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ARITHMETIC FUNCTIONS, PRIME COUNTING FUNCTION AND POLYNOMIALS

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Abstract. In this paper we prove some properties regarding classical arithmetic functions and the prime counting function in connection with polynomials. More specific, this paper deals with composition between arithmetic functions or between the prime counting function and a polynomial and we study when some of these kind of compositions are rational functions or another polynomial. In the proofs of our results we shall use inequalities or estimates of arithmetic functions and the prime counting function as well as some elementary inequalities.

1. Introduction & Main results

The importance of polynomials is well-known in the study of the properties of arithmetic functions like: $\sigma(n) = \sum_{d/n} d$, $d(n) = \sum_{d/n} 1$, Euler's totient function $\phi(n)$ and the prime counting function $\pi(x) = \sum_{p \leqslant x} 1$. Recall that a function R is rational if it can be written in the form $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomial functions in x and $Q(x)$ is not the zero polynomial.

In this paper, we establish some new properties of the functions mentioned above regarding rational functions and we study when the composition of an arithmetic function and a polynomial restricted to the domain of prime numbers is another polynomial. Concerning this matter, we also prove that the composition between the prime counting function and a polynomial restricted to the domain of prime numbers cannot be another polynomial. There are many estimates of the arithmetic functions and the prime counting function in the literature. In the proofs of our results, we shall use the following classical estimates,

THEOREM 1.1. *For the functions $\sigma(n), d(n), \phi(n)$ and $\pi(x)$ defined above, the following properties hold:*

(1) $\sigma(n) < n \log n$, for all $n \geqslant 7$;
(2) $d(n) = o(n^{\varepsilon})$, for all $\varepsilon > 0$;

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(3) $\pi(x) \sim \dfrac{x}{\log x}, x \to \infty$;

(4) $\phi(\sigma(n)) < n\varepsilon$, *for all* $\varepsilon > 0$, *except for a set of density* 0.

(5) $\dfrac{\phi(n)}{n} \geqslant \dfrac{1}{3\log\log n}$, *for all* $n \geqslant 67$.

The proof of this theorem can be found in [8] (part (1), part (5)), [4], [8] (part(2)), [2], [9] (part 4). Part (3) of the theorem is nothing else than the celebrated prime number theorem. If we denote by p_n the n-th prime number, then the prime number theorem can be stated as $p_n \sim n\log n$.

First of all, we state the following folklore

THEOREM 1.2. *The prime counting function,* $\pi(x)$ *cannot be a rational function for all* x *positive integers.*

The standard proof of theorem 1.2 involves algebraic properties of polynomials. In the next section, we give another proof of this theorem based on elementary tools of Real Analysis. This theorem appears as an exercise at page 101 in [4]. In this paper, we will prove other theorems concerning polynomials and arithmetic functions and the prime counting function, namely

THEOREM 1.3. *There do not exist polynomials* $P, Q \in \mathbb{R}[X]$ *such that*

$$\int_0^{\log n} \frac{P(x)}{Q(x)} dx = \frac{n}{\pi(n)},$$

for all positive integers $n \geqslant 1$.

THEOREM 1.4. *Let* $f \in \{\sigma(n), \phi(n), d(n)\}$. *If* $f(P(p)) = Q(p)$ *for any prime number* p, *where* $P, Q \in \mathbb{Z}[X]$ *are monic polynomials, then* $P(X) = X^k$, *where* k *is positive integer.*

THEOREM 1.5. *There do not exists polynomials* $P, Q \in \mathbb{Z}[X]$ *such that* $\pi(P(p)) = Q(p)$, *for any prime number* p.

THEOREM 1.6. *There do not exist polynomials* $P, Q \in \mathbb{R}[X]$ *such that* $g(\sigma(n)) = \dfrac{P(n)}{Q(n)}$, *for all positive integers* $n \geqslant 1$, *where* $g \in \{\phi(n), \sigma(n), d(n)\}$.

2. Proofs of the main results

In this section we prove our main results stated in the previous section. First of all, we begin with the proof of Theorem 1.2, which can be summarised as it follows:

Proof of Theorem 1.2. We assume by contradiction that $\pi(x) = \dfrac{P(x)}{Q(x)}$, for all positive integers x. By the prime number theorem (Theorem 1.1, (*3*)), we have that

$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$. This means that $\lim_{x\to\infty} \frac{P(x)}{xQ(x)} = 0$ which implies that $\deg(P) < \deg(Q) + 1$. On the other hand, since $\lim_{x\to\infty} \pi(x) = \infty$, we have that $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \infty$ which means that $\deg(P) > \deg(Q)$, false. $\square$

REMARK. There exists an algebraic proof of the theorem as mentioned in the previous section which perhaps is already a folkore. It is well-known that if $\pi(n) > \pi(n-1)$, then n is prime. Let us assume that n is a composite number. Thus, $\pi(n) = \pi(n-1)$. We argue by contradiction and assume that $\pi(x) = \frac{P(x)}{Q(x)}$, for all positive integers x. Thus, for n composite we have $\frac{P(n)}{Q(n)} = \frac{P(n-1)}{Q(n-1)}$. Let $S(x) = P(x)Q(x-1) - P(x-1)Q(x)$. From the above assumption, we have that $S(n) = 0$, whenever n is composite. But this means that S has many infinitely zeroes and thus $S \equiv 0$ and we deduce $\pi(n) = \pi(n-1)$, for all n, contradiction.

In [5], L. Panaitopol proved that for every $n \geqslant 1429$, the inequality $\pi(n) > \frac{n}{H_n}$ holds true, where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the harmonic sequence. In fact, this was observed for the first time by Locker-Ernst in [6] which stated that for $n > 50$, a good approximation for $\pi(n)$ is given by n/H_n. The proof given in [5] uses strong approximations for the prime counting functions obtained by Rosser and Schoenfeld in [7]. The proof of the Theorem 1.3 that will be given in what will follow does not use advanced approximations for $\pi(x)$ as described in [5] or [7].

Proof of Theorem 1.3. Suppose that such polynomials exist. Let R be that rational function and put

$$f(x) = \frac{1}{x} \cdot \int_0^{\log x} R(t)dt.$$

Then we have $f(n) = f(n+1)$ whenever $n+1$ is composite. Thus f' vanishes infinitely many times by Rolle's theorem, so there is a sequence c_n between n and $n+1$ whenever $n+1$ is composite such that $f'(c_n) = 0$. Since

$$f(x) + xf'(x) = \frac{1}{x} R(\log x),$$

this gives

$$R(\log(c_n)) = \int_0^{\log(c_n)} R(t)dt,$$

which means that by asymptotic cosiderations that R must be null, contradiction. $\square$

The proof of the next theorem is quite elementary and involves rudiments of Real Analysis and uses the celebrated Dirichlet's theorem on arithmetic progressions.

Proof of Theorem 1.4. First of all, we will prove that $\deg(P) = \deg(Q)$. Assume that $f = \phi(n)$ and $\deg(Q) > \deg(P)$. Since $\phi(n) \leqslant n, n \geqslant 1$ it follows immediately that $Q(p) > P(p)$, so the polynomial $Q(x) - P(x)$ is nonconstant and monic and thus, we have that $\lim_{x\to\infty}(Q(x) - P(x)) = \infty$. But this last assertion contradicts the inequality $Q(p) - P(p) \leqslant 0$, whenever p is prime number. Thus $\deg(Q) \leqslant \deg(P)$. Now, we will prove the converse inequality; $\deg(Q) \geqslant \deg(P)$. Assume by contradiction that $\deg(Q) \leqslant \deg(P) - 1$. Then the polynomial $2P(x) - xQ(x)$ is nonconstant and monic and like we did above, we have that $\lim_{x\to\infty}(2P(x) - xQ(x)) = \infty$, so for sufficiently large prime number p we have $2P(p) - pQ(p) \geqslant 0$ which is equivalent to $\dfrac{Q(p)}{P(p)} \leqslant \dfrac{2}{p}$. On the other hand, we have that

$$\frac{\phi(P(p))}{P(p)} = \prod_{\substack{q \text{ prime} \\ q|P(p)}} \left(1 - \frac{1}{q}\right) > e^{-\sum_{\substack{q \text{ prime} \\ q|P(p)}} \frac{1}{q-1}}.$$

Now, by the prime number theorem stated in the form $p_n \sim n \log n$, there exists a constant $k \in \mathbb{N}^*$ such that $p_n - 1 > \dfrac{n \log n}{k}$, for all $n \geqslant 1$. This means that

$$\sum_{i=1}^{n} \frac{1}{p_i - 1} \leqslant 1 + k \sum_{i=2}^{n} \frac{1}{i \log i}.$$

By Lagrange's mean value theorem applied to the function $x \mapsto \log\log x$, we have

$$\frac{1}{n \log n} < \log\log n - \log\log(n-1),$$

for all $n \geqslant 2$ and it follows that $\sum_{i=2}^{n} \dfrac{1}{i \log i} < 1 + \log\log n$. We obtain

$$\sum_{i=1}^{n} \frac{1}{p_i - 1} < 1 + k + k \log\log n, \ \forall n \geqslant 2.$$

There is a $t \in \mathbb{N}$ which satisfies $p_t < P(p) < p_{t+1}$ and thus we obtain

$$\sum_{\substack{q \text{ prime} \\ q|P(p)}} \frac{1}{q-1} \leqslant \sum_{\substack{q \text{ prime} \\ q \leqslant P(p)}} \frac{1}{q-1} < 1 + k + k \log\log t,$$

It is easy to see that $t \leqslant P(p)$, so

$$\sum_{\substack{q \text{ prime} \\ q|P(p)}} \frac{1}{q-1} \leqslant 1 + k + k \log\log P(p).$$

Using the inequality above and the fact that $\dfrac{\phi(P(p))}{P(p)} > e^{-\sum_{\substack{q \text{ prime} \\ q|P(p)}} \frac{1}{q-1}}$, we obtain that

$$\frac{\phi(P(p))}{P(p)} > e^{1+k}\left(\frac{1}{\log P(p)}\right)^k.$$

On the other hand, since $\dfrac{\phi(P(p))}{P(p)} \leqslant \dfrac{2}{p}$, we have that $p^{\frac{1}{k}} \cdot e^{\frac{k+1}{k}} 2^{-\frac{1}{k}} < \log P(p)$, and if we denote the constant with $A = e^{\frac{k+1}{k}} 2^{-\frac{1}{k}}$ we have $e^{A\sqrt[k]{p}} < P(p)$. Now if we denote with $B(X) = P(X^k)$ we know that $\lim\limits_{x\to\infty} \dfrac{e^{Ax}}{B(x)} = +\infty$, since $A > 0$. This means that for p big enough it holds $e^{A\sqrt[k]{p}} > B(\sqrt[k]{p})$ which is in contradiction with $e^{A\sqrt[k]{p}} < P(p)$ so our assumption $\deg(Q) \leqslant \deg(P) - 1$ fails. Thus $\deg(Q) \leqslant \deg(P)$, so $\deg(Q) = \deg(P)$. Next, we prove that $P(0) = 0$. Assume by contradiction that $P(0) \neq 0$. If we take a prime $q > |P(0)|$, by the property above, we have $P(q) \equiv P(0)(\operatorname{mod} q)$ which means that $(P(q), q) = 1$. According to Dirichlet's theorem, the arithmetic progression $q + rP(q)$ contains many infinitely prime numbers. Let $q_m = q + r_m P(q)$ be the m-th prime in this sequence. We have that

$$P(q + r_m P(q)) \equiv P(q)(\operatorname{mod} P(q)),$$

so $P(q)$ divides $P(q_m)$. Since $\dfrac{\phi(a)}{a} = \prod\limits_{\substack{q \text{ prime} \\ q|a}} \left(1 - \dfrac{1}{q}\right)$ one can easily deduce that for $d|a$ one has $\dfrac{\phi(a)}{a} \leqslant \dfrac{\phi(d)}{d}$. This implies $\dfrac{\phi(P(q_s))}{P(q_s)} \leqslant \dfrac{\phi(P(q))}{P(q)}$ which is equivalent to $\dfrac{Q(q_s)}{P(q_s)} \leqslant \dfrac{\phi(P(q))}{P(q)}$.

Let us note that $\lim\limits_{s\to\infty} q_s = +\infty$. We know that Q and P have the same degree and both are monic polynomials, so $\lim\limits_{x\to+\infty} \dfrac{Q(x)}{P(x)} = 1$. This means that $\lim\limits_{s\to+\infty} \dfrac{Q(q_s)}{P(q_s)} = 1$. Passing to limit when $s \to \infty$ in $\dfrac{\phi(P(q_s))}{P(q_s)} \leqslant \dfrac{\phi(P(q))}{P(q)}$ we obtain $1 \leqslant \dfrac{\phi(P(q))}{P(q)}$ or $P(q) \leqslant \phi(P(q))$. We conclude that $P(q) = 1$. But this can not hold for many infinitely primes q, otherwise $P \equiv 1$, in contradiction with our assumption that P is nonconstant.

Let $P(X) = X^j R(X)$ with $R(0) \neq 0$. We now have $\phi(P(q)) = \phi(q^j R(q))$ and for $q > |R(0)|$ we have $(q, R(q)) = 1$ so

$$\phi(P(q)) = \phi(q^j) \cdot \phi(R(q)) = q^{j-1}(q-1) \cdot \phi(R(q)).$$

Now let $Q(X) = X^i S(X)$ cu $S(0) \neq 0$. We have that $q^{j-1} | q^i S(q)$. If $i < j - 1$ then $q|S(q)$ and since $S(q) \equiv S(0)(\operatorname{mod} q)$ we have $q|S(0)$ for infinitely many primes. This leads to $S(0) = 0$, in contradiction with $S(0) \neq 0$, and therefore $i \geqslant j - 1$. We also have $q - 1 | q^i S(q)$ and since $(q, q-1) = 1$ we get $q - 1 | S(q)$. We know that $S(q) \equiv S(1)(\operatorname{mod}(q-1))$ which combined with $q - 1 | S(q)$ we get $q - 1 | S(1)$, for all primes

$q > |R(0)|$. Thus $S(1) = 0$. This means $Q(X) = X^i(X-1)S_1(X)$ so $\phi(R(q)) = q^{i-j+1}S_1(q)$ and if we denote $X^{i-j+1}S_1(X) = L(x)$ we have a monic polynomial such that $\phi(R(q)) = L(q)$. Applying the same arguments for R and L if R is nonconstant we would have $R(0) = 0$ in contradiction with $R(0) \neq 0$. Thus R is constant and $R \equiv 1$.We conclude that the only solution is $P(X) = X^j$ with $j \geqslant 1$.

Now for the case when $f = \sigma(n)$, we assume that $\deg(P) > \deg(Q)$. Since $\sigma(n) > n, n \geqslant 2$ it follows that $Q(p) > P(p)$ for all p primes. From $\deg(P) > \deg Q$ we deduce that $P(x) - Q(x)$ is nonconstant and monic which implies $\lim\limits_{x\to\infty}(P(x) - Q(x)) = +\infty$. But this last assertion contradicts the inequality $P(p) - Q(p) < 0$, for all primes p. Thus $\deg(P) \leqslant \deg(Q)$.

Now, we will prove the converse inequality $\deg(Q) \leqslant \deg(P)$. Assume by contradiction that $\deg(Q) \geqslant \deg(P) + 1$. Then the polynomial $2Q(x) - xP(x)$ is nonconstant and applying the same arguments as above, we have that $\lim\limits_{x\to\infty}(2Q(x) - xP(x)) = \infty$, so for sufficiently large prime number p we have $2Q(p) - pP(p) \geqslant 0$ which is equivalent to $\dfrac{Q(p)}{P(p)} \geqslant \dfrac{p}{2}$ and further to $\dfrac{\sigma(P(p))}{P(p)} \geqslant \dfrac{p}{2}$.

Since $\lim\limits_{x\to\infty} P(x) = \infty$, there is a p_0 such that for all primes $p \geqslant p_0$ we have $P(p) \geqslant 7$. Using Theorem 1.1 (part (*1*)) we have $\dfrac{\sigma(P(p))}{P(p)} < \log P(p)$.

Combining with $\dfrac{\sigma(P(p))}{P(p)} \geqslant \dfrac{p}{2}$ we obtain $\log P(p) > \dfrac{p}{2}$ for all primes $p \geqslant p_0$.

Now we just have to notice that

$$\lim_{x\to\infty} \frac{\log P(x)}{x} = \lim_{x\to\infty} \frac{P'(x)}{P(x)} = 0.$$

This provides the immediate contradiction to $\dfrac{\log P(p)}{p} > \dfrac{1}{2}$ for all primes $p \geqslant p_0$.

Thus the assumption $\deg(Q) \geqslant \deg(P) + 1$ fails so $\deg(Q) \leqslant \deg(P)$ and corroborating with $\deg(Q) \geqslant \deg(P)$, we conclude that $\deg(Q) = \deg(P)$.

Next, we prove that $P(0) = 0$.

Assume by contradiction that $P(0) \neq 0$. For a fixed prime $q > |P(0)|$, by the property above, we have $P(q) \equiv P(0) (\mathrm{mod}\, q)$ which means that $(P(q), q) = 1$. According to Dirichlet's theorem, the arithmetic progression $q + rP(q)$ contains infinitely many primes. Let q_m be th m-th prime in this sequence. We have that $P(q + r_m P(q)) \equiv P(q) (\mathrm{mod}\, P(q))$, so $P(q)$ divides $P(q_m)$. For a positive integer a since we have

$$\frac{\sigma(a)}{a} = \prod_{\substack{q \text{ prime} \\ q|a, \upsilon_q(a) = \alpha_q}} \left(1 + \frac{1}{q} + ... + \frac{1}{q^{\alpha_q}}\right)$$

(here $\upsilon_q(a)$ is the exponent of q in the decomposition of a in prime factors) one can easily deduce that for $a|d$ one has $\dfrac{\sigma(a)}{a} \leqslant \dfrac{\sigma(d)}{d}$. This implies $\dfrac{\sigma(P(q_s))}{P(q_s)} \geqslant \dfrac{\sigma(P(q))}{P(q)}$ which is equivalent to $\dfrac{Q(q_s)}{P(q_s)} \geqslant \dfrac{\sigma(P(q))}{P(q)}$.

Let us note that $\lim_{s\to\infty} q_s = +\infty$. We know that Q and P have the same degree and both are monic polynomials so $\lim_{x\to+\infty} \frac{Q(x)}{P(x)} = 1$. This means that $\lim_{s\to+\infty} \frac{Q(q_s)}{P(q_s)} = 1$. Passing to limit when $s \to \infty$ in $\frac{\sigma(P(q_s))}{P(q_s)} \geqslant \frac{\sigma(P(q))}{P(q)}$, we obtain $1 \geqslant \frac{\sigma(P(q))}{P(q)}$ so $P(q) \geqslant \sigma(P(q))$ which means $P(q) = 1$ for all prime numbers q so $P \equiv 1$. Thus our assumption was false and let $P(X) = X^j R(X)$ with $R(0) \neq 0$. Now, we have $\sigma(P(q)) = \sigma(q^j R(q))$ and for $q > |R(0)|$ it follows that $(q, R(q)) = 1$ so

$$\sigma(P(q)) = (q^j + q^{j-1} + \ldots + 1)\sigma(R(q)).$$

Let $D(X) = X^j + X^{j-1} + \ldots + 1$. Since Q and D are monic we know that there are $C(X)$ and $T(X)$ in $\mathbb{Z}[X]$ such that $Q(X) = D(X) \cdot C(X) + T(X)$ where $\deg(T) < \deg(D)$. We know that $D(q)|Q(q)$ and it implies $D(q)|T(q)$.

Now since $\deg(T) < \deg(D)$, we obtain $\lim_{x\to\infty} \frac{T(x)}{D(x)} = 0$. This means that $|T(q)| < D(q)$ for all large q and from $D(q)|T(q)$ we can conclude that $T(q) = 0$ for such primes. So T has an infinity of roots thus $T \equiv 0$.

We have

$$(q^j + q^{j-1} + \ldots + 1)\sigma(R(q)) = D(q)\sigma(R(q)) = D(q)C(q)$$

which leads to $\sigma(R(q)) = C(q)$ and since $R(0) \neq 0$ we obtain $R \equiv 1$,by looking at the argument for the previous case. We conclude again that $P(X) = X^j$ with $j \geqslant 1$.

The last case is $f = d(n)$. From Theorem 1.1 (part (2)), we have $Q(p) = o(P^{\varepsilon}(p))$ which is equivalent to the fact that $\lim_{l\to\infty} \frac{Q(p_l)}{P^{\varepsilon}(p_l)} = \lim_{x\to\infty} \frac{Q(x)}{P^{\varepsilon}(x)}$ is finite for all $\varepsilon > 0$. Thus we deduce that Q is constant, otherwise we could have chosen $\varepsilon = \frac{\deg(Q)}{2\deg(P)}$ which provides us a contradiction.

Now let us assume that $P(0) \neq 0$. Again, we employ the well-known property of polynomials which states that for all $a, b \in \mathbb{Z}$ one has $a - b | f(a) - f(b)$. For a fixed prime $q > |P(0)|$, by the property above, we have $P(q) \equiv P(0) (\mathrm{mod}\, q)$ which means that $(P(p), p) = 1$. According to Dirichlet's theorem, the arithmetic progression $q + rP(q)$ contains many infinitely prime numbers. Let q_m be the m-th prime in this sequence. We have that $P(q + r_m P(q)) \equiv P(q) (\mathrm{mod} P(q))$ so $P(q)$ divides $P(q_m)$. Now if $a|b$ and $a < b$ we have $d(a) < d(b)$ so if P is nonconstant $d(P(q)) < d(P(q_m))$ which leads to $Q(q) < Q(q_m)$, which contradicts the fact that Q is constant.

Thus $P(X) = X^j R(X)$ with $R(0) \neq 0$. We now have $d(P(q)) = d(q^j R(q))$ and for $q > |R(0)|$ we have $(q, R(q)) = 1$ so $(P(q)) = d(q^j) \cdot d(R(q)) = (j+1) \cdot d(R(q))$. This means that we have $d(R(q)) = \frac{Q(q)}{j+1}$ so the polynomial R has the property that $d(R(q))$ is constant for all primes q but $R(0) \neq 0$ thus it is constant. So $f(X) = cX^j$ with $j \geqslant 1$. $\square$

REMARK. In the case when $f=\phi(n)$, we would have obtained easier the fact that the polynomials have the same degree using the result from Theorem 1.1 (part *(5)*).

Proof of Theorem 1.5. From Theorem 1.1 (part *(3)*) we know that $\lim_{n\to\infty}\frac{\pi(n)}{n}=0$ so $\lim_{p\to\infty}\frac{\pi(P(p))}{P(p)}=\lim_{p\to\infty}\frac{Q(p)}{P(p)}=0$ which is equivalent to $\lim_{x\to\infty}\frac{Q(x)}{P(x)}=0$. We deduce that $\deg(Q)\leqslant\deg(P)-1$.

Again from Theorem 1.1 (part *(3)*), we have $\lim_{n\to\infty}\frac{\pi(n)\log n}{n}=1$ and thus $\lim_{p\to\infty}\frac{Q(p)\log P(p)}{P(p)}=1$ which is equivalent to $\lim_{x\to\infty}\frac{Q(x)\log P(x)}{P(x)}=1$.

Now since $\deg(Q)\leqslant\deg(P)-1$ the limit $\lim_{x\to\infty}\frac{xQ(x)}{P(x)}$ is finite. Morever we have

$$\lim_{x\to\infty}\frac{\log P(x)}{x}=\lim_{x\to\infty}\frac{P'(x)}{P(x)}=0$$

applying L'Hopital rule and using the fact that $\deg(P')<\deg(P)$. Combining these two observations, by multiplying, we get $\lim_{x\to\infty}\frac{Q(x)\log P(x)}{P(x)}=0$. This ends the proof of the theorem. $\square$

In [9] and [10] many properties are proved about arithmetic functions especially asymptotic estimates of composition of functions like: $\phi(\sigma(n)),\sigma(\sigma(n))$ or $d(\sigma(n))$. In the proof of the theorem that will follow, we shall use our idea from the proof of the Theorem 1.1.

Proof of Theorem 1.6. In all cases we proceed by the way of contradiction. This means that $g(\sigma(n))=\frac{P(n)}{Q(n)},n\geqslant 1$. Firstly, we deal with the case when $g=\phi(n)$. We proceed as in the proof of Theorem 1.1. In [3] and [9] it is proved that $\lim_{n\to\infty}\frac{\phi(\sigma(n))}{n}=0$ except for a set of density 0. This implies that $\lim_{n\to\infty}\frac{P(n)}{nQ(n)}=0$ and thus $\deg(P)<1+\deg(Q)$. On the other hand, since $\lim_{n\to\infty}\phi(\sigma(n))=\infty$, we have that $\lim_{n\to\infty}\frac{P(n)}{Q(n)}=\infty$ which means that $\deg(P)>\deg(Q)$, contradiction.

For $g=\sigma(n)$. We know from [8] that $\limsup_{n\to\infty}\frac{\sigma(n)}{n}=\infty$ so $\limsup_{n\to\infty}\frac{\sigma(\sigma(n))}{n}=\infty$. Thus $\lim_{x\to\infty}\frac{P(x)}{xQ(x)}=\infty$ and we deduce that $\deg(P)>\deg(Q)+1$. Now from Theorem 1.1 (part *(1)*), we have

$$\sigma(\sigma(n))<\sigma(n)\log\sigma(n)<n\log n(\log n+\log\log n)$$

for $n \geqslant 7$. It folows immediately that $\lim\limits_{n\to\infty} \dfrac{\sigma(\sigma(n))}{n^2} = 0$ and thus $\lim\limits_{x\to\infty} \dfrac{P(x)}{x^2 Q(x)} = \infty$ which gives $\deg(P) < \deg(Q) + 2$, and this combined with $\deg(P) > \deg(Q) + 1$ leads to a contradiction.

Finally, when $g = d(n)$, we have (see [8], [4]) that $d(n) \leqslant 2\sqrt{n}, n \geqslant 1$. This means that $d(\sigma(n)) \leqslant 2\sqrt{\sigma(n)}$ so $\lim\limits_{n\to\infty} \dfrac{d(\sigma(n))}{n} = 0$ since $\lim\limits_{n\to\infty} \dfrac{\sigma(n)}{n^2} = 0$ according to Theorem 1.1 (part (*1*)). It follows that $\lim\limits_{x\to\infty} \dfrac{P(x)}{xQ(x)} = 0$ so we have $\deg(P) \leqslant \deg(Q)$.

Now we shall prove that $\limsup\limits_{n\to\infty} d(\sigma(n)) = \infty$. Let $p_1, \ldots, p_k$ the first k prime numbers. According to Dirichlet's theorem, the arithmetic progression $rp_1p_2\ldots p_k - 1$ with $r \in \mathbb{N}^*$ contains an infinity of primes and let q be one of them. We have that $d(\sigma(q)) = d(q+1) = d(rp_1p_2\ldots p_k) \geqslant 2^k$. Letting k be arbitrarly large we get the desired result. Using $\limsup\limits_{n\to\infty} d(\sigma(n)) = \infty$, we deduce $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \infty$ so $\deg(P) > \deg(Q)$. This obviously contradicts $\deg(P) \leqslant \deg(Q)$. $\square$

REMARK. The result $\limsup\limits_{n\to\infty} d(\sigma(n)) = \infty$ appears also in [11, 12]. One can apply the classical inequality $d(m)\sigma(n) \geqslant m$ for $m = \sigma(n) \geqslant n$, and use the liminf result on $\dfrac{\phi(\sigma(n))}{n}$.

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REFERENCES

[1] I. NIVEN, E. ZUCKERMAN, H. MONTGOMERY, *An Introduction to the Theory of Numbers*, John Wiley & Sons (5-th edition), 1991.

[2] L. ALAOGLU, P. ERDOS, *A conjecture in elementary number theory*, Bull. of the AMS **50** (1944), 881–882.

[3] P. ERDOS, A. SARKOZY, C. POMERANCE, *On Locally repeated values of certain arithmetic functions I*, Jour. of Numb. Theory **21** (1985), 319–332.

[4] T. APOSTOL, *Introduction to Analytic Number Theory*, Springer Verlag, 1976.

[5] L. PANAITOPOL, *Several approximations of* $\pi(x)$, Math. Ineq. & Appl. (MIA) **2** (1999), 317–324.

[6] L. LOCKER-ERNST, *Bemerkungen uber die Verteilung Primzahlen*, Elem. der Math. **14** (1959), 1–5.

[7] J. B. ROSSER, L. SCHOENFELD, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

[8] A. GICA, L. PANAITOPOL, *O Introducere în Aritmetică şi Teoria Numerelor*, Editura Universităţii Bucureşti, 2001.

[9] J. SANDOR, L. TOTH, *Extremal orders of compositions of certain arithmetic functions*, Electronic Jour. Comb. Numb. Theory **8** (2008), A34.

[10] J. SANDOR, *On the composition of some arithmetic functions*, II, J. Inequal. Pure and Appl. Math. (JIPAM) **6**, 3 (2005), Art. 73.
[11] J. SANDOR, *On the composition of some arithmetic functions*, Studia Univ. Babes-Bolyai Math. **34** (1989), 7–14.
[12] J. SANDOR, *Selected chapters of geometry, analysis and number theory. Classical topics in new perspectives*, Lambert Acad. Publ., Germany, 2008

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