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ON THE INTERLACING OF CYLINDER FUNCTIONS

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Abstract. Necessary and sufficient conditions for the interlacing of the zeros of cylinder functions and their derivatives of different orders are given.

1. Introduction

The zeros of the Bessel functions have been subjects of studies for more than a century; the field of applications is vast. In the monograph of Watson [9] a number of aspects are discussed, a summary of the most important facts are listed in [1]. More recent results can be found in Refs. [2, 3] where further references are given. In this note I derive a new interlacing theorem for cylinder functions in the form of *necessary and sufficient* conditions. The results are of primary interest from the point of view of the theory of Bessel functions; however, applications might also arise as in Refs. [5, 6], where such relations were useful in inverse scattering problems.

The general solution of the Bessel differential equation (up to a constant multiplier) is given by the cylinder function [9]

$$C_\nu(x) \equiv J_\nu(x)\cos\delta - Y_\nu(x)\sin\delta \tag{1}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively. Considering the Bessel differential equation, a second order linear homogeneous ODE, satisfied by the Bessel functions it is easy to see that $J_\nu(x)$, $Y_\nu(x)$ and $J'_\nu(x)$, $Y'_\nu(x)$ each has an infinity of real zeros, for any given real value of ν. Furthermore, these zeros are all simple with the possible exception of $x=0$. I will use the term *interlace* for two functions if between each consecutive pair of zeros of one function there is one and only one zero of the other. Denote the sth zero of the functions $J_\nu(x)$, $Y_\nu(x)$, $J'_\nu(x)$, $Y'_\nu(x)$, $C_\nu(x)$ and $C'_\nu(x)$ by $j_{\nu,s}$, $y_{\nu,s}$, $j'_{\nu,s}$, $y'_{\nu,s}$, $c_{\nu,s}$ and $c'_{\nu,s}$, respectively, except that $x=0$ is counted as the first zero of $J'_0(x)$ [9].

The following theorem summarizes some known relevant interlacing results.

THEOREM 1. ([9, 8, 4, 5, 6, 7]) *For $\nu \geqslant 0$ the following points hold true.*

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(a) *For* $0 < a \leqslant 2$ *the positive real zeros of* $C_\nu(x)$ *and* $C_{\nu+a}(x)$ *are interlaced. Similarly,* $J'_\nu(x)$, $J'_{\nu+b}(x)$ *and* $Y'_\nu(x)$, $Y'_{\nu+b}(x)$ *are also interlaced if* $0 < b \leqslant 1$, *respectively.*

(b) *If* $0 < c \leqslant 1$ *the inequality sequence*

$$j'_{\nu,s} < y_{\nu,s} < y_{\nu+c,s} < y'_{\nu,s} < j_{\nu,s} < j_{\nu+c,s} < j'_{\nu,s+1} \quad s = 1,2,\ldots \tag{2}$$

holds. For $c > 1$ *this property is destroyed. We also have* $\nu \leqslant j'_{\nu,1}$.

Note that this particular formulation of the interlacing results was obtained only recently [8, 4, 7].

A very important fact (which is a consequence of the Watson formula [9, p. 508 Eq. (3)]) is stated in the following theorem.

THEOREM 2. *$c_{\nu,s}$ and $c'_{\nu,s}$ are continuous increasing functions of the order $\nu > 0$ for all $s = 1,2,\ldots$.*

2. Results

The main result is formulated as follows.

THEOREM 3. *For positive orders ν and μ the positive zeros of*

$$C_\nu(x), C_\mu(x); \qquad J'_\nu(x), J'_\mu(x); \qquad Y'_\nu(x), Y'_\mu(x)$$

are interlaced, respectively, if and only if $|\nu - \mu| \leqslant 2$.

REMARKS. In general, if at least one of ν and μ is negative $C_\nu(x)$ and $C_\mu(x)$ are not interlaced on $(0,\infty)$, i.e. not all the positive real zeros are interlaced, unless the zeros are defined as continuous increasing functions of the order (see Ref. [9], pp. 508–510 on how the zeros disappear when the order is decreased). However, in the particular case of $\delta = 0$ the interlacing of $C_\nu(x)$ and $C_\mu(x)$ is preserved for $\nu, \mu > -1$.

Additional interlacing relations can be proven with the aid of the tools introduced below, e.g. between $J_{\nu+2}(x)$ and $J'_\nu(x)$, but only for specific differences between the orders (which is 2 in this particular example), and thus not in the form of Theorem 3.

In order to prove Theorem 3 two tools are utilized. The first is the conditional transitivity of interlacing relations.

LEMMA 4. *Let f, g and h be continuous functions on some common interval I. Suppose f is interlaced with g and g is interlaced with h on I, where*

$$a(x)f(x) + b(x)g(x) + c(x)h(x) = 0 \tag{3}$$

with some functions a, b, c satisfying $\operatorname{sgn} a(x) = const. \neq 0$, $\operatorname{sgn} b(x) = const. \neq 0$ *and* $\operatorname{sgn} c(x) = const. \neq 0$. *Then f is interlaced with h on I.*

The second tool is a result connecting Wronskians and interlacing.

LEMMA 5. *The Wronskian* $W\left(\sqrt{x}C_\nu(x), \sqrt{x}\bar{C}_\mu(x)\right)$ *has no roots on the interval* $x \in (\min(c_{\nu,1}, \bar{c}_{\mu,1}), \infty)$ *if and only if the positive zeros of the functions* $C_\nu(x)$ *and* $\bar{C}_\mu(x)$ *are interlaced.*

3. Proofs

3.1. Three term recurrence relations

Proof of Lemma 4. Let $\{x_i\}$ and $\{y_i\}$ denote the sets of zeros of f and h on I, respectively. Then the functional equation (3) yields $\operatorname{sgn} g(x_i) = -\operatorname{sgn}(bc)\operatorname{sgn} h(x_i)$ and $\operatorname{sgn} f(y_i) = -\operatorname{sgn}(ab)\ \operatorname{sgn} g(y_i)$. Since f and g are interlaced we have $\operatorname{sgn} g(x_i) = -\operatorname{sgn} g(x_{i+1})$, similarly $\operatorname{sgn} g(y_i) = -\operatorname{sgn} g(y_{i+1})$. Then h (f) must have an odd number of zeros between each consecutive pair of zeros of f (h) implying the two are interlaced on the interval I. □

We prove two interlacing relations using Lemma 4.

COROLLARY 6. *For* $\nu > 0$ *the positive zeros of* $C_\nu(x)$ *and* $C_{\nu+2}(x)$ *are interlaced.*

Proof. Indeed, Lemma 4 yields the statement, since with $I = (0,\infty)$, $f = C_\nu$, $g = C_{\nu+1}$ and $h = C_{\nu+2}$ Eq. (3) can be turned into

$$C_\nu(x) - \frac{2\nu+2}{x}C_{\nu+1}(x) + C_{\nu+2}(x) = 0, \tag{4}$$

which is a known three term recurrence relation. □

For the derivative functions a suitable three term recurrence relation can be found using the well-known ones [1]. From

$$C'_\nu(x) = -C_{\nu+1}(x) + \frac{\nu}{x}C_\nu(x), \qquad C'_{\nu+1}(x) = C_\nu(x) - C_{\nu+2}(x), \tag{5}$$

$$C'_{\nu+1}(x) = C_\nu(x) - \frac{\nu+1}{x}C_{\nu+1}(x), \qquad C'_{\nu+2}(x) = C_{\nu+1}(x) - \frac{\nu+2}{x}C_{\nu+2}(x) \tag{6}$$

we infer that

$$[x^2-(\nu+1)(\nu+2)]C'_\nu(x) + [x^2-\nu(\nu+1)]C'_{\nu+2}(x) = \frac{2(\nu+1)}{x}[x^2-\nu(\nu+2)]C'_{\nu+1}(x) \tag{7}$$

holds.

The first zero of $C'_\nu(x)$ can be at any point of the half line $(0,\infty)$ depending on ν and δ. Eq. (7) implies that the first few zeros of $C'_\nu(x)$ and $C'_{\nu+2}(x)$ may not be interlaced even if $C'_\nu(x)$ and $C'_{\nu+1}(x)$ are interlaced. For $x > \sqrt{(\nu+1)(\nu+2)}$ $C'_\nu(x)$ and $C'_{\nu+2}(x)$ are interlaced if $C'_\nu(x)$ and $C'_{\nu+1}(x)$ are interlaced. However, the first few zeros of $C'_\nu(x)$ and $C'_{\nu+1}(x)$ might still not be interlaced. One can only guarantee interlacing of the derivative functions $C'_\nu(x)$, $C'_{\nu+1}(x)$ and $C'_{\nu+2}(x)$ if $\delta = 0$ or $\delta = \frac{\pi}{2}$.

COROLLARY 7. *The positive zeros of* $J'_\nu(x)$ *and* $J'_{\nu+2}(x)$ *and those of* $Y'_\nu(x)$ *and* $Y'_{\nu+2}(x)$ *are interlaced if* $\nu > 0$.

Proof. The multiplying terms in Eq. (7) are all positive for $x > j'_{\nu+2,1}$ thus Lemma 4 yields that J'_ν and $J'_{\nu+2}$ are interlaced on $(j'_{\nu+2,1}, \infty)$ since J'_ν, $J'_{\nu+1}$ and $J'_{\nu+1}$, $J'_{\nu+2}$ are interlaced (Theorem 1). It remains to show that J'_ν has only one zero $(j'_{\nu,1})$ before $j'_{\nu+2,1}$. We have $j'_{\nu,1} < j'_{\nu+1,1} < j'_{\nu+2,1}$ from Theorem 2, while Theorem 1 implies one further zero $(j'_{\nu,2})$ on $(j'_{\nu+1,1}, j'_{\nu+1,2})$. Analyzing the signs in Eq. (7) yields that this zero must be after $j'_{\nu+2,1}$.

The same reasoning holds for the second order derivative functions. □

The following is a simple corollary of Theorem 2.

COROLLARY 8. *If* $\nu > 0$ *then the previous interlacing relations remain to be true if the difference between the orders is* ε *instead of* 2 *with* $0 < \varepsilon \leqslant 2$.

3.2. Wronskians

To prove the negative parts of Theorem 3 I analyze Wronskians as in [6]. Let

$$\xi_\nu = \sqrt{x} C_\nu(x) = \sqrt{x}[\cos\delta J_\nu(x) - \sin\delta Y_\nu(x)], \tag{8}$$

$$\bar{\xi}_\mu = \sqrt{x} \bar{C}_\mu(x) = \sqrt{x}[\cos\bar{\delta} J_\mu(x) - \sin\bar{\delta} Y_\mu(x)], \tag{9}$$

which functions give rise to the Wronskian

$$W\left(\sqrt{x} C_\nu(x), \sqrt{x}\bar{C}_\mu(x)\right) \equiv W_{\xi_\nu,\bar{\xi}_\mu}(x) = \xi_\nu(x)\bar{\xi}'_\mu(x) - \xi'_\nu(x)\bar{\xi}_\mu(x). \tag{10}$$

Differentiating with respect to x one obtains

$$W'_{\xi_\nu,\bar{\xi}_\mu}(x) = \frac{\mu^2 - \nu^2}{x^2}\xi_\nu(x)\bar{\xi}_\mu(x), \tag{11}$$

which holds because of the differential equation

$$x^2\left[\frac{d^2}{dx^2} + 1\right]\xi_\nu(x) = \left(\nu^2 - \frac{1}{4}\right)\xi_\nu(x), \tag{12}$$

inferred from the Bessel equation.

From Eq. (11) follows that the set of local extrema of $W_{\xi_\nu,\bar{\xi}_\mu}(x)$ is $\{w_{\nu\mu,s}\}_{s=1}^\infty = \{c_{\nu,s}\}_{s=1}^\infty \cup \{\bar{c}_{\mu,s}\}_{s=1}^\infty$. At these positions the Wronskian takes

$$\mathrm{extr}_s W_{\xi_\nu,\bar{\xi}_\mu}(x) \equiv W_{\xi_\nu,\bar{\xi}_\mu}(w_{\nu\mu,s}) = \begin{cases} -\xi'_\nu(c_{\nu,t})\bar{\xi}_\mu(c_{\nu,t}) \\ +\xi_\nu(\bar{c}_{\mu,t})\bar{\xi}'_\mu(\bar{c}_{\mu,t}), \end{cases} \tag{13}$$

where the exact value of t depends on the interlacing of $C_\nu(x)$ and $\bar{C}_\mu(x)$.

Now we are ready to prove Lemma 5.

Proof of Lemma 5. Let $\bar{c}_{\mu,1} < c_{\nu,1}$. Without loss of generality suppose $\operatorname{sgn} C_\nu(0+) = \operatorname{sgn} C_\mu(0+) = 1$ (otherwise to satisfy the equation we take the opposite of the respective functions, whose zeros coincide with the original ones). Please note that $\operatorname{sgn} C'_\nu(c_{\nu,n}) = (-1)^n$.

Suppose the zeros of $C_\nu(x)$ and $C_\mu(x)$ are interlaced implying $\operatorname{sgn} C_\nu(\bar{c}_{\mu,n}) = (-1)^{n+1}$ and $\operatorname{sgn} \bar{C}_\mu(c_{\nu,n}) = (-1)^n$. Then every odd (even) numbered extremum is at a zero of $\bar{C}_\mu(x)$ ($C_\nu(x)$). From Eq. (13) and the signs of the constituent functions it follows that

$$\operatorname{sgn} \operatorname{extr}_n W_{\xi_\nu,\bar{\xi}_\mu}(x) = -1 \tag{14}$$

independent of n implying for $W_{\xi_\nu,\bar{\xi}_\mu}(x)$ no zeros on $(\min(c_{\nu,1}, \bar{c}_{\mu,1}), \infty)$.

Since $\{w_{\nu\mu,s}\}_{s=1}^\infty = \{c_{\nu,s}\}_{s=1}^\infty \cup \{\bar{c}_{\mu,s}\}_{s=1}^\infty$ it is apparent that the converse of the statement is true as well. $\square$

This lemma will now be used to derive the breaking conditions (negative parts) for Theorem 3.

In what follows I will use some asymptotic properties of the Bessel functions. From the definitions of $J_\nu(x)$ and $Y_\nu(x)$, i.e.

$$J_\nu(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}, \qquad Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{15}$$

it is inferred that for $\nu > 0$

$$C_\nu(x) = \sin\delta \left(\frac{\Gamma(\nu)2^\nu}{\pi} + o(1)\right) x^{-\nu}, \qquad x \to 0. \tag{16}$$

The asymptotics of the Wronskian can be derived from the asymptotics of the Bessel functions, namely

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + o(1), \qquad x \to \infty, \tag{17}$$

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + o(1), \qquad x \to \infty, \tag{18}$$

therefore

$$W_{\xi_\nu,\bar{\xi}_\mu}(x) = \frac{2}{\pi} \sin\left(\frac{\mu-\nu}{2}\pi + \delta - \bar{\delta}\right) + o(1), \qquad x \to \infty \tag{19}$$

meaning that the Wronskian converges to a constant at infinity.

LEMMA 9. *Let ν, $\mu > 0$. Then the interlacing of $C_\nu(x)$ and $\bar{C}_\mu(x)$ breaks down in the following cases:*

a) $\bar{C}_\mu(x) \equiv C_\mu(x)$ *with* $|\nu - \mu| > 2$;

b) $C_\nu(x) \equiv J_\nu(x)$ *and* $\bar{C}_\mu(x) \equiv Y_\mu(x)$ *with* $|\nu - \mu| > 1$ *provided that* $y_{\mu,1} < j_{\nu,1}$.

Proof. a) The proof is elementary in view of Lemma 5. One only needs to show that the Wronskian associated to the given cylinder functions has at least one zero between its first extremum and infinity.

From Eq. (16) it follows, that independently of ν $\operatorname{sgn} C_\nu(0+) = \operatorname{sgn}\sin\delta$. Let $\mu < \nu$. Since $\operatorname{sgn} C_\nu(0+) = \operatorname{sgn} C_\mu(0+)$ and $c_{\mu,1} < c_{\nu,1}$ the first extremum of $W_{\xi_\nu,\bar{\xi}_\mu}(x)$ is positive. (For $\sin\delta = 0$ we have two Bessel functions of the first kind and $\operatorname{sgn} J_\nu(0+) = \operatorname{sgn} J_\mu(0+)$ still holds.) In Eq. (19) we have $\delta - \bar{\delta} = 0$, thus if $4k < \nu - \mu < 2 + 4k$ ($k \in \mathbb{Z}^+$) the Wronskian is positive at the first extremum and negative at infinity, which assumes an odd number of zeros on this interval. By Lemma 5 in this case $C_\nu(x)$ and $C_\mu(x)$ are not interlaced.

It is easy to see now that by increasing ν (to reach the uncovered regions of the previous argumentation) the interlacing is not recovered. Let $S > 0$ be such that for $n < S$ $c_{\mu,n} < c_{\nu,n} < c_{\mu,n+1}$ but $c_{\mu,S} < c_{\nu,S} < c_{\mu,S+1} < c_{\mu,S+2} < c_{\nu,S+1}$, i.e. only the first S zeros of $C_\nu(x)$ and $C_\mu(x)$ are interlaced. Because of Theorem 2 interlacing cannot be recovered by increasing ν ($c_{\mu,S+2} < c_{\nu,S+1} < c_{\nu+\varepsilon,S+1}$, $\forall\varepsilon > 0$).

b) (This part was already proven in [6]; however, in a more complicated way.) Let $\mu < \nu$. In this case the first extremum is negative since $\operatorname{sgn} J_\nu(0+) = -\operatorname{sgn} Y_\mu(0+)$, while $\delta - \bar{\delta} = -\frac{\pi}{2}$ in Eq. (19) implies for $1 + 4k < \nu - \mu < 3 + 4k$ ($k \in \mathbb{Z}^+$) that the Wronskian converges to a positive number. Therefore the Wronskian has at least one zero. For the uncovered regions of $|\mu - \nu| > 1$ the same kind of reasoning works as the one we used in case a. □

From the proof one can see that "shifted interlacing" occurs on every (a,b) interval where $W_{\xi_\nu,\xi_\mu}(x)$ has no zeros. By "shifted interlacing" we mean $c_{\mu,s} < c_{\nu,s+d} < c_{\mu,s}$ for $s = s_1, s_2, \ldots, s_n$ with some fixed $d \neq 0$ shift (ordinary interlacing is defined by $d = 0$). Especially important is the interval (z,∞) with z being the greatest zero of the Wronskian.

LEMMA 10. *Let ν, $\mu > 0$. Then the interlacing of $C'_\nu(x)$ and $C'_\mu(x)$ breaks down for $|\nu - \mu| > 2$, either $C \equiv J$ or $C \equiv Y$.*

Proof. Let $\mu < \nu$. Using Lemma 9a and the recurrence relation

$$C'_\nu(x) = -C_{\nu+1}(x) + \frac{\nu}{x} C_\nu(x) \tag{20}$$

I will show that the interlacing

$$c'_{\mu,1} < c'_{\nu,1} < c'_{\mu,2} < \ldots \tag{21}$$

is certainly broken for $|\nu - \mu| > 2$.

From the recurrence relation (20) we infer that the zeros of $C'_\nu(x)$ converge to those of $C_{\nu+1}(x)$, moreover they can be identified with one another since $C'_\nu(x)$ and $C_{\nu+1}(x)$ are interlaced (Theorem 1) and also the zeros of both functions are well separated asymptotically (see Eq. (17)). That is either $c_{\nu+1,s} \approx c'_{\nu,s}$ or $c_{\nu+1,s} \approx c'_{\nu,s+1}$ for big s's.

Let now $\nu = \mu + 2 + K$ with some $4k < K < 2 + 4k$ $(k \in \mathbb{Z}^+)$. Then the Wronskian $W_{\xi_{\nu+1},\xi_{\mu+1}}(x)$ has an odd number of zeros implying for the zeros of $C_{\nu+1}(x)$ and $C_{\mu+1}(x)$ shifted interlacing on (z,∞). Because of the asymptotic identification between $C'_\nu(x)$ and $C_{\nu+1}(x)$ the shifted interlacing, that is a broken interlacing, also holds for the zeros of $C'_\nu(x)$ (with perhaps a different threshold index).

For the uncovered regions of $2 + 4k < K < 4 + 4k$ $(k \in \mathbb{Z}^+)$ the same kind of argument works that was used in Lemma 9. $\square$

The necessary and sufficient interlacing conditions of Theorem 3 are yielded by the combination of Corollary 8, Lemma 9 and 10.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover Publications, New York, 360–371, 1972.

[2] Á. ELBERT, *Some recent results on the zeros of Bessel functions and orthogonal polynomials*, J. Comp. Appl. Math. **133** (2001), 66–83.

[3] A. LAFORGIA AND PIERPAOLO NATALINI, *Zeros of Bessel functions: monotonicity, concavity, inequalities*, Le Matematiche **62** (2007), 255–270.

[4] H. Y. LIU AND J. ZOU, *Zeros of the Bessel and spherical Bessel functions and their applications for uniqueness in inverse acoustic obstacle scattering*, IMA J. Appl. Math. **72** (2007), 817–831.

[5] H. Y. LIU AND J. ZOU, *Zeros of the Bessel and spherical Bessel functions and their applications for uniqueness in inverse acoustic obstacle scattering problems, Technical Report CUHK-2007-02 (342)*, The Chinese University of Hong Kong, Hong Kong, 2007.

[6] T. PÁLMAI AND B. APAGYI, *On nonsingular potentials of Cox-Thompson inversion scheme*, J. Math. Phys. **51** (2010), 022114.

[7] T. PÁLMAI AND B. APAGYI, *Interlacing of positive real zeros of Bessel functions*, J. Math. Anal. Appl. **375** (2011), 320–322.

[8] J. SEGURA, *Interlacing of the zeros of contiguous hypergeometric functions*, Numer. Algor. **49** (2008), 387–407.

[9] G. N. WATSON, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1944.

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