

SOME NEW BOUNDS FOR THE GENERALIZED TRIANGLE INEQUALITY

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Abstract. In this paper we present some new inequalities in normed linear spaces which much improve the triangle inequality. Our results refine and generalize the corresponding ones obtained by Mitani et al. [On sharp triangle inequalities in Banach spaces, J. Math. Anal. Appl. 336 (2007) 1178-1186].

1. Introduction

The triangle inequality is one of the most fundamental and extensively used inequalities in analysis and other fields of mathematics. This inequality has been studied by many authors(see e.g. [4, 6, 7] and references cited therein). Recently Mitani et al. [9] proved the following sharp triangle inequality and its reverse inequality with n elements in a normed linear space.

THEOREM 1.1. *For any nonzero elements x_1, x_2, \dots, x_n in a normed linear space X , we have*

$$\begin{aligned} & \sum_{k=2}^n \left(k - \left\| \sum_{i=1}^k \frac{x_i^*}{\|x_i^*\|} \right\| \right) (\|x_k^*\| - \|x_{k+1}^*\|) \\ & \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \\ & \leq \sum_{k=2}^n \left(k - \left\| \sum_{i=n-k+1}^n \frac{x_i^*}{\|x_i^*\|} \right\| \right) (\|x_{n-k+1}^*\| - \|x_{n-k}^*\|), \end{aligned} \tag{1.1}$$

where $x_1^*, x_2^*, \dots, x_n^*$ is a rearrangement of x_1, x_2, \dots, x_n with $\|x_1^*\| \geq \|x_2^*\| \geq \dots \geq \|x_n^*\|$ and $x_{n+1}^* = x_0^* = 0$.

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In the case $n = 2$, for any nonzero elements $x, y \in X$ with $\|x\| \leq \|y\|$ we have the following

$$\begin{aligned} \|x+y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\| &\leq \|x\| + \|y\| \\ &\leq \|x+y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|y\|. \end{aligned} \tag{1.2}$$

The first inequality in (1.2) was given earlier in Hudzik and Landes [3]. The inequality (1.2) is also found in a recent paper of Maligranda [6]. In 2008, Maligranda [7] used (1.2) to obtain the estimation

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \alpha[x, y] \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \tag{1.3}$$

of the *angular distance* $\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ between two nonzero elements x and y in X which was defined by Clarkson in [1]. The right hand of estimate (1.3) is a refinement of the Massera-Schaffer inequality proved in 1958 (see [8, Lemma 5.1]): for nonzero vectors x and y in X we have $\alpha[x, y] \leq \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}$, which is stronger than the Dunkl-Williams inequality $\alpha[x, y] \leq \frac{4\|x-y\|}{\|x\| + \|y\|}$ proved in [2]. In the same paper, Dunkl and Williams proved that the constant 4 can be replaced by 2 if X is an inner product space. Kirk and Smiley [5] proved that if the inequality $\alpha[x, y] \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$ holds for all nonzero $x, y \in X$, then X is an inner product space.

The purpose of this paper is to establish some new sharp inequalities related to the generalized triangle inequality in normed linear spaces. To proceed in this direction we first introduce the following notations.

Let $n \geq 2$ be an integer, $\mathbf{q} = (q_1, \dots, q_n)$ be a n -tuple of nonnegative real numbers and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, where X is a normed linear space. Consider the *partial functionals*

$$\mathcal{F}_{nk}(\mathbf{x}, \mathbf{q}) = \sum_{i=1}^k q_i \|x_i\| - \left\| \sum_{i=1}^k q_i x_i \right\| \geq 0$$

and

$$\mathcal{F}'_{nk}(\mathbf{x}, \mathbf{q}) = \sum_{i=n-k+1}^n q_i \|x_i\| - \left\| \sum_{i=n-k+1}^n q_i x_i \right\| \geq 0$$

for all $k \in \{2, 3, \dots, n\}$. When $k = n$ we shall write $\mathcal{F}_n(\mathbf{x}, \mathbf{q})$ instead of $\mathcal{F}_{nn}(\mathbf{x}, \mathbf{q})$ and $\mathcal{F}'_{nn}(\mathbf{x}, \mathbf{q})$.

2. Main results

In this section we shall show the following inequalities which are more general than the inequalities in Theorem 1.1. Our proof is different from the proof given in [9].

THEOREM 2.1. *Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be two n -tuples of non-negative real numbers such that $q_i \neq 0$ for each $i \in \{1, 2, \dots, n\}$ and $p_1/q_1 \geq \dots \geq p_n/q_n$. Then, we have*

$$(0 \leq) \sum_{k=2}^n \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) \mathcal{T}_{nk}(\mathbf{x}, \mathbf{q}) \leq \mathcal{T}_n(\mathbf{x}, \mathbf{p}) \tag{2.1}$$

$$\leq \sum_{k=2}^n \left(\frac{p_{n-k+1}}{q_{n-k+1}} - \frac{p_{n-k}}{q_{n-k}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}), \tag{2.2}$$

where $p_{n+1}/q_{n+1} = p_0/q_0 = 0$.

Proof. First, we prove the inequality (2.1) by induction. For $n = 2$ it follows from the triangle inequality that

$$\begin{aligned} \|p_1x_1 + p_2x_2\| &= \left\| \frac{p_2}{q_2}(q_1x_1 + q_2x_2) + \left(p_1 - \frac{p_2q_1}{q_2} \right) x_1 \right\| \\ &\leq \frac{p_2}{q_2} \|q_1x_1 + q_2x_2\| + p_1 \|x_1\| - \frac{p_2q_1}{q_2} \|x_1\| \\ &= p_1 \|x_1\| + \frac{p_2}{q_2} (\|q_1x_1 + q_2x_2\| - q_1 \|x_1\|) \\ &= p_1 \|x_1\| + p_2 \|x_2\| + \frac{p_2}{q_2} (\|q_1x_1 + q_2x_2\| - q_1 \|x_1\| - q_2 \|x_2\|) \end{aligned}$$

which establishes (2.1) for the case $n = 2$. Therefore let $n \geq 3$. For simplicity, we put $m = p_n/q_n$ and $M = p_1/q_1$. Assume that the inequality (2.1) holds true for $n - 1$. Then by the triangle inequality,

$$\begin{aligned} \left\| \sum_{i=1}^n p_i x_i \right\| &= \left\| m \sum_{i=1}^n q_i x_i + \sum_{i=1}^{n-1} (p_i - m q_i) x_i \right\| \\ &\leq m \left\| \sum_{i=1}^n q_i x_i \right\| + \left\| \sum_{i=1}^{n-1} (p_i - m q_i) x_i \right\|. \end{aligned} \tag{2.3}$$

Now, let $\mathbf{x}' = (x'_1, \dots, x'_{n-1})$, $\mathbf{p}' = (p'_1, \dots, p'_{n-1})$ and $\mathbf{q}' = (q'_1, \dots, q'_{n-1})$, where $x'_i = x_i$, $p'_i = p_i - m q_i \geq 0$ and $q'_i = q_i > 0$ for all $i \in \{1, 2, \dots, n - 1\}$. It is easy to see that $p'_1/q'_1 \geq \dots \geq p'_{n-1}/q'_{n-1}$ and $\mathcal{T}_{(n-1)k}(\mathbf{x}', \mathbf{q}') = \mathcal{T}_{nk}(\mathbf{x}, \mathbf{q})$ for all $k \in \{2, 3, \dots, n - 1\}$.

From the inductive assumption on (2.1) we have

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} p'_i x'_i \right\| &\leq \sum_{i=1}^{n-1} p'_i \|x'_i\| - \sum_{k=2}^{n-1} \left(\frac{p'_k}{q'_k} - \frac{p'_{k+1}}{q'_{k+1}} \right) \mathcal{T}_{(n-1)k}(\mathbf{x}', \mathbf{q}') \\ &= \sum_{i=1}^{n-1} (p_i - mq_i) \|x_i\| - \sum_{k=2}^{n-1} \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) \mathcal{T}_{nk}(\mathbf{x}, \mathbf{q}), \end{aligned}$$

where we take $p'_n/q'_n = 0$. This together with (2.3) implies that

$$\begin{aligned} \left\| \sum_{i=1}^n p_i x_i \right\| &\leq m \left\| \sum_{i=1}^n q_i x_i \right\| + \sum_{i=1}^{n-1} p_i \|x_i\| - m \sum_{i=1}^{n-1} q_i \|x_i\| + p_n \|x_n\| - mq_n \|x_n\| \\ &\quad - \sum_{k=2}^{n-1} \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) \mathcal{T}_{nk}(\mathbf{x}, \mathbf{q}) \\ &= \sum_{i=1}^n p_i \|x_i\| - \sum_{k=2}^n \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) \mathcal{T}_{nk}(\mathbf{x}, \mathbf{q}). \end{aligned}$$

This proves the inequality (2.1).

Next we shall show the inequality (2.2). By the triangle inequality, we have

$$\begin{aligned} \left\| \sum_{i=1}^n q_i x_i \right\| &= \left\| \frac{1}{M} \sum_{i=1}^n p_i x_i + \sum_{i=2}^n \left(q_i - \frac{p_i}{M} \right) x_i \right\| \\ &\leq \frac{1}{M} \left\| \sum_{i=1}^n p_i x_i \right\| + \left\| \sum_{i=2}^n \left(q_i - \frac{p_i}{M} \right) x_i \right\|. \end{aligned} \tag{2.4}$$

Take $\mathbf{y} = (y_1, \dots, y_{n-1})$, $\mathbf{t} = (t_1, \dots, t_{n-1})$ and $\mathbf{s} = (s_1, \dots, s_{n-1})$ where $y_i = x_{n-i+1}$, $t_i = q_{n-i+1} - p_{n-i+1}/M \geq 0$ and $s_i = q_{n-i+1} > 0$ for all $i \in \{1, 2, \dots, n-1\}$. It is easy to verify that $t_1/s_1 \geq \dots \geq t_{n-1}/s_{n-1}$ and $\mathcal{T}_{(n-1)k}(\mathbf{y}, \mathbf{s}) = \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q})$ for all $k \in \{2, 3, \dots, n-1\}$. Applying (2.1) for \mathbf{y} , \mathbf{t} and \mathbf{s} instead of \mathbf{x} , \mathbf{p} and \mathbf{q} , respectively, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} t_i y_i \right\| &\leq \sum_{i=1}^{n-1} t_i \|y_i\| - \sum_{k=2}^{n-1} \left(\frac{t_k}{s_k} - \frac{t_{k+1}}{s_{k+1}} \right) \mathcal{T}_{(n-1)k}(\mathbf{y}, \mathbf{s}) \\ &= \sum_{i=1}^{n-1} \left(q_{n-i+1} - \frac{p_{n-i+1}}{M} \right) \|x_{n-i+1}\| - \frac{1}{M} \sum_{k=2}^{n-1} \left(\frac{p_{n-k}}{q_{n-k}} - \frac{p_{n-k+1}}{q_{n-k+1}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}) \\ &= \sum_{i=2}^n \left(q_i - \frac{p_i}{M} \right) \|x_i\| + \frac{1}{M} \sum_{k=2}^{n-1} \left(\frac{p_{n-k+1}}{q_{n-k+1}} - \frac{p_{n-k}}{q_{n-k}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}) \end{aligned}$$

where we take $t_n/s_n = 0$. This together with (2.4) implies that

$$\begin{aligned} \left\| \sum_{i=1}^n q_i x_i \right\| &\leq \frac{1}{M} \left\| \sum_{i=1}^n p_i x_i \right\| + \sum_{i=2}^n q_i \|x_i\| - \frac{1}{M} \sum_{i=2}^n p_i \|x_i\| \\ &\quad + q_1 \|x_1\| - \frac{1}{M} p_1 \|x_1\| + \frac{1}{M} \sum_{k=2}^{n-1} \left(\frac{p_{n-k+1}}{q_{n-k+1}} - \frac{p_{n-k}}{q_{n-k}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}), \end{aligned}$$

or equivalently,

$$\begin{aligned} \sum_{i=1}^n p_i \|x_i\| - \left\| \sum_{i=1}^n p_i x_i \right\| &\leq M \left(\sum_{i=1}^n q_i \|x_i\| - \left\| \sum_{i=1}^n q_i x_i \right\| \right) \\ &\quad + \sum_{k=2}^{n-1} \left(\frac{p_{n-k+1}}{q_{n-k+1}} - \frac{p_{n-k}}{q_{n-k}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}) \\ &= \sum_{k=2}^n \left(\frac{p_{n-k+1}}{q_{n-k+1}} - \frac{p_{n-k}}{q_{n-k}} \right) \mathcal{T}'_{nk}(\mathbf{x}, \mathbf{q}). \end{aligned}$$

This completes the proof. \square

For $\mathbf{p} = (1/n, \dots, 1/n)$ in Theorem 2.1 we obtain the following inequality

$$\begin{aligned} (0 \leq) \sum_{k=2}^n \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \left(\sum_{i=1}^k q_i \|x_i\| - \left\| \sum_{i=1}^k q_i x_i \right\| \right) \\ \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \tag{2.5} \\ \leq \sum_{k=2}^n \left(\frac{1}{q_{n-k+1}} - \frac{1}{q_{n-k}} \right) \left(\sum_{i=n-k+1}^n q_i \|x_i\| - \left\| \sum_{i=n-k+1}^n q_i x_i \right\| \right), \end{aligned}$$

where $q_n \geq \dots \geq q_1 > 0$ and $1/q_{n+1} = 1/q_0 = 0$.

We end this section with the following sharp triangle inequality.

THEOREM 2.2. *For any nonzero elements x_1, x_2, \dots, x_n in a normed linear space X , we have*

$$\begin{aligned} \sum_{k=2}^n \left(\sum_{i=1}^k \frac{\|x_i\|}{\|x_i^*\|} - \left\| \sum_{i=1}^k \frac{x_i}{\|x_i^*\|} \right\| \right) (\|x_k^*\| - \|x_{k+1}^*\|) \\ \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \tag{2.6} \\ \leq \sum_{k=2}^n \left(\sum_{i=n-k+1}^n \frac{\|x_i\|}{\|x_i^*\|} - \left\| \sum_{i=n-k+1}^n \frac{x_i}{\|x_i^*\|} \right\| \right) (\|x_{n-k+1}^*\| - \|x_{n-k}^*\|), \end{aligned}$$

where $x_1^*, x_2^*, \dots, x_n^*$ is a rearrangement of x_1, x_2, \dots, x_n with $\|x_1^*\| \geq \|x_2^*\| \geq \dots \geq \|x_n^*\|$ and $x_{n+1}^* = x_0^* = 0$.

Proof. It is immediate consequence of the inequality (2.5) by choosing $q_i = 1/\|x_i^*\|$ for all $i \in \{1, 2, \dots, n\}$. \square

The particular case of Theorem 2.2 is Theorem 1.1 due to Mitani et al. [9]. The following examples show that neither our refinement (2.6) nor refinement (1.1) of the generalized triangle inequality is always better.

EXAMPLE 2.3. Let X be the normed space \mathbb{R} with the norm $\|x\| = |x|$. Then, for $x_1 = -1, x_2 = 2, x_n = 1$ and $x_i = (-1)^i$ for all $3 \leq i < n$ we have $2 = \|x_1^*\| > \|x_2^*\| = \dots = \|x_n^*\| = 1$. By the elementary computations we obtain

$$\begin{aligned} L_1 &:= \sum_{k=2}^n \left(k - \left\| \sum_{i=1}^k \frac{x_i^*}{\|x_i^*\|} \right\| \right) (\|x_k^*\| - \|x_{k+1}^*\|) \\ &= n - \left\| \sum_{i=1}^n \frac{x_i^*}{\|x_i^*\|} \right\| = \begin{cases} n & \text{if } n \text{ even} \\ n - 1 & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

and

$$\begin{aligned} L_2 &:= \sum_{k=2}^n \left(\sum_{i=1}^k \frac{\|x_i\|}{\|x_i^*\|} - \left\| \sum_{i=1}^k \frac{x_i}{\|x_i^*\|} \right\| \right) (\|x_k^*\| - \|x_{k+1}^*\|) \\ &= \sum_{i=1}^n \frac{\|x_i\|}{\|x_i^*\|} - \left\| \sum_{i=1}^n \frac{x_i}{\|x_i^*\|} \right\| = \begin{cases} n - 1 & \text{if } n \text{ even} \\ n - 2 & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

Thus, $L_2 < L_1$.

EXAMPLE 2.4. Let $X = \mathbb{R}^2$ with the norm of $x = (a, b)$ given by $\|x\| = |a| + |b|$. Taking $x_1 = (-1, 0)$, $x_2 = (3/4, 3/4)$, $x_n = (0, -1)$ and $x_i = (0, (-1)^{i-1})$ for all $3 \leq i < n$, we have $3/2 = \|x_1^*\| > \|x_2^*\| = \dots = \|x_n^*\| = 1$. Therefore,

$$\frac{x_1}{\|x_1^*\|} = \left(\frac{-2}{3}, 0 \right), \frac{x_1^*}{\|x_1^*\|} = \left(\frac{1}{2}, \frac{1}{2} \right), \frac{x_i}{\|x_i^*\|} = x_i \text{ and } \frac{x_i^*}{\|x_i^*\|} = x_i^* \text{ for all } i = 2, 3, \dots, n.$$

Thus

$$L_1 = n - \left\| \sum_{i=1}^n \frac{x_i^*}{\|x_i^*\|} \right\| = n - 1$$

and

$$L_2 = \sum_{i=1}^n \frac{\|x_i\|}{\|x_i^*\|} - \left\| \sum_{i=1}^n \frac{x_i}{\|x_i^*\|} \right\| = \begin{cases} n - (2/3) & \text{if } n \text{ even} \\ n - (1/6) & \text{if } n \text{ odd} \end{cases},$$

which imply $L_1 < L_2$.

Similar examples show that the second inequalities in (1.1) and (2.6) are not comparable.

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