from inequality (3.11), and we clearly see that inequality (3.6) becomes equality if and only if $x = y$. \square

Acknowledgements. The authors wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1966.
- [2] H. ALZER AND S. L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math., **172**, 2 (2004), 289–312.
- [3] G. D. ANDERSON, S. L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals and modular equations*, Pacific J. Math., **192**, 1 (2000), 1–37.
- [4] G. D. ANDERSON, S. L. QIU AND M. VUORINEN, *Bounds for the Hersch-Pfluger and Belinskii distortion functions*, Computational Methods and Function Theory 1997 (Nicosia), 9–22, Ser. Approx. Decompos., 11, World Sci. Publ., River Edge, NJ, 1999.
- [5] G. D. ANDERSON, S. L. QIU AND M. VUORINEN, *Precise estimates for differences of the Gaussian hypergeometric function*, J. Math. Anal. Appl., **215**, 1 (1997), 212–234.
- [6] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Topics in special functions II*, Confrom. Geom. Dyn., **11** (2007), 250–270.
- [7] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.
- [8] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for plane quasiconformal mappings*, The Mathematical Legacy of Wilhelm Magnus: Groups, Geometry and Special Functions (Brooklyn, NY, 1992), 1–27, Contemp. Math., 169, Amer. Math. Soc., Providence, RI, 1994.
- [9] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, SIAM J. Math. Anal., **23**, 2 (1992), 512–524.
- [10] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for complete elliptic integrals and their ratios*, SIAM J. Math. Anal., **21**, 2 (1990), 536–549.
- [11] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Distortion functions for plane quasiconformal mappings*, Israel J. Math., **62**, 1 (1988), 1–16.
- [12] R. W. BARNARD, K. PEARCE AND K. C. RICHARDS, *A monotonicity property involving ${}_3F_2$ and comparisons of the classical approximations of elliptical arc length*, SIAM J. Math. Anal., **32**, 2 (2000), 403–419.
- [13] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM*, John Wiley & Sons, New York, 1987.
- [14] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers Group, Dordrecht, 2003.
- [15] S. PONNUSAMY AND M. VUORINEN, *Univalence and convexity properties for Gaussian hypergeometric functions*, Rocky Mountain J. Math., **31**, 1 (2001), 327–353.
- [16] S. L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Some inequalities for the Hersch-Pfluger distortion function*, J. Inequal. Appl., **4**, 2 (1999), 115–139.
- [17] S. L. QIU AND M. VUORINEN, *Special functions in geometric function theory*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, 621–659, Elsevier Sci. B. V., Amsterdam, 2005.

- [18] S. L. QIU AND M. VUORINEN, *Landen inequalities for hypergeometric functions*, Nagoya Math. J., **154** (1999), 31–56.
- [19] S. L. QIU AND M. VUORINEN, *Infinite products and normalized quotients of hypergeometric functions*, SIAM J. Math. Anal., **30**, 5 (1999), 1057–1075.
- [20] M. VUORINEN, *Conformal Geometry and Quasiregular Mappings*, Springer-Verlag, Berlin, 1988.

(Received October 2, 2010)

Songliang Qiu
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018, China
e-mail: sl_qiu@zstu.edu.cn

Yefang Qiu
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018, China
e-mail: qiuyefang861013@126.com

Miaokun Wang
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018, China
e-mail: wmk000@126.com

Yuming Chu
Department of Mathematics
Huzhou Teachers College
Huzhou 313000, China
e-mail: chuyuming@hutc.zj.cn