

SOME HARDY INEQUALITIES ON HALF SPACES FOR GRUSHIN TYPE OPERATORS

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Abstract. We prove some sharp Hardy type inequalities on half spaces for Grushin type operators like $\Delta_x + (1 + \gamma)^2|x|^{2\gamma}\Delta_y$ with $\gamma > 0$.

1. Introduction

The Hardy inequality in \mathbb{R}^N reads that, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \quad (1.1)$$

and the constant $\frac{(N-2)^2}{4}$ in (1.1) is sharp. Recently, it has been proved by S. Filippas, A. Tertikas and J. Tidblom ([5]) that the following Hardy inequality is valid for $f \in C_0^\infty(\mathbb{R}_+^N)$

$$\int_{\mathbb{R}_+^N} |\nabla u(x)|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}_+^N} \frac{u(x)^2}{|x|^2} dx, \quad (1.2)$$

where $\mathbb{R}_+^N = \{(x_1, \dots, x_n) | x_n > 0\}$, and the constant $\frac{N^2}{4}$ is sharp. This shows that the Hardy constant jumps from $\frac{(N-2)^2}{4}$ to $\frac{N^2}{4}$, when the singularity of the potential reaches the boundary.

The aim of this note is to prove similar Hardy type inequality on half spaces for Grushin type operators like $\Delta_x + (1 + \gamma)^2|x|^{2\gamma}\Delta_y$ with $\gamma > 0$. It has been proved by D'Ambrosio ([1]) that for $\alpha < Q - 2$ and $u \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, there holds

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\nabla_\gamma u|^2}{\rho(x,y)^\alpha} dx dy \geq \frac{(Q-2-\alpha)^2}{4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{2+\alpha}(x,y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x,y)} dx dy, \quad (1.3)$$

and the constant $\frac{(Q-2-\alpha)^2}{4}$ in (1.3) is sharp, where $\nabla_\gamma = (\nabla_x, (1 + \gamma)|x|^\gamma \nabla_y)$, $Q = m + (1 + \gamma)n$ and $\rho(x,y) = (|x|^{2+2\gamma} + |y|^2)^{\frac{1}{2+2\gamma}}$. In this note we shall show when the singularity is on the boundary, the Hardy constant also jumps. In fact, we have the following:

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THEOREM 1.1. *There holds, for all $u \in C_0^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$,*

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma u|^2 dx dy \geq \frac{Q^2}{4} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2}{\rho^2(x,y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x,y)} dx dy \tag{1.4}$$

and the constant $\frac{Q^2}{4}$ in (1.4) is sharp.

However, it seems that the method used in [5] can not be applied to Grushin type operator. So in order to prove Theorem 1.1, we use a different technique which is motivated by V. Maz'ya and T. Shaposhnikova (see [7], Theorem 6.1).

As an application of Theorem 1.1, we obtain the following Hardy inequalities with weights.

THEOREM 1.2. *Let $\alpha < Q - 2$. There holds, for all $u \in C_0^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$,*

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{|\nabla_\gamma u|^2}{\rho^\alpha} \geq \left(\frac{(Q - \alpha)^2}{4} + \alpha \right) \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2}{\rho^{2+\alpha}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \tag{1.5}$$

and the constant $\frac{(Q-\alpha)^2}{4} + \alpha$ in (1.5) is sharp.

We also obtain some sharp Rellich inequalities on $\mathbb{R}_+^m \times \mathbb{R}^n$.

THEOREM 1.3. *Let $\alpha < Q - 2$. There holds, for all $u \in C_0^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$,*

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q + \alpha + 2)(Q - \alpha - 2)}{4} \right)^2 \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}} \tag{1.6}$$

and the constant $\left(\frac{(Q+\alpha+2)(Q-\alpha-2)}{4} \right)^2$ in (1.6) is sharp.

Our method can also be applied to the half space $\mathbb{R}^m \times \mathbb{R}_+^n$. To this end, we have

THEOREM 1.4. *Let $\alpha < Q - 2$. There holds, for all $u \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}_+^n)$,*

$$\int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{|\nabla_\gamma u|^2}{\rho^\alpha} \geq \left(\frac{(Q + 2\gamma - \alpha)^2}{4} + (1 + \gamma)\alpha \right) \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2}{\rho^{2+\alpha}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \tag{1.7}$$

and the constant $\frac{(Q+2\gamma-\alpha)^2}{4} + (1 + \gamma)\alpha$ in (1.7) is sharp.

THEOREM 1.5. *Let $\alpha < Q + 2\gamma - 2$. There holds, for all $u \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}_+^n)$,*

$$\int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q + 2\gamma + \alpha + 2)(Q + 2\gamma - \alpha - 2)}{4} \right)^2 \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}} \tag{1.8}$$

and the constant $\left(\frac{(Q+2\gamma+\alpha+2)(Q+2\gamma-\alpha-2)}{4} \right)^2$ in (1.8) is sharp.

2. The proofs

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [1, 3, 4] for more precise information about Grushin type operators. Recall that Grushin type operator is the operator defined on $\mathbb{R}^{m+n} = \mathbb{R}_x^m \times \mathbb{R}_y^n$ by

$$\Delta_\gamma = \Delta_x + (\gamma + 1)^2 |x|^{2\gamma} \Delta_y, \quad \gamma > 0.$$

Δ_γ is elliptic for $x \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}_y^n$. This operator belongs to the wide class of subelliptic operators introduced and studied by Franchi and Lanconelli in [4].

Observe that the operator Δ_γ possesses a natural family of anisotropic dilations, namely

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{\gamma+1} y), \quad \lambda > 0.$$

One easily checks that Δ_γ is homogeneous of degree two with respect to $\{\delta_\lambda\}_{\lambda>0}$. The Jacobian of the dilations δ_λ is λ^Q , where $Q = m + (\gamma + 1)n$ is the homogeneous dimension. For simplicity, we will write $\lambda(x, y)$ to denote $\delta_\lambda(x, y)$. The operator Δ_γ can be written in the form “sum of squares” $\Delta_\gamma = \sum_{i=1}^{m+n} Y_i^2$ by choosing

$$Y_i = \frac{\partial}{\partial x_i} \quad \text{for } i = 1, \dots, m$$

$$Y_{j+m} = (1 + \gamma) |x|^\gamma \frac{\partial}{\partial y_j} \quad \text{for } j = 1, \dots, n.$$

Denote by $\nabla_\gamma = (Y_1, \dots, Y_{m+n}) = (\nabla_x, (1 + \gamma) |x|^\gamma \nabla_y)$. Then

$$\Delta_\gamma = \langle \nabla_\gamma, \nabla_\gamma \rangle$$

and ∇_γ is homogeneous of degree one with respect to $\{\delta_\lambda\}_{\lambda>0}$.

The anisotropic dilation structure on \mathbb{R}^{m+n} introduces homogeneous norm

$$\rho(x, y) = (|x|^{2(\gamma+1)} + |y|^2)^{\frac{1}{2(\gamma+1)}}.$$

With this norm, we can define the ball centered at origin with radius R

$$B_R = \{\xi \in \mathbb{R}^{m+n} : \rho(\xi) < R\}$$

and the unit sphere $\Sigma = \partial B_1 = \{(x, y) \in \mathbb{R}^{m+n} : \rho(x, y) = 1\}$. A function f on \mathbb{R}^{m+n} is said to be radial if $f(x, y) = f(\rho)$. If f is radial, then (see [1])

$$|\nabla_\gamma f| = \frac{|x|^\gamma}{\rho^\gamma} |f'(\rho)| \tag{2.1}$$

and

$$\Delta_\gamma f = |\nabla_\gamma \rho|^2 \left(f'' + \frac{Q-1}{\rho} f' \right) = \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \left(f'' + \frac{Q-1}{\rho} f' \right). \tag{2.2}$$

Next, we recall the polar coordinate for Grushin type operators which is similar to that on homogeneous groups ([6], Proposition (1.15)). Given any $(0, 0) \neq \xi = (x, y) \in \mathbb{R}^{m+n}$, set $x^* = \frac{x}{\rho(x,y)}$, $y^* = \frac{y}{\rho(x,y)^{1+\gamma}}$ and $\xi^* = (x^*, y^*)$. Then $\rho(\xi^*) = 1$. Moreover, there exists a unique Radon measure σ on Σ such that for all $f \in L^1(\mathbb{R}^{m+n})$,

$$\int_{\mathbb{R}^{m+n}} f(x, y) dx dy = \int_0^\infty \int_\Sigma f(\lambda(x^*, y^*)) \lambda^{Q-1} d\sigma d\lambda. \tag{2.3}$$

In fact, (2.3) can be obtained through a transformation given by D’Ambrosio et al ([2], see also [1], page 728).

Finally, we define two smooth functions $H(t)$ and $\Phi(t)$ which satisfy

$$H(t) = \begin{cases} 0, & t \leq 1; \\ 1, & t \geq 2 \end{cases} \tag{2.4}$$

and

$$\Phi(t) = \begin{cases} 1, & t \leq 1; \\ 0, & t \geq 2. \end{cases} \tag{2.5}$$

Without loss of generality, we assume $0 \leq H(t) \leq 1$ and $0 \leq \Phi(t) \leq 1$ for all $t \in \mathbb{R}$.

Before the proof of main results, we need the following Lemma.

LEMMA 2.1. *Let $Q \geq 3$. The following Hardy inequality*

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} |\nabla_\gamma u|^2 dx dy \geq \frac{(Q-2)^2}{4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^2(x, y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x, y)} dx dy \tag{2.6}$$

is valid for all $u \in C_0^\infty(\mathbb{R}^{m+n})$ which satisfies the condition

$$u(x, y) = \tilde{u}(x_1, \dots, x_{m-3}, \sqrt{x_{m-2}^2 + x_{m-1}^2 + x_m^2}, y). \tag{2.7}$$

Furthermore, the constant $\frac{(Q-2)^2}{4}$ is also sharp.

Proof. Since the inequality (2.6) is valid for all functions in $C_0^\infty(\mathbb{R}^{m+n})$, it is also valid for functions in $C_0^\infty(\mathbb{R}^{m+n})$ satisfying the condition (2.7). So to finish the proof, it is enough to show the the constant $\frac{(Q-2)^2}{4}$ is sharp.

Let $F_\varepsilon(\rho) = H(\frac{\rho}{\varepsilon})\rho^{\frac{2-Q}{2}}$ and $F_{\varepsilon,R}(\rho) = \Phi(\frac{\rho}{R})F_\varepsilon(\rho) = \Phi(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})\rho^{\frac{2-Q}{2}}$ if $\rho(x, y) > 0$, where $0 < \varepsilon < 1$ and $R > 2$. By the definition of $H(t)$, $F_{\varepsilon,R}(\rho) \equiv 0$ if $0 < \rho \leq \varepsilon$. So we may set $F_{\varepsilon,R}(0) = 0$ so that $F_{\varepsilon,R}(\rho(x, y)) \in C_0^\infty(\mathbb{R}^{m+n})$. We consider $F_{\varepsilon,R}(\rho)$ as the test function. It is easy to check

$$F_{\varepsilon,R}(\rho) = \begin{cases} \rho^{\frac{2-Q}{2}} H(\frac{\rho}{\varepsilon}), & \rho \leq 2\varepsilon; \\ \rho^{\frac{2-Q}{2}}, & 2\varepsilon < \rho < R; \\ \rho^{\frac{2-Q}{2}} \Phi(\frac{\rho}{R}), & \rho \geq R. \end{cases}$$

We have, using the polar coordinate (2.3),

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{F_{\varepsilon,R}^2(\rho)}{\rho^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} dx dy = \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma$$

and

$$\begin{aligned} & \int_{\mathbb{R}^m \times \mathbb{R}^n} |\nabla_\gamma F_{\varepsilon,R}(\rho)|^2 dx dy = \int_0^\infty |F'_{\varepsilon,R}|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \int_0^\infty \left| \frac{1}{R} \Phi'\left(\frac{\rho}{R}\right) F_\varepsilon(\rho) + \Phi\left(\frac{\rho}{R}\right) F'_\varepsilon(\rho) \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{1}{R^2} \int_0^\infty \left| \Phi'\left(\frac{\rho}{R}\right) F_\varepsilon \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma + \int_0^\infty \left| \Phi\left(\frac{\rho}{R}\right) F'_\varepsilon \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ & \quad + \frac{2}{R} \int_0^\infty \Phi\left(\frac{\rho}{R}\right) \Phi'\left(\frac{\rho}{R}\right) F_\varepsilon(\rho) F'_\varepsilon(\rho) \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &=: (I) + (II) + (III), \end{aligned}$$

where

$$\begin{aligned} (I) &= \frac{1}{R^2} \int_0^\infty \left| \Phi'\left(\frac{\rho}{R}\right) F_\varepsilon \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{1}{R^2} \int_R^{2R} \left| \Phi'\left(\frac{\rho}{R}\right) H\left(\frac{\rho}{\varepsilon}\right) \rho^{\frac{2-Q}{2}} \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{1}{R^2} \int_R^{2R} \left| \Phi'\left(\frac{\rho}{R}\right) \rho^{\frac{2-Q}{2}} \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &\leq \max_{t \in \mathbb{R}} |\Phi'(t)|^2 \cdot \frac{1}{R^2} \int_R^{2R} \rho d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma = \max_{t \in \mathbb{R}} |\Phi'(t)|^2 \cdot \frac{3}{2} \int_\Sigma |x^*|^{2\gamma} d\sigma, \end{aligned} \tag{2.8}$$

$$\begin{aligned} (II) &= \int_0^\infty \left| \Phi\left(\frac{\rho}{R}\right) F'_\varepsilon \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) \left| \frac{2-Q}{2} H\left(\frac{\rho}{\varepsilon}\right) \rho^{-\frac{Q}{2}} + \frac{1}{\varepsilon} H'\left(\frac{\rho}{\varepsilon}\right) \rho^{\frac{2-Q}{2}} \right|^2 \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{(Q-2)^2}{4} \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ & \quad + \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \Phi^2\left(\frac{\rho}{R}\right) \left| H'\left(\frac{\rho}{\varepsilon}\right) \right|^2 \rho d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ & \quad + \frac{2-Q}{\varepsilon} \int_\varepsilon^{2\varepsilon} \Phi^2\left(\frac{\rho}{R}\right) H\left(\frac{\rho}{\varepsilon}\right) H'\left(\frac{\rho}{\varepsilon}\right) d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{(Q-2)^2}{4} \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ & \quad + \left[\frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left| H'\left(\frac{\rho}{\varepsilon}\right) \right|^2 \rho d\rho + \frac{2-Q}{\varepsilon} \int_\varepsilon^{2\varepsilon} H\left(\frac{\rho}{\varepsilon}\right) H'\left(\frac{\rho}{\varepsilon}\right) d\rho \right] \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \end{aligned}$$

$$\begin{aligned} &\leq \frac{(Q-2)^2}{4} \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &\quad + \left(\frac{3}{2} \max_{t \in \mathbb{R}} |H'(t)|^2 + (Q-2) \max_{t \in \mathbb{R}} |H'(t)|\right) \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} (III) &= \frac{2}{R} \int_0^\infty \Phi\left(\frac{\rho}{R}\right) \Phi'\left(\frac{\rho}{R}\right) F_\varepsilon(\rho) F'_\varepsilon(\rho) \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= \frac{2}{R} \int_R^{2R} \Phi\left(\frac{\rho}{R}\right) \Phi'\left(\frac{\rho}{R}\right) \rho^{\frac{2-Q}{2}} \cdot \frac{2-Q}{2} \rho^{-\frac{Q}{2}} \rho^{Q-1} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &\leq \max_{t \in \mathbb{R}} |\Phi'(t)| \cdot \frac{Q-2}{R} \int_R^{2R} d\rho \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma \\ &= (Q-2) \max_{t \in \mathbb{R}} |\Phi'(t)| \cdot \int_\Sigma |x^*|^{2\gamma} d\sigma. \end{aligned} \tag{2.10}$$

Therefore, by (2.8)-(2.10),

$$\frac{\int_{\mathbb{R}^m \times \mathbb{R}^n} |\nabla_\gamma F_{\varepsilon,R}(\rho)|^2 dx dy}{\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{F_{\varepsilon,R}^2(\rho)}{\rho^2} |x|^{2\gamma} dx dy} \leq \frac{(Q-2)^2}{4} + \frac{C_{H,\Phi}}{\int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho},$$

where

$$C_{H,\Phi} = \frac{3}{2} \max_{t \in \mathbb{R}} |\Phi'(t)|^2 + (Q-2) \max_{t \in \mathbb{R}} |\Phi'(t)| + \frac{3}{2} \max_{t \in \mathbb{R}} |H'(t)|^2 + (Q-2) \max_{t \in \mathbb{R}} |H'(t)|.$$

To finish the proof, it is enough to show

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho = +\infty.$$

In fact,

$$\int_0^\infty \Phi^2\left(\frac{\rho}{R}\right) H^2\left(\frac{\rho}{\varepsilon}\right) \rho^{-1} d\rho \geq \int_{2\varepsilon}^R \rho^{-1} d\rho = \ln R - \ln 2\varepsilon \rightarrow +\infty \ (\varepsilon \rightarrow 0^+).$$

The desired result follows. \square

REMARK 2.2. Since the test function $F_{\varepsilon,R}(\rho)$ in Lemma 2.1 is radial, one can see the constant $\frac{(Q-2)^2}{4}$ is also sharp when the function u is restricted to be in $C_0^\infty(\mathbb{R}^{m+n})$ and satisfy

$$u(x, y) = \tilde{u}(x, y_1, \dots, y_{n-3}, \sqrt{y_{n-2}^2 + y_{n-1}^2 + y_n^2}).$$

Now we can prove the Theorem 1.1

Proof of Theorem 1.1. Consider the sharp Hardy inequality on $\mathbb{R}_x^{m+2} \times \mathbb{R}_y^n$: for $v \in C_0^\infty(\mathbb{R}_x^{m+2} \times \mathbb{R}_y^n)$, there holds

$$\int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} |\nabla_\gamma v|^2 \geq \frac{(m + (1 + \gamma)n)^2}{4} \int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} \frac{v^2}{\rho^2(x, y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x, y)}. \tag{2.11}$$

Next we assume the function $v = \tilde{v}(x_1, \dots, x_{m-1}, \sqrt{x_m^2 + x_{m+1}^2 + x_{m+2}^2}, y)$. By Lemma 2.1, the constant in (2.11) is also sharp when v is limited to such functions. Set $x'_m = \sqrt{x_m^2 + x_{m+1}^2 + x_{m+2}^2}$. We get, by (2.11),

$$\begin{aligned} \int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} |\nabla_\gamma \tilde{v}|^2 &= |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} (|\nabla_x \tilde{v}|^2 + |\partial_{x'_m} \tilde{v}|^2 + (1 + \gamma)^2 |(x, x'_m)|^{2\gamma} |\nabla_y \tilde{v}|^2) x_m^2 \\ &\geq \frac{(m + (1 + \gamma)n)^2}{4} |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{\tilde{v}^2}{\rho^2} \frac{|(x, x'_m)|^{2\gamma}}{\rho^{2\gamma}} x_m^2, \end{aligned}$$

where $|\mathbb{S}^2|$ is the volume of \mathbb{S}^2 , the unit sphere in \mathbb{R}^3 . Replacing x'_m by x_m yields

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma \tilde{v}|^2 x_m^2 \geq \frac{Q^2}{4} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{\tilde{v}^2}{\rho^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} x_m^2. \tag{2.12}$$

Set $u = x_m \tilde{v}$. Then

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma u|^2 = \int_{\mathbb{R}_+^m \times \mathbb{R}^n} x_m^2 |\nabla_\gamma \tilde{v}|^2 + \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \tilde{v}^2 + \int_{\mathbb{R}_+^m \times \mathbb{R}^n} 2x_m \tilde{v} \frac{\partial \tilde{v}}{\partial x_m}.$$

Since

$$\int_0^{+\infty} 2x_m \tilde{v} \frac{\partial \tilde{v}}{\partial x_m} dx_m = \int_0^{+\infty} x_m \frac{\partial \tilde{v}^2}{\partial x_m} dx_m = - \int_0^{+\infty} \tilde{v}^2 dx_m,$$

we have

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma u|^2 = \int_{\mathbb{R}_+^m \times \mathbb{R}^n} x_m^2 |\nabla_\gamma \tilde{v}|^2.$$

By (2.12),

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma u|^2 \geq \frac{Q^2}{4} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2}{\rho^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}}.$$

The desired result follows. \square

Proof of Theorem 1.2. Replacing u by $\rho^{-\frac{\alpha}{2}} u$ in Theorem 1.1, we obtain

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma (\rho^{-\frac{\alpha}{2}} u)|^2 dx dy \geq \frac{Q^2}{4} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2}{\rho^{2+\alpha}(x, y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x, y)} dx dy. \tag{2.13}$$

On the other hand, by (2.1) and (2.2),

$$\begin{aligned} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} |\nabla_\gamma (\rho^{-\frac{\alpha}{2}} u)|^2 &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \left(\frac{|\nabla_\gamma u|^2}{\rho^\alpha} + u^2 |\nabla_\gamma \rho^{-\frac{\alpha}{2}}|^2 + 2\rho^{-\frac{\alpha}{2}} u \langle \nabla_\gamma \rho^{-\frac{\alpha}{2}}, \nabla_\gamma u \rangle \right) \\ &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \left(\frac{|\nabla_\gamma u|^2}{\rho^\alpha} + \frac{\alpha^2}{4} \frac{u^2}{\rho^{2+\alpha}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} + \frac{1}{2} \langle \nabla_\gamma \rho^{-\alpha}, \nabla_\gamma u^2 \rangle \right) \\ &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \left(\frac{|\nabla_\gamma u|^2}{\rho^\alpha} + \frac{\alpha^2}{4} \frac{u^2}{\rho^{2+\alpha}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} - \frac{u^2}{2} \Delta_\gamma \rho^{-\alpha} \right) \\ &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \left(\frac{|\nabla_\gamma u|^2}{\rho^\alpha} + \frac{\alpha^2 - 2\alpha(\alpha + 2 - Q)}{4} \frac{u^2}{\rho^{2+\alpha}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \right). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{|\nabla_\gamma u|^2}{\rho^\alpha} &\geq \frac{Q^2 - \alpha^2 + 2\alpha(\alpha + 2 - Q)}{4} \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{2+\alpha} \rho^{2\gamma}} \\ &= \left(\frac{(Q - \alpha)^2}{4} + \alpha \right) \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{2+\alpha} \rho^{2\gamma}} \end{aligned}$$

and the constant $\frac{(Q-\alpha)^2}{4} + \alpha$ is sharp since $\frac{Q^2}{4}$ is the sharp constant in (2.13).

To prove Theorem 1.3, we need the following sharp Rellich inequalities:

LEMMA 2.3. *Let $\alpha < Q - 4$. There holds, for all $u \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n)$,*

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q + \alpha)(Q - \alpha - 4)}{4} \right)^2 \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}} \tag{2.14}$$

and the constant $\left(\frac{(Q+\alpha)(Q-\alpha-4)}{4} \right)^2$ in (2.14) is sharp.

Proof. A simple calculation shows $-2u\Delta_\gamma u = 2|\nabla_\gamma u|^2 - \Delta_\gamma u^2$. Therefore,

$$-2 \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u\Delta_\gamma u}{\rho^{\alpha+2}} = 2 \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\nabla_\gamma u|^2}{\rho^{\alpha+2}} - \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Delta_\gamma u^2}{\rho^{\alpha+2}}.$$

Through integration by parts, we have, using (2.2),

$$- \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Delta_\gamma u^2}{\rho^{\alpha+2}} = (\alpha + 2)(Q - \alpha - 4) \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2.$$

Therefore, by (1.3), we obtain

$$\begin{aligned} -2 \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u\Delta_\gamma u}{\rho^{\alpha+2}} &= 2 \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\nabla_\gamma u|^2}{\rho^{\alpha+2}} - \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Delta_\gamma u^2}{\rho^{\alpha+2}} \\ &\geq \frac{(Q - \alpha - 4)^2}{2} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2 + (\alpha + 2)(Q - \alpha - 4) \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2 \\ &= \frac{(Q + \alpha)(Q - \alpha - 4)}{4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{(Q + \alpha)(Q - \alpha - 4)}{4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2 &\leq - \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{u\Delta_\gamma u}{\rho^{\alpha+2}} \\ &\leq \left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|u|^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \right)^{\frac{1}{2}}. \end{aligned}$$

Canceling and raising both sides to the power 2, we obtain (2.14).

Now we show the the constant $\frac{(Q+\alpha)^2(Q-\alpha-4)^2}{16}$ is sharp. Let $G_{\varepsilon,R}(\rho) = \Phi(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})\rho^{\frac{4+\alpha-Q}{2}}$ if $\rho(x,y) > 0$, where $0 < \varepsilon < 1$ and $R > 2$. Set $G_{\varepsilon,R}(0) = 0$ so that $G_{\varepsilon,R}(\rho(x,y)) \in C_0^\infty(\mathbb{R}^{m+n})$ since $G_{\varepsilon,R}(\rho) \equiv 0$ if $0 < \rho \leq \varepsilon$. We consider $G_{\varepsilon,R}(\rho)$ as the test function and compute

$$\begin{aligned} & \Delta_\gamma G_{\varepsilon,R}(\rho) \\ &= \left(G''_{\varepsilon,R} + \frac{Q-1}{\rho} G'_{\varepsilon,R} \right) |\nabla_\gamma \rho|^2 \\ &= -\frac{(Q+\alpha)(Q-\alpha-4)}{4} \Phi\left(\frac{\rho}{R}\right) H\left(\frac{\rho}{\varepsilon}\right) \rho^{\frac{\alpha-Q}{2}} |\nabla_\gamma \rho|^2 \\ & \quad + (3+\alpha) \left(\frac{1}{\varepsilon} \Phi\left(\frac{\rho}{R}\right) H'\left(\frac{\rho}{\varepsilon}\right) + \frac{1}{R} \Phi'\left(\frac{\rho}{R}\right) H\left(\frac{\rho}{\varepsilon}\right) \right) \rho^{\frac{2+\alpha-Q}{2}} |\nabla_\gamma \rho|^2 \\ & \quad + \left(\frac{1}{\varepsilon^2} \Phi\left(\frac{\rho}{R}\right) H''\left(\frac{\rho}{\varepsilon}\right) + 2\frac{1}{\varepsilon R} \Phi'\left(\frac{\rho}{R}\right) H'\left(\frac{\rho}{\varepsilon}\right) + \frac{1}{R^2} \Phi''\left(\frac{\rho}{R}\right) H\left(\frac{\rho}{\varepsilon}\right) \right) \rho^{\frac{4+\alpha-Q}{2}} |\nabla_\gamma \rho|^2. \end{aligned}$$

Since $\varepsilon < 1 < R/2$, $\Phi'(\frac{\rho}{R})H'(\frac{\rho}{\varepsilon}) \equiv 0$ by the definition of H and Φ . Therefore, by Minkowski inequality,

$$\begin{aligned} \left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Delta_\gamma G_{\varepsilon,R}|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \right)^{\frac{1}{2}} & \leq \frac{(Q+\alpha)(Q-\alpha-4)}{4} \left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^Q} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & \quad + \left(\frac{(3+\alpha)^2}{\varepsilon^2} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H'(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-2}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & \quad + \left(\frac{(3+\alpha)^2}{R^2} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi'(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-2}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & \quad + \left(\frac{1}{\varepsilon^4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H''(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-4}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & \quad + \left(\frac{1}{R^4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi''(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-4}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & = : \frac{(Q+\alpha)(Q-\alpha-4)}{4} \left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon,R}^2}{\rho^{\alpha+4}} |\nabla_\gamma \rho|^2 \right)^{\frac{1}{2}} \\ & \quad + (A)^{\frac{1}{2}} + (B)^{\frac{1}{2}} + (C)^{\frac{1}{2}} + (D)^{\frac{1}{2}}, \end{aligned}$$

where

$$\begin{aligned} (A) &= \frac{(3+\alpha)^2}{\varepsilon^2} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H'(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-2}} |\nabla_\gamma \rho|^2 dx dy \\ &= \frac{(3+\alpha)^2}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} |H'(\frac{\rho}{\varepsilon})|^2 \rho d\rho \int_\Sigma |\nabla_\gamma \rho|^2 d\sigma \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(3+\alpha)^2}{\varepsilon^2} \max_{t \in \mathbb{R}} |H'(t)|^2 \int_{\varepsilon}^{2\varepsilon} \rho d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&= \frac{3}{2} (3+\alpha)^2 \max_{t \in \mathbb{R}} |H'(t)|^2 \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma, \\
(B) &= \frac{(3+\alpha)^2}{R^2} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi'(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-2}} |\nabla_{\gamma} \rho|^2 dx dy \\
&= \frac{(3+\alpha)^2}{R^2} \int_R^{2R} |\Phi'(\frac{\rho}{R})|^2 \rho d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&\leq \frac{(3+\alpha)^2}{R^2} \max_{t \in \mathbb{R}} |\Phi'(t)|^2 \int_R^{2R} \rho d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&= \frac{3}{2} (3+\alpha)^2 \max_{t \in \mathbb{R}} |\Phi'(t)|^2 \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma, \\
(C) &= \frac{1}{\varepsilon^4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H''(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-4}} |\nabla_{\gamma} \rho|^2 dx dy \\
&= \frac{1}{\varepsilon^4} \int_{\varepsilon}^{2\varepsilon} |H''(\frac{\rho}{\varepsilon})|^2 \rho^3 d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&\leq \frac{1}{\varepsilon^4} \max_{t \in \mathbb{R}} |H''(t)|^2 \int_{\varepsilon}^{2\varepsilon} \rho^3 d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&= \frac{15}{4} \max_{t \in \mathbb{R}} |H''(t)|^2 \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma
\end{aligned}$$

and

$$\begin{aligned}
(D) &= \frac{1}{R^4} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi''(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^{Q-4}} |\nabla_{\gamma} \rho|^2 dx dy \\
&= \frac{1}{R^4} \int_R^{2R} |\Phi''(\frac{\rho}{R})|^2 \rho^3 d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&\leq \frac{1}{R^4} \max_{t \in \mathbb{R}} |\Phi''(t)|^2 \int_R^{2R} \rho^3 d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\
&= \frac{15}{4} \max_{t \in \mathbb{R}} |\Phi''(t)|^2 \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Delta_{\gamma} G_{\varepsilon, R}|^2}{|\nabla_{\gamma} \rho|^2 \rho^{\alpha}} \right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon, R}^2}{\rho^{\alpha+4}} |\nabla_{\gamma} \rho|^2 \right)^{\frac{1}{2}}} \leq \frac{(Q+\alpha)(Q-\alpha-4)}{4} + \frac{(A)^{\frac{1}{2}} + (B)^{\frac{1}{2}} + (C)^{\frac{1}{2}} + (D)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon, R}^2}{\rho^{\alpha+4}} |\nabla_{\gamma} \rho|^2 \right)^{\frac{1}{2}}} \\
&\leq \frac{(Q+\alpha)(Q-\alpha-4)}{4} + \frac{\tilde{C}_{H, \Phi}}{\left(\int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon, R}^2}{\rho^{\alpha+4}} |\nabla_{\gamma} \rho|^2 \right)^{\frac{1}{2}}},
\end{aligned}$$

where

$$\begin{aligned} \tilde{C}_{H,\Phi} &= \frac{3}{2}(3 + \alpha)^2 \left(\max_{t \in \mathbb{R}} |H'(t)|^2 + \max_{t \in \mathbb{R}} |\Phi'(t)|^2 \right) \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\ &\quad + \frac{15}{4} \left(\max_{t \in \mathbb{R}} |\Phi''(t)|^2 + \max_{t \in \mathbb{R}} |H''(t)|^2 \right) \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma. \end{aligned}$$

So to finish the proof, it is enough to show

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon,R}^2}{\rho^{\alpha+4}} |\nabla_{\gamma} \rho|^2 = +\infty.$$

In fact,

$$\begin{aligned} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{G_{\varepsilon,R}^2}{\rho^{\alpha+4}} |\nabla_{\gamma} \rho|^2 &= \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{|\Phi(\frac{\rho}{R})H(\frac{\rho}{\varepsilon})|^2}{\rho^Q} |\nabla_{\gamma} \rho|^2 \\ &\geq \int_{2\varepsilon}^R \rho^{-1} d\rho \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \\ &= (\ln R - \ln 2\varepsilon) \int_{\Sigma} |\nabla_{\gamma} \rho|^2 d\sigma \rightarrow +\infty \quad (\varepsilon \rightarrow 0^+). \end{aligned}$$

The desired result follows. \square

REMARK 2.4. Since the test function $G_{\varepsilon,R}(\rho)$ in Lemma 2.3 is radial, the constant $\left(\frac{(Q+\alpha)(Q-\alpha-4)}{4}\right)^2$ is also sharp when the function u is restricted to be in $C_0^\infty(\mathbb{R}^{m+n})$ and satisfy

$$u(x, y) = \tilde{u}(x_1, \dots, x_{m-3}, \sqrt{x_{m-2}^2 + x_{m-1}^2 + x_m^2}, y)$$

or

$$u(x, y) = \tilde{u}(x, y_1, \dots, y_{n-3}, \sqrt{y_{n-2}^2 + y_{n-1}^2 + y_n^2})$$

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.1. By Lemma 2.3, we consider the sharp Rellich inequality on $\mathbb{R}_x^{m+2} \times \mathbb{R}_y^n$: for $v \in C_0^\infty(\mathbb{R}_x^{m+2} \times \mathbb{R}_y^n)$ and $\alpha < m + 2 + n(1 + \gamma) - 4 = Q - 2$, where $Q = m + n(1 + \gamma)$, there holds

$$\int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} \frac{|\Delta_{\gamma} v|^2}{|\nabla_{\gamma} \rho|^2 \rho^{\alpha}} \geq \left(\frac{(Q + 2 + \alpha)(Q - \alpha - 2)}{4} \right)^2 \int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} \frac{v^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}. \tag{2.15}$$

Now we assume the function $v = \tilde{v}(x_1, \dots, x_{m-1}, \sqrt{x_m^2 + x_{m+1}^2 + x_{m+2}^2}, y)$. By Remark 2.4, the constant in (2.15) is also sharp when v is limited to such functions. Set $\tilde{x}_m = \sqrt{x_m^2 + x_{m+1}^2 + x_{m+2}^2}$. We compute

$$\begin{aligned} &\int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} \frac{|\Delta_{\gamma} \tilde{v}|^2}{|\nabla_{\gamma} \rho|^2 \rho^{\alpha}} \\ &= |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{\tilde{x}_m^2 \left| \sum_{i=1}^{m-1} \frac{\partial^2 \tilde{v}}{\partial x_i^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{x}_m^2} + \frac{2}{\tilde{x}_m} \frac{\partial \tilde{v}}{\partial \tilde{x}_m} + (1 + \gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right|^2}{|\nabla_{\gamma} \rho|^2 \rho^{\alpha}}. \end{aligned}$$

Replacing \tilde{x}_m by x_m in the equality above, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{m+2} \times \mathbb{R}^n} \frac{|\Delta_\gamma \tilde{v}|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &= |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{x_m^2 \left| \sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + \frac{2}{x_m} \frac{\partial \tilde{v}}{\partial x_m} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &\geq \left(\frac{(Q+2+\alpha)(Q-\alpha-2)}{4} \right)^2 |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{x_m^2 \tilde{v}^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}. \end{aligned} \tag{2.16}$$

Set $u = x_m \tilde{v}$. Then

$$\begin{aligned} & \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j^2} \\ &= x_m \left[\sum_{i=1}^{m-1} \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right] + \frac{\partial^2 (x_m \tilde{v})}{\partial x_m^2} \\ &= x_m \left[\sum_{i=1}^{m-1} \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right] + x_m \frac{\partial^2 \tilde{v}}{\partial x_m^2} + 2 \frac{\partial \tilde{v}}{\partial x_m} \\ &= x_m \left[\sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + \frac{2}{x_m} \frac{\partial \tilde{v}}{\partial x_m} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right]. \end{aligned} \tag{2.17}$$

By (2.16) and (2.17),

$$\begin{aligned} & |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{\left| \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j^2} \right|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &\geq \left(\frac{(Q+2+\alpha)(Q-\alpha-2)}{4} \right)^2 |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{x_m^2 \tilde{v}^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}} \\ &= \left(\frac{(Q+2+\alpha)(Q-\alpha-2)}{4} \right)^2 |\mathbb{S}^2| \int_{\mathbb{R}^{m-1}} \int_0^\infty \int_{\mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}, \end{aligned}$$

i.e.

$$\int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q+2+\alpha)(Q-\alpha-2)}{4} \right)^2 \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}. \tag{2.18}$$

The constant $\left(\frac{(Q+2+\alpha)(Q-\alpha-2)}{4} \right)^2$ in (2.18) is sharp since the same constant in (2.15) is sharp. This completes the proof of Theorem 1.3. \square

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.1 and Theorem 1.2. Consider the sharp Hardy inequality on $\mathbb{R}_x^m \times \mathbb{R}_y^{n+2}$:

$$\int_{\mathbb{R}^m \times \mathbb{R}^{n+2}} |\nabla_\gamma v|^2 \geq \frac{(m+(1+\gamma)n+2\gamma)^2}{4} \int_{\mathbb{R}^m \times \mathbb{R}^{n+2}} \frac{v^2}{\rho^2(x,y)} \frac{|x|^{2\gamma}}{\rho^{2\gamma}(x,y)}, \tag{2.19}$$

where $v \in C_0^\infty(\mathbb{R}_x^m \times \mathbb{R}_y^{n+2})$. Next we assume the function

$$v = \tilde{v}(x, y_1, \dots, y_{n-1}, \sqrt{y_n^2 + y_{n+1}^2 + y_{n+2}^2}).$$

By Remark 2.2, the constant in (2.19) is also sharp when v is limited to such functions.

Following the proof of Theorem 1.1, we have, replacing $\sqrt{y_n^2 + y_{n+1}^2 + y_{n+2}^2}$ by y_n ,

$$\int_{\mathbb{R}^m \times \mathbb{R}_+^n} |\nabla_\gamma \tilde{v}|^2 y_n^2 \geq \frac{(Q + 2\gamma)^2}{4} \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{\tilde{v}^2 |x|^{2\gamma}}{\rho^2 \rho^{2\gamma} y_n^2}. \tag{2.20}$$

Set $u = y_n \tilde{v}$. By (2.20),

$$\int_{\mathbb{R}^m \times \mathbb{R}_+^n} |\nabla_\gamma u|^2 = \int_{\mathbb{R}^m \times \mathbb{R}_+^n} |\nabla_\gamma \tilde{v}|^2 y_n^2 \geq \frac{(Q + 2\gamma)^2}{4} \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^2 \rho^{2\gamma}} \tag{2.21}$$

and the constant appeared in (2.21) is sharp. Therefore, we prove inequality (1.7) for $\alpha = 0$.

Next, replacing u by $\rho^{-\frac{\alpha}{2}} u$ in (2.21), we have, following the proof of Theorem 1.2,

$$\begin{aligned} \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{|\nabla_\gamma u|^2}{\rho^\alpha} &= \int_{\mathbb{R}^m \times \mathbb{R}_+^n} |\nabla_\gamma (\rho^{-\frac{\alpha}{2}} u)|^2 - \frac{\alpha^2 - 2\alpha(\alpha + 2 - Q)}{4} \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^{2+\alpha} \rho^{2\gamma}} \\ &\geq \frac{(Q + 2\gamma)^2 - \alpha^2 + 2\alpha(\alpha + 2 - Q)}{4} \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^{2+\alpha} \rho^{2\gamma}} \\ &= \left(\frac{(Q + 2\gamma - \alpha)^2}{4} + (1 + \gamma)\alpha \right) \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^{2+\alpha} \rho^{2\gamma}}. \end{aligned}$$

The desired result follows. \square

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.3. Consider the sharp Rellich inequality on $\mathbb{R}_x^m \times \mathbb{R}_y^{n+2}$: for $v \in C_0^\infty(\mathbb{R}_x^m \times \mathbb{R}_y^{n+2})$ and $\alpha < m + (n + 2)(1 + \gamma) - 4 = Q + 2\gamma - 2$, there holds

$$\int_{\mathbb{R}^m \times \mathbb{R}^{n+2}} \frac{|\Delta_\gamma v|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q + 2\gamma + \alpha + 2)(Q + 2\gamma - \alpha - 2)}{4} \right)^2 \int_{\mathbb{R}^m \times \mathbb{R}^{n+2}} \frac{v^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}.$$

Here we also denote by $Q = m + n(1 + \gamma)$. Set

$$v = \tilde{v}(x, y_1, \dots, y_{n-1}, \sqrt{y_n^2 + y_{n+1}^2 + y_{n+2}^2}).$$

By Remark 2.4, the constant $\left(\frac{(Q + 2\gamma + \alpha + 2)(Q + 2\gamma - \alpha - 2)}{4} \right)^2$ is also sharp when v is limited

to such functions. Therefore, if we set $\tilde{y}_n = \sqrt{y_n^2 + y_{n+1}^2 + y_{n+2}^2}$, then

$$\begin{aligned} & \int_{\mathbb{R}^m \times \mathbb{R}^{n+2}} \frac{|\Delta_\gamma \tilde{v}|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &= |\mathbb{S}^2| \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\tilde{y}_n^2 \left| \sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \left(\sum_{j=1}^{n-1} \frac{\partial^2 \tilde{v}}{\partial y_j^2} + \frac{\partial^2 \tilde{v}}{\partial y_n^2} + \frac{2}{y_n} \frac{\partial \tilde{v}}{\partial y_n} \right) \right|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &\geq \left(\frac{(Q+2\gamma+\alpha+2)(Q+2\gamma-\alpha-2)}{4} \right)^2 |\mathbb{S}^2| \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\tilde{y}_n^2 \tilde{v}^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}. \end{aligned}$$

Replacing \tilde{y}_m by y_m in the equality above, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{y_n^2 \left| \sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \left(\sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} + \frac{2}{y_n} \frac{\partial \tilde{v}}{\partial y_n} \right) \right|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &\geq \left(\frac{(Q+2\gamma+\alpha+2)(Q+2\gamma-\alpha-2)}{4} \right)^2 \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{y_n^2 \tilde{v}^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}. \end{aligned} \tag{2.22}$$

Set $u = y_n \tilde{v}$. We compute

$$\begin{aligned} & \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j^2} \\ &= y_n \left[\sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^{n-1} \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right] + (1+\gamma)^2 |x|^{2\gamma} \frac{\partial^2 (y_n \tilde{v})}{\partial y_n^2} \\ &= y_n \left[\sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^{n-1} \frac{\partial^2 \tilde{v}}{\partial y_j^2} \right] + (1+\gamma)^2 |x|^{2\gamma} \left(y_n \frac{\partial^2 \tilde{v}}{\partial y_n^2} + 2 \frac{\partial \tilde{v}}{\partial y_n} \right) \\ &= y_n \left[\sum_{i=1}^m \frac{\partial^2 \tilde{v}}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \left(\sum_{j=1}^n \frac{\partial^2 \tilde{v}}{\partial y_j^2} + \frac{2}{y_n} \frac{\partial \tilde{v}}{\partial y_n} \right) \right]. \end{aligned} \tag{2.23}$$

By (2.22) and (2.23), we have

$$\begin{aligned} & \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\left| \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} + (1+\gamma)^2 |x|^{2\gamma} \sum_{j=1}^n \frac{\partial^2 u}{\partial y_j^2} \right|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \\ &\geq \left(\frac{(Q+2\gamma+\alpha+2)(Q+2\gamma-\alpha-2)}{4} \right)^2 \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}, \end{aligned}$$

i.e.

$$\int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{|\Delta_\gamma u|^2}{|\nabla_\gamma \rho|^2 \rho^\alpha} \geq \left(\frac{(Q+2\gamma+\alpha+2)(Q+2\gamma-\alpha-2)}{4} \right)^2 \int_{\mathbb{R}^m \times \mathbb{R}_+^n} \frac{u^2 |x|^{2\gamma}}{\rho^{4+\alpha+2\gamma}}.$$

The desired result follows. \square

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