Mathematical
Inequalities
& Applications
Volume 15, Number 2 (2012), 289–299

ON STRONG $(\alpha,\mathbb{F})$–CONVEXITY

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(*Communicated by C. Niculescu*)

Abstract. In this paper, strongly (α,T)-convex functions, i.e., functions $f:D\to\mathbb{R}$ satisfying the functional inequality

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)-t\alpha\big((1-t)(x-y)\big)-(1-t)\alpha\big(t(y-x)\big)$$

for $x,y\in D$ and $t\in T\cap[0,1]$ are investigated. Here D is a convex set in a linear space, α is a nonnegative function on $D-D$, and $T\subseteq\mathbb{R}$ is a nonempty set. The main results provide various characterizations of strong (α,T)-convexity in the case when T is a subfield of $\mathbb{R}$.

1. Introduction

Let $I\subset\mathbb{R}$ be an interval and c be a positive number. A function $f:I\to\mathbb{R}$ is called *strongly convex with modulus* c if

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2,$$

for all $x,y\in I$ and $t\in[0,1]$. Strongly convex functions have been introduced by Polyak [14] and they play an important role in optimization theory and mathematical economics. Many properties of them can be found in the literature (see, e.g. [13], [15], [22], [8]). It is known, for instance, that a function $f:I\to\mathbb{R}$ is strongly convex with modulus c if and only if for every $x_0\in\operatorname{int}I$ there exists an $a\in\mathbb{R}$ such that

$$f(x)\geqslant c(x-x_0)^2+a(x-x_0)+f(x_0),\qquad x\in I,$$

i.e., f has a quadratic support at x_0. If f is differentiable and strongly convex with modulus c then its derivative f' is "strongly monotone" in the sense: $(f'(x)-f'(y))(x-y)\geqslant 2c(x-y)^2$, $x,y\in I$ (cf. [15, p. 268]).

In this paper we introduce the class of strongly (α,T)-convex functions (which is much wider then the class of strongly convex functions) and present, among other, some generalizations of the results mentioned above.

Mathematics subject classification (2010): Primary 39B62, 26B25.
Keywords and phrases: strong convexity, α-convexity.
This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402 and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund.

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Paper MIA-15-24

Let X be a real linear space. For a nonempty convex subset $D \subseteq X$, denote

$$D^* := D - D := \{x - y : x, y \in D\}.$$

Given a nonnegative *even* function $\alpha : D^* \to \mathbb{R}_+$ and a nonempty set $T \subseteq \mathbb{R}$ such that $T \cap [0,1]$ is nonempty, we say that a map $f : D \to \mathbb{R}$ is *strongly* (α, T)*-convex*, if for all $x, y \in D$ and $t \in T \cap [0,1]$,

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t\alpha\big((1-t)(x-y)\big) - (1-t)\alpha\big(t(y-x)\big) \quad (1)$$

holds. If (1) holds with $T = \{1/2\}$, i.e., for all $x, y \in D$,

$$f\big(\tfrac{x+y}{2}\big) \leqslant \tfrac{f(x)+f(y)}{2} - \alpha\big(\tfrac{x-y}{2}\big), \quad (2)$$

then the function f is called *strongly* α*-Jensen convex.* If $T \supseteq [0,1]$, then the function f is termed *strongly* α*-convex.* By the nonnegativity of α, we have that strongly α-Jensen convex, strongly (α, T)-convex, and strongly α-convex functions are always convex in the same sense, respectively. More generally, if $\alpha, \beta : D^* \to \mathbb{R}_+$ and $\alpha \geqslant \beta$, then strong convexity with respect to α in some sense implies strong convexity with respect to β in the same sense. Note also that for $\alpha(x) = cx^2$ strong α-convexity coincides with strong convexity with modulus c.

2. Strengthening the strong Jensen convexity

In the next theorem and corollary, which are particular cases of the theorem in [6], the strong α-Jensen convexity property will be strengthened. We provide their proof here because it is much simpler and more transparent than in the general case.

THEOREM 1. *Let $f : D \to \mathbb{R}$ be a strongly α-Jensen convex function. Then f is strongly $\widetilde{\alpha}$-Jensen convex on D, where*

$$\widetilde{\alpha}(u) := \sup\big\{n^2\alpha\big(\tfrac{u}{n}\big) \mid n \in \mathbb{N}\big\} \qquad (u \in D^*). \quad (3)$$

Proof. Assume that $f : D \to \mathbb{R}$ is strongly α-Jensen convex and let $n \in \mathbb{N}$. Let $x, y \in D$. Consider the segment $[x, y]$ and divide it into $2n$ pieces of equal subsegments. For this, we define the system of points $x_0 = x$, $x_1, \dots, x_{2n-1}$, $x_{2n} = y$ in the following way

$$x_i := x + \tfrac{i}{2n}(y - x) \qquad (i \in \{0, \dots, 2n\}). \quad (4)$$

We have the following two obvious identities:

$$x_{i+1} - x_{i-1} = \tfrac{y-x}{n} \qquad \text{and} \qquad x_i = \tfrac{x_{i-1}+x_{i+1}}{2}. \qquad (i \in \{1, \dots, 2n-1\}).$$

Therefore, by the strong α-Jensen convexity of f, we get

$$f(x_i) \leqslant \tfrac{f(x_{i-1})+f(x_{i+1})}{2} - \alpha\big(\tfrac{x-y}{2n}\big) \quad (5)$$

for all $i \in \{1,\dots,2n-1\}$. Multiplying (5) by

$$\begin{aligned} i, &\quad \text{if} \qquad i \in \{1,\dots,n\} \\ 2n-i, &\quad \text{if} \qquad i \in \{n+1,\dots,2n-1\} \end{aligned}$$

and adding the inequalities so obtained, we get that

$$\begin{aligned} &\sum_{i=1}^{n} i f(x_i) + \sum_{i=n+1}^{2n-1} (2n-i) f(x_i) \\ &\qquad \leqslant \sum_{i=1}^{n} \tfrac{i}{2}\big(f(x_{i-1}) + f(x_{i+1})\big) + \sum_{i=n+1}^{2n-1} \tfrac{2n-i}{2}\big(f(x_{i-1}) + f(x_{i+1})\big) \\ &\qquad\quad - \left(\sum_{i=1}^{n} i + \sum_{i=n+1}^{2n-1} (2n-i)\right) \alpha\big(\tfrac{x-y}{2n}\big). \end{aligned} \tag{6}$$

The coefficient of $f(x_i)$ in inequality (6) is the following:

$$\begin{aligned} i - \tfrac{i+1}{2} - \tfrac{i-1}{2} &= 0, \qquad \text{if} \qquad 1 \leqslant i \leqslant n-1, \\ n - \tfrac{n-1}{2} - \tfrac{2n-(n+1)}{2} &= 1, \qquad \text{if} \qquad i = n, \\ (2n-i) - \tfrac{2n-i-1}{2} - \tfrac{2n-i+1}{2} &= 0, \qquad \text{if} \qquad n+1 \leqslant i \leqslant 2n-1. \end{aligned}$$

The terms $f(x_0) = f(x)$ and $f(x_{2n}) = f(y)$ appear only on the right hand side of (6) with coefficients

$$\tfrac{1}{2} \qquad \text{and} \qquad \tfrac{2n-(2n-1)}{2} = \tfrac{1}{2},$$

respectively. Finally, the coefficient of the error function is

$$\sum_{i=1}^{n} i + \sum_{i=n+1}^{2n-1} (2n-i) = \tfrac{n(n+1)}{2} + \tfrac{(n-1)n}{2} = n^2.$$

Thus, (6) reduces to,

$$f\big(\tfrac{x+y}{2}\big) = f(x_n) \leqslant \tfrac{f(x_0)+f(x_{2n})}{2} - n^2\alpha\big(\tfrac{x-y}{2n}\big) = \tfrac{f(x)+f(y)}{2} - n^2\alpha\big(\tfrac{x-y}{2n}\big).$$

Therefore, we get that f is strongly $\widetilde{\alpha}$-Jensen convex, where $\widetilde{\alpha}$ is defined by (3). This completes the proof.

EXAMPLE 2. Assume that a function $f : I \to \mathbb{R}$ is strongly $\sin^2$-Jensen convex, i.e.,

$$f\big(\tfrac{x+y}{2}\big) \leqslant \tfrac{f(x)+f(y)}{2} - \sin^2\big(\tfrac{x-y}{2}\big) \qquad (x,y \in I),$$

then, by Theorem 1, f is also $\widetilde{\sin^2}$-Jensen convex, where, for $u \in \mathbb{R}$,

$$\begin{aligned} \widetilde{\sin^2}(u) &= \sup\{n^2 \sin^2(\tfrac{u}{n}) \mid n \in \mathbb{N}\} \\ &= u^2 \sup\{\tfrac{n^2}{u^2} \sin^2(\tfrac{u}{n}) \mid n \in \mathbb{N}\} = u^2 \lim_{n\to\infty} \frac{\sin^2\big(\frac{u}{n}\big)}{\big(\frac{u}{n}\big)^2} = u^2. \end{aligned}$$

This means that f is also strongly Jensen convex with modulus 1, i.e.,

$$f\left(\tfrac{x+y}{2}\right) \leqslant \tfrac{f(x)+f(y)}{2} - \left(\tfrac{x-y}{2}\right)^2 \qquad (x,y \in I).$$

COROLLARY 3. *Assume that there exists* $u \in \frac{1}{2}D^* \setminus \{0\}$ *such that*

$$\limsup_{n\to\infty} n^2 \alpha\left(\tfrac{u}{n}\right) = \infty. \tag{7}$$

Then there is no strongly α*-Jensen convex function on* D.

Proof. Assume that there exists a strongly α-Jensen convex function $f : D \to \mathbb{R}$ and let $u \in \frac{1}{2}D^* \setminus \{0\}$ such that (7) holds. By Theorem 1, we get that f is $\widetilde{\alpha}$-Jensen convex on D. The definition of $\widetilde{\alpha}$ implies that,

$$\widetilde{\alpha}(u) \geqslant \limsup_{n\to\infty} n^2 \alpha\left(\tfrac{u}{n}\right) = \infty. \tag{8}$$

Hence, we get that $-\widetilde{\alpha}(u) = -\infty$, which means that the error term in (2) is equal to $-\infty$ for some $x, y \in D$. This is a contradiction resulting the statement.

REMARK 4. If X is a normed space, ε, p are positive constants, and $\alpha(u) := \varepsilon\|u\|^p$ for $u \in D^*$, then condition (7) holds if and only if $p < 2$.

3. On strong $(\alpha, \mathbb{F})$-convexity

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Given a real linear space X, a function $\varphi : X \to \mathbb{R}$ is called $\mathbb{F}$*-linear*, if it is *additive*, i.e., for all $x, y \in X$,

$$\varphi(x+y) = \varphi(x) + \varphi(y),$$

and it is $\mathbb{F}$*-homogeneous*, i.e., for all $x \in X$ and for all $\lambda \in \mathbb{F}$,

$$\varphi(\lambda x) = \lambda \varphi(x).$$

A function $\psi : X \to \mathbb{R}$ is called $\mathbb{F}$*-sublinear*, if it is *subadditive*, i.e., for all $x, y \in X$,

$$\psi(x+y) \leqslant \psi(x) + \psi(y),$$

and it is *positively* $\mathbb{F}$*-homogeneous*, i.e., for all $x \in X$ and for all $\lambda \in \mathbb{F} \cap \mathbb{R}_+$,

$$\psi(\lambda x) = \lambda \psi(x).$$

The $\mathbb{F}$*(-algebraic) dual* $X'_{\mathbb{F}}$ of the space X is defined by

$$X'_{\mathbb{F}} := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is } \mathbb{F}\text{-linear}\}.$$

As it is well-known, additive functions are automatically $\mathbb{Q}$-linear, therefore we always have the inclusion $X'_{\mathbb{F}} \subseteq X'_{\mathbb{Q}}$.

The following result is a consequence of the standard separation/sandwich theorems (cf. Mazur–Orlicz [7], Holmes [4], Nikodem–Páles–Wąsowicz [12]) or of the Rodé theorem ([19]).

THEOREM 5. *Let $\psi : X \to \mathbb{R}$ be an $\mathbb{F}$-sublinear function, then there exists a $\varphi \in X'_{\mathbb{F}}$, such that $\varphi \leqslant \psi$.*

Let $D \subseteq X$ be a nonempty set. We say that D is $\mathbb{F}$-*algebraically open* if, for all $x_0 \in D$ and $h \in X$, there exits an $\varepsilon \in]0,+\infty[$ such that $[x_0, x_0+th] \subset D$ for all $t \in [0,\varepsilon] \cap \mathbb{F}$. In what follows, assume that $D \subseteq X$ is a nonempty $\mathbb{F}$-algebraically open convex set. In the sequel, strongly $(0,\mathbb{F})$-convex functions are simply called $\mathbb{F}$-convex.

PROPOSITION 6. *[Cf.* [2]*] Let $f : D \to \mathbb{R}$ be an $\mathbb{F}$-convex function, $x_0 \in D$ and $h \in X$, then the mapping*

$$t \mapsto \frac{f(x_0+th)-f(x_0)}{t}$$

is nondecreasing on the set $\{t \in \mathbb{F} \setminus \{0\} \mid x_0+th \in D\}$.

The generalized $\mathbb{F}$-*directional derivative* of an $\mathbb{F}$-convex function $f : D \to \mathbb{R}$ at $x_0 \in D$ in a direction $h \in X$, denoted by $f'_{\mathbb{F}}(x_0,h)$, is defined as follows

$$f'_{\mathbb{F}}(x_0,h) := \lim_{\substack{t \to 0^+ \\ t \in \mathbb{F}}} \frac{f(x_0+th)-f(x_0)}{t}.$$

Note that this generalized $\mathbb{F}$-directional derivative has similar properties as the standard directional derivative.

PROPOSITION 7. *[Cf.* [2]*] Let $f : D \to \mathbb{R}$ be an $\mathbb{F}$-convex function and $x_0 \in D$ be an arbitrary element of D, then the mapping $h \mapsto f'_{\mathbb{F}}(x_0,h)$ is $\mathbb{F}$-sublinear.*

Our main result is contained in the following theorem.

THEOREM 8. *For any function $f : D \to \mathbb{R}$, the following conditions are equivalent:*

(i) f is strongly $(\alpha,\mathbb{F})$-convex.

(ii) f is $\mathbb{F}$-directionally differentiable at every point of D, and for all $x_0 \in D$, the map $h \mapsto f'_{\mathbb{F}}(x_0,h)$ is $\mathbb{F}$-sublinear on X, furthermore for all $x_0, x \in D$,

$$f(x) \geqslant f(x_0) + f'_{\mathbb{F}}(x_0, x-x_0) + \alpha(x-x_0). \tag{9}$$

(iii) For all $x_0 \in D$, there exits an element $\varphi \in X'_{\mathbb{F}}$ such that

$$f(x) \geqslant f(x_0) + \varphi(x-x_0) + \alpha(x-x_0) \quad \textit{for all} \quad x \in D. \tag{10}$$

Proof. (i) $\Rightarrow$ (ii) It is evident that f is $\mathbb{F}$-convex, which implies that f is $\mathbb{F}$-directionally differentiable at every point of D, moreover for all $x_0 \in D$ the map $h \mapsto$

$f'_{\mathbb{F}}(x_0,h)$ is $\mathbb{F}$-sublinear on X. To prove (9), let $x_0,x \in D$ be arbitrary. Since D is $\mathbb{F}$-algebraically open, there exists an $\varepsilon \in]0,1[$ such that $x+\frac{t}{1-t}(x-x_0)=x_0+\frac{x-x_0}{1-t} \in D$ for all $t \in [0,\varepsilon]\cap\mathbb{F}$. Let $h:=\frac{x-x_0}{1-t}$. By the strong $(\mathbb{F},\alpha)$-convexity of f, we get that

$$f(x_0+th) \leqslant (1-t)f(x_0)+tf(x_0+h)-t\alpha\big((1-t)h\big)-(1-t)\alpha(-th) \qquad \text{for all} \quad t \in [0,\varepsilon]\cap\mathbb{F}.$$

Rearranging the above inequality, we have that

$$f(x_0+h) \geqslant f(x_0)+\frac{f(x_0+th)-f(x_0)}{t}+\alpha\big((1-t)h\big)+\frac{1-t}{t}\alpha(-th) \qquad \text{for all} \quad t \in]0,\varepsilon]\cap\mathbb{F}.$$

By the nonnegativity of α,

$$f(x_0+h) \geqslant f(x_0)+\frac{f(x_0+th)-f(x_0)}{t}+\alpha\big((1-t)h\big) \qquad \text{for all} \quad t \in]0,\varepsilon]\cap\mathbb{F}.$$

Substituting $h=\frac{x-x_0}{1-t}$, we obtain that, for all $t \in]0,\varepsilon]\cap\mathbb{F}$,

$$f\Big(x_0+\frac{x-x_0}{1-t}\Big) \geqslant f(x_0)+\frac{1}{1-t}\,\frac{f\big(x_0+\frac{t}{1-t}(x-x_0)\big)-f(x_0)}{t/(1-t)}+\alpha(x-x_0) \tag{11}$$

holds. By the $\mathbb{F}$-convexity of f, the mapping $s \mapsto f\big(x_0+s(x-x_0)\big)$ is continuous on $[0,\varepsilon]\cap\mathbb{F}$, whence we get

$$\lim_{\substack{t\to 0^+\\ t\in\mathbb{F}}} f\Big(x_0+\frac{x-x_0}{1-t}\Big)=f(x),$$

furthermore, the limit

$$\lim_{\substack{t\to 0^+\\ t\in\mathbb{F}}} \frac{f\big(x_0+\frac{t}{1-t}(x-x_0)\big)-f(x_0)}{t/(1-t)}$$

exists and equals $f'_{\mathbb{F}}(x_0,x-x_0)$. Thus, taking the limit $t\to 0^+$ for $t\in\mathbb{F}$ in (11), we get (9), which completes the proof of (ii).

(ii) $\Rightarrow$ (iii) Assume that f is $\mathbb{F}$-directionally differentiable at every point of D and for all $x_0 \in D$, $h \mapsto f'_{\mathbb{F}}(x_0,h)$ is $\mathbb{F}$-sublinear. By Theorem 5, there exists an element $\varphi \in X'_{\mathbb{F}}$, such that

$$f'_{\mathbb{F}}(x_0,h) \geqslant \varphi(h) \qquad \text{for all} \quad h \in X.$$

This and (9) implies that (10) holds.

(iii) $\Rightarrow$ (i) Let $x,y \in D$, $t \in [0,1]\cap\mathbb{F}$, and set $x_0:=tx+(1-t)y$. Then, by (iii), we have

$$\begin{aligned} f(x) &\geqslant f(tx+(1-t)y)+\varphi\big((1-t)(x-y)\big)+\alpha\big((1-t)(x-y)\big),\\ f(y) &\geqslant f(tx+(1-t)y)+\varphi\big(t(y-x)\big)+\alpha\big(t(y-x)\big). \end{aligned}$$

Multiplying the first inequality by t and the second inequality by $1-t$ and adding up the inequalities so obtained, we get (1).

COROLLARY 9. *Let $f : D \to \mathbb{R}$ be a strongly $(\alpha,\mathbb{F})$-convex function. Then $\alpha(0) = 0$, α is $\mathbb{F}$-directionally differentiable at 0 and*

$$\alpha'_{\mathbb{F}}(0,h) = 0 \qquad (h \in X).$$

Proof. If f is strongly $(\alpha,\mathbb{F})$-convex then property (ii) of Theorem 8 holds. Let $x_0 \in D$ be fixed and $h \in X$. Since D is $\mathbb{F}$-algebraically open, there exists an $\varepsilon \in]0,1[\cap\mathbb{F}$ such that $x_0 + \varepsilon h \in D$. Then, substituting $x = x_0 + th$ into (9) (where $t \in]0,\varepsilon[\cap\mathbb{F}$), we get

$$f(x_0+th) \geqslant f(x_0) + f'_{\mathbb{F}}(x_0,th) + \alpha(th). \tag{12}$$

Taking $h = 0$, by the nonnegativity of α, it follows that $\alpha(0) = 0$. On the other hand, rearranging the above inequality,

$$\frac{f(x_0+th) - f(x_0)}{t} \geqslant f'_{\mathbb{F}}(x_0,h) + \frac{\alpha(th)}{t}.$$

By taking the limit $t \to 0+$ and using the nonnegativity of α again, it follows that

$$\lim_{\substack{t \to 0^+ \\ t \in \mathbb{F}}} \frac{\alpha(th)}{t} = 0. \tag{13}$$

Therefore, $\alpha'_{\mathbb{F}}(0,h) = 0$, proving that the $\mathbb{F}$-directional derivative of α at the origin is zero.

In the following result, we strengthen $(\alpha,\mathbb{F})$-convexity.

COROLLARY 10. *Let $f : D \to \mathbb{R}$ be a strongly $(\alpha,\mathbb{F})$-convex function. Then f is strongly $(\widehat{\alpha},\mathbb{F})$-convex, where $\widehat{\alpha} : D^* \to \mathbb{R}$ is defined by*

$$\widehat{\alpha}(u) := \sup\left\{ \tfrac{\alpha(tu)}{t} \mid t \in]0,1] \cap \mathbb{F} \right\}. \tag{14}$$

Proof. Let $f : D \to \mathbb{R}$ be a strongly $(\alpha,\mathbb{F})$-convex function and x_0, x be arbitrary elements of D. By Proposition 6, the mapping $t \mapsto \frac{f(x_0+t(x-x_0))-f(x_0)}{t}$ is nondecreasing on $]0,1]\cap\mathbb{F}$. Thus, for all $t \in]0,1]\cap\mathbb{F}$, we have that

$$f(x) - f(x_0) \geqslant \frac{f(x_0+t(x-x_0)) - f(x_0)}{t}.$$

By Theorem 8, we get that

$$f(x_0+t(x-x_0)) - f(x_0) \geqslant f'_{\mathbb{F}}(x_0,t(x-x_0)) + \alpha(t(x-x_0)) \qquad (t \in]0,1]\cap\mathbb{F}).$$

Combining the above inequalities, we obtain

$$f(x) - f(x_0) \geqslant f'_{\mathbb{F}}(x_0,x-x_0) + \tfrac{\alpha(t(x-x_0))}{t} \qquad (t \in]0,1]\cap\mathbb{F}).$$

Using Theorem 8, this means that the function f is also strongly $(\widehat{\alpha},\mathbb{F})$-convex, which completes the proof.

4. The $\mathbb{F}$-subdifferential of $(\alpha,\mathbb{F})$-convex functions

For any $f : D \to \mathbb{R}$ function and $x_0 \in D$, define the $\mathbb{F}$-*subdifferential* of f at x_0 by

$$\partial_{\mathbb{F}} f(x_0) := \{\varphi \in X'_{\mathbb{F}} \mid f(x) \geqslant f(x_0) + \varphi(x - x_0) \quad \text{for all} \quad x \in D\}. \tag{15}$$

Obviously, $\partial_{\mathbb{F}} f(\cdot)$ can be considered as a set-valued mapping defined on D with values in $2^{X'_{\mathbb{F}}}$.

For $\mathbb{F}$-convex functions, the $\mathbb{F}$-subdifferential $\partial_{\mathbb{F}} f(x_0)$ can also be expressed in terms of the $\mathbb{F}$-directional derivative of f at x_0.

PROPOSITION 11. *[Cf.* [2]*] Let $f : D \to \mathbb{R}$ be an $\mathbb{F}$-convex function. Then, for all $x_0 \in D$,*

$$\partial_{\mathbb{F}} f(x_0) = \{\varphi \in X'_{\mathbb{F}} \mid f'_{\mathbb{F}}(x_0, h) \geqslant \varphi(h) \quad \textit{for all} \quad h \in X\}. \tag{16}$$

To describe the properties of the $\mathbb{F}$-subdifferential of strongly $(\alpha,\mathbb{F})$-convex functions, we need to recall and define certain generalized monotonicity concepts whose original versions were introduced by Minty [9] and R. T. Rockafellar ([16], [18], [17]) in order to characterize the subdifferentials of convex functions.

We say that a set-valued mapping $\Phi : D \to 2^{X'_{\mathbb{Q}}}$ is α-*monotone* if

$$\varphi(y - x) + \alpha(y - x) + \psi(x - y) + \alpha(x - y) \leqslant 0 \tag{17}$$

holds for every $x, y \in D$, $\varphi \in \Phi(x)$, and $\psi \in \Phi(y)$. We call a set-valued mapping $\Phi : D \to 2^{X'_{\mathbb{Q}}}$ α-*cyclically monotone* if the inequality

$$\sum_{j=0}^{n} \Big(\varphi_j(x_{j+1} - x_j) + \alpha(x_{j+1} - x_j)\Big) \leqslant 0 \tag{18}$$

is fulfilled for every $n \in \mathbb{N}$, $x_j \in D$ $(j \in \{0, 1, \ldots, n, n+1\})$ with $x_{n+1} = x_0$, and $\varphi_j \in \Phi(x_j)$ $(j \in \{0, 1, \ldots, n\})$. Obviously, by taking $n = 2$ in the above definition, α-cyclical monotonicity implies α-monotonicity, however, the reversed implication may not be valid. In the particular case when α is identically zero, we simply speak about *monotone* and *cyclically monotone* set-valued maps. Observe that, by the nonnegativity of α, the properties α-monotonicity and α-cyclical monotonicity imply monotonicity and cyclical monotonicity, respectively.

We say that a mapping $\Phi : D \to 2^{X'_{\mathbb{F}}}$ is $\mathbb{F}$-*maximal monotone* if Φ is monotone and, for any monotone mapping $\Psi : D \to 2^{X'_{\mathbb{F}}}$ fulfilling $\Phi(x) \subseteq \Psi(x)$ for all $x \in D$, also $\Phi(x) = \Psi(x)$ holds for every $x \in D$. In particular, $\mathbb{Q}$-maximal monotone mappings are called *maximal monotone*.

In the next result we summarize the properties of the $\mathbb{F}$-subdifferential of strongly $(\alpha,\mathbb{F})$-convex functions.

THEOREM 12. *Let $f : D \to \mathbb{R}$ be an $(\alpha,\mathbb{F})$-convex function. Then, for every $x_0 \in D$, $\partial_{\mathbb{F}} f(x_0)$ is a nonempty convex subset in $X'_{\mathbb{F}}$ which is closed with respect to the*

pointwise convergence and

$$\partial_{\mathbb{F}} f(x_0) = \{\varphi \in X'_{\mathbb{F}} \mid f(x) \geqslant f(x_0) + \varphi(x-x_0) + \alpha(x-x_0) \quad \textit{for all} \quad x \in D\}. \tag{19}$$

Furthermore, the map $\partial_{\mathbb{F}} f : D \to 2^{X'_{\mathbb{F}}}$ is $\mathbb{F}$-maximal monotone and α-cyclically monotone.

Proof. The convexity and closedness (with respect to the pointwise convergence) of $\partial_{\mathbb{F}} f(x_0)$ directly follows from its definition. The inclusion $\supseteq$ in (19) is a consequence of the definition (15). To prove the reversed inclusion, let $\varphi \in \partial_{\mathbb{F}} f(x_0)$ be arbitrary. Then, by Proposition 11, $f'_{\mathbb{F}}(x_0,h) \geqslant \varphi(h)$ holds for all $h \in X$. On the other hand, by the second assertion of Theorem 8, for all $x \in D$, we have

$$f(x) \geqslant f(x_0) + f'_{\mathbb{F}}(x_0, x-x_0) + \alpha(x-x_0).$$

Hence, for all $x \in D$,

$$f(x) \geqslant f(x_0) + \varphi(x-x_0) + \alpha(x-x_0).$$

This proves that φ also belongs to the right hand side of (19) and hence (19) holds with equality.

By the third assertion of Theorem 8, the right hand side of (19) is nonempty, which yields the nonemptiness of $\partial_{\mathbb{F}} f(x_0)$.

The $\mathbb{F}$-maximal monotonicity is a consequence of [2, Theorem 5.4].

To prove the α-cyclic monotonicity of f, let $n \in \mathbb{N}$, $x_j \in D$ for $j \in \{0,1,\dots,n,n+1\}$ with $x_{n+1} = x_0$, and $\varphi_j \in \partial_{\mathbb{F}} f(x_j)$ for $j \in \{0,1,\dots,n\}$. Then, by the third assertion of Theorem 8,

$$f(x_{j+1}) \geqslant f(x_j) + \varphi(x_{j+1}-x_j) + \alpha(x_{j+1}-x_j) \qquad (j \in \{0,1,\dots,n\}).$$

Adding up these inequalities for $j \in \{0,1,\dots,n\}$ and using $x_{n+1} = x_0$, the inequality (18) follows immediately proving the α-cyclic monotonicity of f.

The following statement is analogous to [16, Theorem 1].

THEOREM 13. *If $\Phi : D \to 2^{X'_{\mathbb{F}}}$ is a nonempty-valued α-cyclically monotone set-valued map, then there exists a strongly $(\alpha,\mathbb{F})$-convex function $f : D \to \mathbb{R}$ such that $\Phi(x) \subseteq \partial_{\mathbb{F}} f(x)$ for every $x \in D$.*

Proof. Let $x_0 \in D$ be fixed. For each $x \in D$, let $S(x)$ denote the set of all finite sums of the form

$$\sum_{j=0}^{n-1} \Big(\varphi_j(x_{j+1}-x_j) + \alpha(x_{j+1}-x_j)\Big),$$

where $n \in \mathbb{N}$, $x_j \in D$ $(j \in \{1,\dots,n\})$ such that $x_n = x$, and $\varphi_j \in \Phi(x_j)$ for $j \in \{0,1,\dots,n-1\}$.

For any $\varphi \in \Phi(x)$, the α-cyclical monotonicity of Φ yields

$$\sum_{j=0}^{n-1} \Big(\varphi_j(x_{j+1}-x_j) + \alpha(x_{j+1}-x_j)\Big) + \varphi(x_0-x) + \alpha(x_0-x) \leqslant 0.$$

Thus $-(\varphi(x_0-x)+\alpha(x_0-x))$ is an upper bound for $S(x)$. Hence, we may define a function $f:D\to\mathbb{R}$ by

$$f(x)=\sup S(x) \qquad (x\in D).$$

In order to prove the desired inclusion $\Phi(x)\subseteq\partial_{\mathbb{F}}f(x)$, consider arbitrary elements $x,y\in D$, $\varphi\in\Phi(x)$, and $\varepsilon>0$. Then there exist $n\in\mathbb{N}$, $x_j\in D$ $(j\in\{1,\dots,n\})$ with $x_n=x$, and $\varphi_j\in\Phi(x_j)$ $(j\in\{0,1,\dots,n-1\})$ such that

$$\sum_{j=0}^{n-1}\Big(\varphi_j(x_{j+1}-x_j)+\alpha(x_{j+1}-x_j)\Big)>f(x)-\varepsilon.$$

The definition of $f(y)$ and the above inequality yields

$$\begin{aligned}f(y)&\geqslant\sum_{j=0}^{n-1}\Big(\varphi_j(x_{j+1}-x_j)+\alpha(x_{j+1}-x_j)\Big)+\varphi(y-x)+\alpha(y-x)\\&\geqslant f(x)-\varepsilon+\varphi(y-x)+\alpha(y-x).\end{aligned}$$

Letting ε tend to 0, we get

$$f(y)\geqslant f(x)+\varphi(y-x)+\alpha(y-x). \tag{20}$$

Hence, $\varphi\in\partial_{\mathbb{F}}f(x)$ which proves the inclusion $\Phi(x)\subseteq\partial_{\mathbb{F}}f(x)$. In particular, for every $x\in D$, by the nonemptiness of $\Phi(x)$, there exists an element $\varphi\in X'_{\mathbb{F}}$ such that (20) holds for all $y\in D$ showing that the third assertion of Theorem 8 is valid. Thus, by Theorem 8, we obtain that f is strongly $(\alpha,\mathbb{F})$-convex.

An immediate consequence of Theorem 13 is the following result.

COROLLARY 14. *If $\Phi:D\to 2^{X'_{\mathbb{F}}}$ is an $\mathbb{F}$-maximal α-cyclically monotone mapping, then there exists a strongly $(\alpha,\mathbb{F})$-convex function $f:D\to\mathbb{R}$ such that $\Phi(x)=\partial_{\mathbb{F}}f(x)$ for every $x\in D$.*

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(Received October 27, 2010)

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