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MORE JORDAN TYPE INEQUALITIES

D. AHARONOV AND U. ELIAS

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Abstract. The function $\tan(\pi x/2)/(\pi x/2)$ is expanded into a Laurent series of $1-x^2$, where the coefficients are given explicitly as combinations of zeta function of even integers. This is used to achieve a sequence of upper and lower bounds which are very precise even at the poles at $x=\pm 1$.

Similar results are obtained for other trigonometric functions with poles.

1. Introduction

Jordan's inequality,

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1, \qquad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2},$$

has numerous extensions and generalizations. (To simplify the notation we agree that for $x=0$ the value of $\sin x/x$ is 1). An exhaustive review of the literature is available in [6]. One of the many types of generalizations is a sequence of the two sided inequalities. The first of them is the inequality

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2-4x^2), \qquad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \tag{1.1}$$

by Ling Zhu [10]. It is followed by

$$\begin{aligned} \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2) + \frac{12-\pi^2}{16\pi^5}(\pi^2-4x^2)^2 &\leqslant \frac{\sin x}{x} \\ \leqslant \frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2-4x^2) + \frac{\pi-3}{\pi^5}(\pi^2-4x^2)^2, \end{aligned} \tag{1.2}$$

by [11]; Finally Ling Zhu verifies [12] the general

$$\sum_{n=0}^{N} a_n(\pi^2-4x^2)^n + a_{N+1}(\pi^2-4x^2)^{N+1} \leqslant \frac{\sin x}{x} \leqslant \sum_{n=0}^{N} a_n(\pi^2-4x^2)^n + b_{N+1}(\pi^2-4x^2)^{N+1}, \tag{1.3}$$

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where a_n are given by a recursion formula. See also [6, Section 3.7].

Later (1.3) was extended in [13] to

$$\sum_{n=0}^{N} a_{2n,r}(r^2-x^2)^n+\alpha_{N+1,r}(r^2-x^2)^{N+1} \leqslant \frac{\sin x}{x} \leqslant \sum_{n=0}^{N} a_{2n,r}(r^2-x^2)^n+\beta_{N+1,r}(r^2-x^2)^{N+1} \tag{1.4}$$

for $|x| \leqslant r \leqslant \pi/2$. The proofs of inequalities (1.1)–(1.4) are based on a monotone version of the l'Hospital rule: *If $g'(x) \neq 0$ and $f'(x)/g'(x)$ is increasing on (a,b), so is the quotient $(f(x)-f(a))/(g(x)-g(a))$.*

The two sided inequalities (1.3) are closely related to the infinite expansion of [8]

$$\frac{\sin x}{x} = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n R_n}{n!\pi^{2n}}(\pi^2-4x^2)^n \tag{1.5}$$

where $R_k = \sum_{n=k}^{\infty} \frac{(-1)^n n!}{(2n+1)!(n-k)!}\left(\frac{\pi}{2}\right)^{2n}$ and $(-1)^k R_k > 0$. See also [6, Section 2.3]. Identity (1.5) is verified by a straightforward rearrangement of a double power series. By direct integration one achieves also

$$\cos x = \frac{1}{4\pi}(\pi^2-4x^2) + \sum_{n=1}^{\infty} \frac{(-1)^n R_n}{8(n+1)!\pi^{2n}}(\pi^2-4x^2)^{n+1}.$$

Similar ideas were developed for Bessel functions in a paper by Baricz and Wu [2] and are summarized in [3]. In [3, p. 145] the author uses the notation $\mathscr{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x)$, normalized so that $\mathscr{J}_p(0) = 1$. Without normalization the result of Baricz and Wu are formulated as

$$\begin{aligned}\sum_{k=0}^{N} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2-x^2)^k + \alpha_p(r^2-x^2)^{N+1} &\leqslant x^{-p}J_p(x) \\ &\leqslant \sum_{k=0}^{N} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2-x^2)^k + \beta_p(r^2-x^2)^{N+1}\end{aligned} \tag{1.6}$$

for all N and for $|x| \leqslant r$, where $0 < r \leqslant j_{p+1,1}$ and $j_{p+1,1}$ denotes the first positive zero of $J_{p+1}(x)$.

The aim of this study is to develop similar ideas for the the trigonometric functions which have singularities, namely, the $\tan, \cot, \sec$ and cosec functions. Our attention was drawn to this subject in an attempt to refine the inequality of Becker and Stark [4]

$$\frac{8/\pi^2}{1-x^2} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{1}{1-x^2}, \qquad -1 < x < 1. \tag{1.7}$$

Our main result (Theorem 1) is that the function $\frac{\tan(\pi x/2)}{\pi x/2}$ admits a sequence of upper and lower bounds and these inequalities are closely related to a Laurent series expansion in powers of $1-x^2$. The same approach will be applied also to $\cot, \sec$ and cosec.

In spite of the similarity, there are several essential differences between our result and those of Ling Zhu for $\sin x/x$ and of Baricz for the Bessel functions.

(a) Bessel functions and, consequently $\sin x = x^{-1/2}J_{1/2}(x)$, are solutions of simple linear second order differential equations. This is a useful tool in [3]. Unfortunately, we do not know any simple linear differential equation for the tan function. On the contrary, the results for $\tan(\pi x/2)/(\pi x/2)$ are influenced by its poles at $x = \pm 1$.

(b) We start our study with the infinite series and the upper and lower bounds appear as consequences of this alternating series. We do not use the monotone l'Hospital rule. Our method uses partial fractions expansions.

In analogy with (1.4) and (1.6) we also expand $\dfrac{\tan(\pi x/2)}{\pi x/2}$ into a Laurent series of $r^2 - x^2$, with $0 < r < 1$. This expansion has different properties from that which corresponds to $r = 1$.

2. Inequalities and series for the *tan* function

In this section we present an infinite series expansion for the tan function and two sets of upper and lower bounds, one with an even number of terms and one with odd number of terms.

THEOREM 2.1. *(a) We have the infinite expansion*

$$\frac{\tan(\pi x/2)}{\pi x/2} = \frac{8}{\pi^2}\left[\frac{1}{1-x^2} + \sum_{k=0}^{\infty}(-1)^k \frac{T_{k+1}}{4^{k+1}}(1-x^2)^k\right], \tag{2.1}$$

where

$$T_p = (-1)^p\left[2\sum_{m=1}^{[p/2]}\binom{2p-2m-1}{p-1}\zeta(2m) - \binom{2p-1}{p-1}\right], \qquad p = 1,2,\dots. \tag{2.2}$$

All the coefficients T_p are positive. The infinite series in (2.1) converges for $|x| < 3$.

(b) For all even number $2N$ and $-1 < x < 1$, we have the pair of inequalities

$$\frac{8}{\pi^2}\left[\frac{1}{1-x^2} + \sum_{k=0}^{2N-1}(-1)^k \frac{T_{k+1}}{4^{k+1}}(1-x^2)^k + H_{2N+1}(1-x^2)^{2N}\right] \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{8}{\pi^2}\left[\frac{1}{1-x^2} + \sum_{k=0}^{2N}(-1)^k \frac{T_{k+1}}{4^{k+1}}(1-x^2)^k\right], \tag{2.3}$$

where

$$H_{2N+1} = \frac{\pi^2}{8} - \left(1 + \frac{T_1}{4} - \frac{T_2}{4^2} + - \cdots - \frac{T_{2N}}{4^{2N}}\right)$$

and $0 < H_{2N+1} < \dfrac{T_{2N+1}}{4^{2N+1}}$. The left hand side inequality of (2.3) is strict for $x = 0$.

(c) For all odd number $2N+1$ and $-1<x<1$, we have

$$\frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{k=0}^{2N+1}(-1)^k\frac{T_{k+1}}{4^{k+1}}(1-x^2)^k\right] \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{k=0}^{2N}(-1)^k\frac{T_{k+1}}{4^{k+1}}(1-x^2)^k - H_{2N+2}(1-x^2)^{2N+1}\right], \tag{2.4}$$

where

$$H_{2N+2}=-\frac{\pi^2}{8}+\left(1+\frac{T_1}{4}-\frac{T_2}{4^2}+-\cdots+\frac{T_{2N+1}}{4^{2N+1}}\right)$$

and $0<H_{2N+2}<\dfrac{T_{2N+2}}{4^{2N+2}}$. *The right hand side inequality of (2.4) is strict for* $x=0$.

EXAMPLES. The first coefficients are $T_1=1$, $T_2=2\zeta(2)-\binom{3}{1}=\pi^2/3-3$ and $T_3=-2\binom{3}{2}\zeta(2)+\binom{5}{2}=-6\zeta(2)+10=10-\pi^2$. Accordingly,

$$H_1=\frac{\pi^2}{8}-1\,, \qquad H_2=-\left(\frac{\pi^2}{8}-1-\frac{T_1}{4}\right)=\frac{10-\pi^2}{8}\,.$$

The first two special cases of (2.3) (for $2N=0$) and (2.4) (for $2N+1=1$) are

$$\frac{8/\pi^2}{1-x^2}+\frac{\pi^2-8}{\pi^2}\leqslant\frac{\tan(\pi x/2)}{\pi x/2}\leqslant\frac{8/\pi^2}{1-x^2}+\frac{2}{\pi^2}\,, \tag{2.5}$$

and

$$\frac{8/\pi^2}{1-x^2}+\frac{2}{\pi^2}-\frac{\pi^2-9}{6\pi^2}(1-x^2)\leqslant\frac{\tan(\pi x/2)}{\pi x/2}\leqslant\frac{8/\pi^2}{1-x^2}+\frac{2}{\pi^2}-\frac{10-\pi^2}{\pi^2}(1-x^2). \tag{2.6}$$

Proof of Theorem 2.1. For sake of convenience we divide the proof into several steps.

(i) The infinite expansion for tan. Our starting point is the partial fraction expansion $\tan z=\sum_{n=0}^{\infty}\dfrac{8z}{(2n+1)^2\pi^2-4z^2}$ which holds in every domain which contains no poles. It follows that

$$\frac{\tan(\pi x/2)}{\pi x/2}=\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-x^2}=\frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{n=1}^{\infty}\frac{1}{4n(n+1)+(1-x^2)}\right]. \tag{2.7}$$

Each term of the series may be expanded into a geometric series, provided that $|1-x^2|<4n(n+1)$ for $n=1,2,\dots$, i.e., for $|x|<3$. This yields

$$\begin{aligned}\frac{\tan(\pi x/2)}{\pi x/2} &= \frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k(1-x^2)^k}{(4n(n+1))^{k+1}}\right)\right]\\ &= \frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}(n+1)^{k+1}}\right)\frac{(-1)^k}{4^{k+1}}(1-x^2)^k\right].\end{aligned} \tag{2.8}$$

The expansion (2.1) will follow if we show that the sums $T_p=\sum_{n=1}^{\infty}\frac{1}{n^p(n+1)^p}$ are given explicitly by (2.2) for $p=1,2,\dots$. The proof is based on the following identity [7, Ex. 39, p. 246]: *If* $xy=x+y$, *then for every* $m,k\geqslant 1$,

$$x^m y^k=\sum_{j=1}^{m}\binom{m+k-j-1}{k-1}x^j+\sum_{j=1}^{k}\binom{m+k-j-1}{m-1}y^j. \tag{2.9}$$

(2.9) may be verified by mathematical induction on m and k. We take in (2.9) $m=k=p$, $x=-\frac{1}{n}$, $y=\frac{1}{n+1}$, and conclude that

$$\frac{(-1)^p}{n^p(n+1)^p}=\sum_{j=1}^{p}\binom{2p-j-1}{p-1}\left[\frac{(-1)^j}{n^j}+\frac{1}{(n+1)^j}\right]. \tag{2.10}$$

To calculate T_p we sum (2.10) for $n=1,2,\dots$ and separate $j=1$ from $j=2,\dots,p$:

$$\begin{aligned}(-1)^pT_p &= (-1)^p\sum_{n=1}^{\infty}\sum_{j=1}^{p}\binom{2p-j-1}{p-1}\left[\frac{(-1)^j}{n^j}+\frac{1}{(n+1)^j}\right]\\ &= \binom{2p-2}{p-1}\sum_{n=1}^{\infty}\left[-\frac{1}{n}+\frac{1}{n+1}\right]+\sum_{j=2}^{p}\binom{2p-j-1}{p-1}\left[(-1)^j\zeta(j)+\big(\zeta(j)-1\big)\right]\\ &= \binom{2p-2}{p-1}\cdot(-1)+\sum_{j=2}^{p}\binom{2p-j-1}{p-1}\left[\big((-1)^j+1\big)\zeta(j)-1\big)\right]\\ &= 2\sum_{2\leqslant 2m\leqslant p}\binom{2p-2m-1}{p-1}\zeta(2m)-\sum_{j=1}^{p}\binom{2p-j-1}{p-1}.\end{aligned}$$

Since $\sum_{j=1}^{p}\binom{2p-j-1}{p-1}=\binom{2p-1}{p-1}$, formula (2.2) is verified and part (a) of the theorem follows.

We remark that similar sums $\sum_{n=1}^{\infty}\frac{1}{(n-1/2)^p(n+1/2)^p}$ are mentioned in [9, p. 70, (373)] for $p=1,2,3$ and 4.

(ii) The inequalities. Let us write the series (2.1) as

$$\frac{\tan(\pi x/2)}{\pi x/2}=\frac{8}{\pi^2}\left[\frac{1}{1-x^2}+\sum_{k=0}^{m}(-1)^k\frac{T_{k+1}}{4^{k+1}}(1-x^2)^k+R_{m+1}(x)\right],$$

with the remainder term

$$R_{m+1}(x) = (-1)^{m+1}(1-x^2)^{m+1}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{T_{\ell+m+2}}{4^{\ell+m+2}}(1-x^2)^{\ell}.$$

For each x, $0 \leqslant x \leqslant 1$, the series

$$F_{m+1}(x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{T_{\ell+m+2}}{4^{\ell+m+2}}(1-x^2)^{\ell}$$

and its derivative series

$$F'_{m+1}(x) = 2x\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\ell\frac{T_{\ell+m+2}}{4^{\ell+m+2}}(1-x^2)^{\ell-1}$$

are alternating series with smaller and smaller terms. Indeed,

$$\frac{1}{2^p} < T_p = \sum_{n=1}^{\infty}\frac{1}{n^p(n+1)^p} < \frac{1}{2^p}\zeta(p) < \frac{1}{2^p}\zeta(2) < \frac{1}{2^{p-1}},$$

thus

$$\frac{T_{\ell+m+2}}{4^{\ell+m+2}} > \frac{1}{2^{\ell+m+2}4^{\ell+m+2}} > \frac{1}{2^{\ell+m+2}4^{\ell+m+3}} > \frac{T_{\ell+m+3}}{4^{\ell+m+3}}$$

and similarly

$$\ell\frac{T_{\ell+m+2}}{4^{\ell+m+2}} > (\ell+1)\frac{T_{\ell+m+3}}{4^{\ell+m+3}}.$$

Thus, by properties of alternating series, $F_{m+1}(x)$ and $F'_{m+1}(x)$ are positive functions for $0 \leqslant x \leqslant 1$. Consequently $F_{m+1}(x)$ increases for $0 \leqslant x \leqslant 1$, and is bounded from below and from above by

$$0 < \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{T_{\ell+m+2}}{4^{\ell+m+2}} = F_{m+1}(0) \leqslant F_{m+1}(x) \leqslant F_{m+1}(1) = \frac{T_{m+2}}{4^{m+2}}.$$

The lower bound of $F_{m+1}(x)$, $\frac{T_{m+2}}{4^{m+2}} - \frac{T_{m+3}}{4^{m+3}} + - \cdots$ may be written as a finite sum: For $x = 0$, (2.1) yields

$$1 = \frac{8}{\pi^2}\left[1 + \sum_{k=0}^{\infty}(-1)^k\frac{T_{k+1}}{4^{k+1}}\right], \tag{2.11}$$

so

$$\begin{aligned} 0 < H_{m+2} &:= \frac{T_{m+2}}{4^{m+2}} - \frac{T_{m+3}}{4^{m+3}} + - \cdots \\ &= (-1)^{m+1}\left[\frac{\pi^2}{8} - \left(1 + \frac{T_1}{4} - \frac{T_2}{4^2} + - \cdots + (-1)^m\frac{T_{m+1}}{4^{m+1}}\right)\right]. \end{aligned} \tag{2.12}$$

Consequently $R_{m+1}(x) = (-1)^{m+1}(1-x^2)^{m+1}F_{m+1}(x)$ is bounded for $m = 2N$ and for $m = 2N-1$, respectively, by

$$\begin{aligned} -(1-x^2)^{2N+1}\frac{T_{2N+2}}{4^{2N+2}} &\leqslant R_{2N+1}(x) \leqslant -(1-x^2)^{2N+1}H_{2N+2} < 0, \\ 0 < (1-x^2)^{2N}H_{2N+1} &\leqslant R_{2N}(x) \leqslant (1-x^2)^{2N}\frac{T_{2N+1}}{4^{2N+1}}. \end{aligned}$$

and parts (b) and (c) of the theorem follow.

The remainder term satisfies $|R_{m+1}(x)| \leqslant (1-x^2)^{m+1}\frac{T_{m+2}}{4^{m+2}} \leqslant \frac{(1-x^2)^{m+1}}{2^{3m+5}}$, hence it decreases rapidly, in particular near the poles $x=\pm 1$. For example, for $x=0.9$ and $m=4$ we have $R_5(0.9) < 0.19^5/2^{17} \approx 2\cdot 10^{-9}$.

The bounds in (2.4) and (2.3) are also very precise near the poles $x=\pm 1$. Indeed, the difference between the m-th upper and lower bounds (m odd or even) is less then $\frac{T_{m+3}}{4^{m+3}}(1-x^2)^{m+1} < \frac{1}{2^{3m+8}}(1-x^2)^{m+1}$.

It is interestiong to compare the convergence of the series (2.1) with that of the Taylor series [14, Section 3:14]

$$\frac{\tan(\pi x/2)}{\pi x/2} = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty}\lambda(2k+2)x^{2k}, \qquad |x|<1, \tag{2.13}$$

where $\lambda(p) = \sum_{n=0}^{\infty} 1/(2n+1)^p$ denotes the Dirichlet lambda function. Since $1 < \lambda(p) < 1+2/3^p$, the remainder of the series (2.13) after m terms is

$$\frac{8}{\pi^2}\sum_{k=m+1}^{\infty}\lambda(2k+2)x^{2k} = \frac{8}{\pi^2}\frac{x^{2m+2}}{1-x^2}(1+o(1)).$$

Here $o(1)$ is uniformly small for large values of m. On the other hand, we saw above that the remainder of the series (2.1) after m terms is $\frac{8}{\pi^2}(1-x^2)^{m+1}\frac{O(1)}{2^{3m}}$.

After extracting m-th root from the two remainders for large m, we compare, up to $1+o(1)$, x^2 with $(1-x^2)/8$. The consequence is that after m terms, for large m, the Taylor series (2.13) yields a smaller remainder for $|x|<1/3$, while for $1/3<|x|<1$, the series (2.1) has a smaller remainder. □

Due to (1.4) and (1.6) it is also natural to expand $\frac{\tan(\pi x/2)}{\pi x/2}$ into power series of r^2-x^2. Formally we follow (2.7) and (2.8):

$$\begin{aligned}\frac{\tan(\pi x/2)}{\pi x/2} &= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-x^2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{\left((2n+1)^2-r^2\right)+(r^2-x^2)}\\ &= \frac{8}{\pi^2}\left[\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1-r}{2})^{k+1}(n+\frac{1+r}{2})^{k+1}}\right)\frac{(-1)^k}{4^{k+1}}(r^2-x^2)^k\right]. \qquad (2.14)\\ &= \frac{8}{\pi^2}\sum_{k=0}^{\infty}(-1)^k\frac{\tilde{T}_{k+1}(r)}{4^{k+1}}(r^2-x^2)^k,\end{aligned}$$

with

$$\tilde{T}_p(r) = \sum_{n=0}^{\infty}\frac{1}{(n+\frac{1-r}{2})^p(n+\frac{1+r}{2})^p}. \tag{2.15}$$

Note that $\tilde{T}_p(r)$ differs from T_p by it's range of summation and it is not defined for $r=1$.

Equations (2.14) and (2.8) are superficially similar. However, they describe different phenomena. (2.8) describes the function $\tan(\pi x/2)/(\pi x/2)$ in the interval $(-3,3)$, including its singularities at $x=\pm 1$. Expansion (2.14) is valid when $|r^2-x^2|<(2n+1)^2-r^2$ for all $n=0,1,2,\dots$. Assuming $r<1$, this holds when $|r^2-x^2|<1-r^2$, i.e.,

$$2r^2-1<x^2<1,$$

which includes the points $x=\pm r$ and their neighborhoods. Note that in the complex plane $|r^2-z^2|<C$ is the interior of a lemniscate with focal points $z=\pm r$.

Inequalities analogous to (2.3) and (2.4) may be deduced from expansion (2.14). We do not attempt to calculate the sums (2.15) explicitly as functions of r. Rather we shall obtain a recursive formula for the coefficients $\tilde{T}_p(r)$.

The function $\tan x$ satisfies the first order, quadratic Riccati differential equation $y'=1+y^2$. It easy to see that $u(x)=\dfrac{\tan(\pi x/2)}{\pi x/2}$ satisfies $x\dfrac{du}{dx}=1-u+\dfrac{\pi^2}{4}x^2u^2$ for $|x|<1$. If we let $z=r^2-x^2$ and $u(x)=v(z)$, then $v(z)$ satisfies

$$-2(r^2-z)\frac{dv}{dz}=1-v+\frac{\pi^2}{4}(r^2-z)v^2$$

for $-1+r^2<z<r^2$. Substitution of $v(z)=\dfrac{8}{\pi^2}\sum_{k=0}^{\infty}(-1)^k\dfrac{\tilde{T}_{k+1}(r)}{4^{k+1}}z^k$, $v(0)=\dfrac{\tan(\pi r/2)}{\pi r/2}$ and comparison of coefficients lead to the following quadratic recursion formula for $\tilde{T}_k(r)$, $0<r<1$:

$$\begin{aligned}
\tilde{T}_1(r)&=\frac{\pi^2}{2}\,\frac{\tan(\pi r/2)}{\pi r/2},\\
r^2\tilde{T}_2(r)&=\pi^2-2\tilde{T}_1(r)+r^2\tilde{T}_1^2(r),\\
(k+1)r^2\tilde{T}_{k+2}(r)&=-(4k+2)\tilde{T}_{k+1}+r^2\sum_{j=0}^{k}\tilde{T}_{j+1}\tilde{T}_{k-j+1}+4\sum_{j=0}^{k-1}\tilde{T}_{j+1}\tilde{T}_{k-j},\ k=1,2,\dots
\end{aligned}\tag{2.16}$$

3. Inequalities for the secant function

Expansions and inequalities of similar type are available for the cot, sec and cosec functions. In this section we present the results for the secant function. We remind, in analogy with inequalities (1.7) and (4.1), that

$$\frac{1}{1-x^2}\leqslant\sec\frac{\pi x}{2}\leqslant\frac{4/\pi}{1-x^2},\qquad -1<x<1.\tag{3.1}$$

To prove (3.1), it is sufficient to show that the even function $(1-x^2)\sec(\pi x/2)$ increases for $0\leqslant x\leqslant 1$. This indeed holds, since by (1.7)

$$\frac{d}{dx}\big((1-x^2)\sec(\pi x/2)\big)=2x\sec(\pi x/2)\left[\frac{\pi^2}{8}(1-x^2)\frac{\tan(\pi x/2)}{\pi x/2}-1\right]\geqslant 0.\tag{3.2}$$

The constants in (3.1) follow now by the limit values for $x=0$ and as $x\to 1^-$.

Our aim is to generalize the right hand side of (3.1) into an infinite sequence of inequalities.

THEOREM 3.1. *(a) We have the infinite expansion*

$$\sec(\pi x/2)=\frac{4}{\pi}\left[\frac{1}{1-x^2}-\sum_{k=0}^{\infty}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k\right], \tag{3.3}$$

where

$$S_p=(-1)^p\left[\binom{2p-2}{p-2}-\binom{2p-2}{p-1}+2\sum_{2\leqslant 2m\leqslant p}\left[\binom{2p-2m-2}{p-2}-\binom{2p-2m-2}{p-1}\right]\eta(2m)\right] \tag{3.4}$$

for $p=1,2,\dots$ and η denotes the Dirichlet eta function. All the coefficients S_p are positive.

(b) For all even number $2N$ and $-1<x<1$, we have the pair of inequalities

$$\begin{aligned}\frac{4}{\pi}\left[\frac{1}{1-x^2}-\sum_{k=0}^{2N}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k\right]&\leqslant \sec(\pi x/2)\\ &\leqslant \frac{4}{\pi}\left[\frac{1}{1-x^2}-\sum_{k=0}^{2N-1}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k-J_{2N+1}(1-x^2)^{2N}\right],\end{aligned} \tag{3.5}$$

where

$$J_{2N+1}=-\frac{\pi}{4}+\left(1-\frac{S_1}{4}+\frac{S_2}{4^2}-+\cdots+\frac{S_{2N}}{4^{2N}}\right)$$

and $0<J_{2N+1}<\dfrac{S_{2N+1}}{4^{2N+1}}$. The right hand side inequality of (3.5) is strict for $x=0$.

(c) For all odd number $2N+1$ and $-1<x<1$, we have

$$\begin{aligned}\frac{4}{\pi}\left[\frac{1}{1-x^2}-\sum_{k=0}^{2N}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k+J_{2N+2}(1-x^2)^{2N+1}\right]&\\ \leqslant \sec(\pi x/2)\leqslant \frac{4}{\pi}\left[\frac{1}{1-x^2}-\sum_{k=0}^{2N+1}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k\right],&\end{aligned} \tag{3.6}$$

where

$$J_{2N+2}=\frac{\pi}{4}-\left(1-\frac{S_1}{4}+\frac{S_2}{4^2}-+\cdots-\frac{S_{2N+1}}{4^{2N+1}}\right)$$

and $0<J_{2N+2}<\dfrac{S_{2N+2}}{4^{2N+2}}$. The left hand side inequality of (3.6) is strict for $x=0$.

EXAMPLES. The first coefficients are $S_1 = 1$, $S_2 = -1 + 2\eta(2) = \pi^2/6 - 1$ and $S_3 = 2 - 2\eta(2) = 2 - \pi^2/6$. Accordingly,

$$J_1 = -\frac{\pi}{4} + 1\,, \qquad J_2 = \frac{\pi}{4} - 1 + \frac{1}{4} = \frac{\pi - 3}{4}\,.$$

For $2N = 0$ we get on $-1 < x < 1$

$$\frac{4}{\pi}\left[\frac{1}{1-x^2} - \frac{1}{4}\right] \leqslant \sec(\pi x/2) \leqslant \frac{4}{\pi}\left[\frac{1}{1-x^2} - 1 + \frac{\pi}{4}\right]$$

and for $2N + 1 = 1$,

$$\frac{4}{\pi}\left[\frac{1}{1-x^2} - \frac{1}{4} + \frac{\pi - 3}{4}(1-x^2)\right] \leqslant \sec(\pi x/2) \leqslant \frac{4}{\pi}\left[\frac{1}{1-x^2} - \frac{1}{4} + \frac{\pi^2 - 6}{96}(1-x^2)\right]\,.$$

Proof. We start with the partial fraction expansion $\sec z = \pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(n+1/2)^2\pi^2 - z^2}$. Then for $-1 < x < 1$,

$$\begin{aligned}
\sec(\pi x/2) &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2 - x^2} \\
&= \frac{4}{\pi}\left[\frac{1}{1-x^2} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n+1)}{4n(n+1) + (1-x^2)}\right] \\
&= \frac{4}{\pi}\left[\frac{1}{1-x^2} - \sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n+1)}{n^{k+1}(n+1)^{k+1}}\right)(-1)^k\frac{(1-x^2)^k}{4^{k+1}}\right] \\
&= \frac{4}{\pi}\left[\frac{1}{1-x^2} - \sum_{k=0}^{\infty}(-1)^k\frac{S_{k+1}}{4^{k+1}}(1-x^2)^k\right],
\end{aligned}$$

where $S_p = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n+1)}{n^p(n+1)^p} > 0$ for $p = 1, 2, \ldots$.

The coefficients S_p may be presented explicitly by finite sums of the Dirichlet η function. Let us multiply (2.10) by $2n + 1 \equiv 2(n+1) - 1$ and separate $j = 1$ from $2 \leqslant j \leqslant p$:

$$\begin{aligned}
\frac{(-1)^p(2n+1)}{n^p(n+1)^p} &= -\binom{2p-2}{p-1}\left[\frac{1}{n} + \frac{1}{n+1}\right] \\
&\quad + \sum_{j=2}^{p}\binom{2p-j-1}{p-1}\left[2\frac{(-1)^j}{n^{j-1}} + \frac{(-1)^j}{n^j} + 2\frac{1}{(n+1)^{j-1}} - \frac{1}{(n+1)^j}\right].
\end{aligned}$$

We multiply the last equation by $(-1)^{n-1}$ and sum it for $n = 1, 2, \ldots$. With the Dirichlet eta function

$$\eta(p) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^p} = -\eta(p) + 1, \qquad p \geqslant 1, \tag{3.7}$$

we get

$$
\begin{aligned}
&(-1)^p S_p \\
&= -\binom{2p-2}{p-1}\left[\eta(1)+\left(-\eta(1)+1\right)\right] \\
&\quad + \sum_{j=2}^{p}\binom{2p-j-1}{p-1}\Big[2(-1)^j\eta(j-1)+(-1)^j\eta(j)+2\left(-\eta(j-1)+1\right)-\left(-\eta(j)+1\right)\Big] \\
&= -\binom{2p-2}{p-1}+\sum_{j=2}^{p}\binom{2p-j-1}{p-1}+\sum_{j=2}^{p}\binom{2p-j-1}{p-1}\Big[2\left((-1)^j-1\right)\eta(j-1)+\left((-1)^j+1\right)\eta(j)\Big] \\
&= -\binom{2p-2}{p-1}+\binom{2p-2}{p}+\sum_{2\leqslant 2m\leqslant p}2\Big[\binom{2p-2m-1}{p-1}-2\binom{2p-2m-2}{p-1}\Big]\eta(2m) \\
&= \binom{2p-2}{p-2}-\binom{2p-2}{p-1}+2\sum_{2\leqslant 2m\leqslant p}\Big[\binom{2p-2m-2}{p-2}-\binom{2p-2m-2}{p-1}\Big]\eta(2m).
\end{aligned}
$$

Here we used the facts that $\sum_{j=2}^{p}\binom{2p-j-1}{p-1}=\binom{2p-2}{p}=\binom{2p-2}{p-2}$ and that

$$
\binom{2p-2m-1}{p-1}-2\binom{2p-2m-2}{p-1}=\binom{2p-2m-2}{p-2}-\binom{2p-2m-2}{p-1}.
$$

Thus, (3.4) is proved.

The inequalities (3.5) and (3.6) are verified as in Theorem 2.1. □

The coefficients T_p and S_p are easily related. In (3.2) we substitute $\sec(\pi x/2)$ and $\dfrac{\tan(\pi x/2)}{\pi x/2}$, respectively, by the series (2.1) and (3.3) and compare powers of $1-x^2$, and the result is

$$
S_{n+1}=T_{n+1}+\sum_{k=0}^{n-1}T_{k+1}S_{n-k}\,.
$$

4. Inequalities for other trigonometric functions

We outline some other results which may be verified by similar arguments. For the cotangent function, let us recall the inequality [5, Eq. (17)]

$$
\frac{2x^2}{1-x^2}\leqslant 1-\pi x\cot(\pi x)\leqslant\frac{\pi^2}{3}\frac{x^2}{1-x^2}\,. \tag{4.1}
$$

It is possible generalize the left hand side of (4.1) into the infinite expansion

$$
1-\pi x\cot(\pi x)=2x^2\left[\frac{1}{1-x^2}+\sum_{k=0}^{\infty}(-1)^k\frac{C_{k+1}}{4^{k+1}}(1-x^2)^k\right], \tag{4.2}
$$

$$C_p = \sum_{n=2}^{\infty} \frac{2^{2p}}{(n^2-1)^p} = \sum_{n=1}^{\infty} \frac{2^{2p}}{n^p(n+2)^p}$$
$$= (-1)^p \left[\sum_{2\leqslant 2m\leqslant p} 2^{2m+1} \binom{2p-2m-1}{p-1} \zeta(2m) - \binom{2p-1}{p-1} - 2^{2p-1} \right], \quad p \geqslant 1. \tag{4.3}$$

All C_p-s are positive. The infinite series in (4.2) converges for $|x| < 2$. For example, $C_1 = 3$, $C_2 = -11 + 8\zeta(2) = \frac{4}{3}\pi^2 - 11$, $C_3 = 42 - 24\zeta(2) = 42 - 4\pi^2$.

Inequalities in the style of Theorems 2.1 and 3.1 may be proved for $1 - \pi x \cot(\pi x)$. (4.2) is obtained from the expansion $\pi x \cot(\pi x) = 1 - 2x^2 \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$. For the calculation of C_p in (4.3), it is useful to take in (2.9), $m = k = p$, $x = \frac{-2}{n}, y = \frac{2}{n+2}$ and sum the result for $n = 1, 2, \ldots$. For the summation of the series, one has to use the binomial identity $\sum_{j=1}^{p} \binom{2p-j-1}{p-1} 2^j = 2^{2p-1}$, [7, p. 167, Eq. 5.20].

For the cosecant function we remind, in analogy with (1.7), (3.1) and (4.1), the double inequality [5, Eq. (19)]

$$\frac{(\pi^2/6)x^2}{1-x^2} \leqslant \pi x \operatorname{cosec}(\pi x) - 1 \leqslant \frac{2x^2}{1-x^2}, \qquad -1 < x < 1. \tag{4.4}$$

The right hand side of (4.4) may be generalized into the expansion

$$\pi x \operatorname{cosec}(\pi x) - 1 = 2x^2 \left[\frac{1}{1-x^2} - \sum_{k=0}^{\infty} (-1)^k \frac{D_{k+1}}{4^{k+1}} (1-x^2)^k \right], \tag{4.5}$$

where

$$D_p = \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2p}}{(n^2-1)^p} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2p}}{n^p(n+2)^p}$$
$$= (-1)^p \left[\sum_{2\leqslant 2m\leqslant p} 2^{2m+1} \binom{2p-2m-1}{p-1} \eta(2m) - 2^{2p-1} + \binom{2p-1}{p-1} \right]. \tag{4.6}$$

Here $D_1 = 1$, $D_2 = -5 + 8\eta(2) = \frac{2}{3}\pi^2 - 5$, $D_3 = 22 - 24\eta(2) = 22 - 2\pi^2$. Expansion (4.5) follows from the partial fraction expansion $\operatorname{cosec} z = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - z^2}$. D_p is calculated similarly to C_p.

If we substitute the series (4.2) and (4.5) into the identity

$$x \frac{d}{dx} \left((1-x^2) \pi x \operatorname{cosec}(\pi x) \right) = \pi x \operatorname{cosec}(\pi x) \left[-2x^2 + (1-x^2)(1 - \pi x \cot(\pi x)) \right]$$

and compare powers of $1 - x^2$, it follows that C_n and D_n are related by

$$nD_n + 4nD_{n-1} = C_n + 2C_{n-1} + \sum_{k=1}^{n-1} D_k C_{n-k} + \sum_{k=1}^{n-2} D_k C_{n-k-1}.$$

5. Inequalities for Bessel functions

Finally we show that our methods are applicable also to Bessel functions and they provide a short proof of (1.6), based on other principles than those of Baricz and Wu in [2].

An expansion of the form

$$x^{-p}J_p(x) = \sum_{k=0}^{\infty} c_k (r^2 - x^2)^k \tag{5.1}$$

holds for all x in the complex plane. To see this, put $z = r^2 - x^2$, i.e., $x = \sqrt{r^2 - z}$. Since $x^{-p}J_p(x)$ is an even analytic function in the whole complex plane, it follows that $x^{-p}J_p(x)\Big|_{x=\sqrt{r^2-z}}$ is analytic for all z and may be expanded as $\sum_{k=0}^{\infty} c_k z^k$. Consequently (5.1) holds for all x. To calculate the c_k-s, let us apply the identity [1, Eq. 9.1.30]

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left(x^{-p}J_p(x)\right) = (-1)^m x^{-(p+m)} J_{p+m}(x), \qquad m = 1, 2, \dots$$

to expansion (5.1). Since $\left(\frac{1}{x}\frac{d}{dx}\right)^m (r^2 - x^2)^k = (-2)^m k(k-1)\cdots(k-m+1)(r^2-x^2)^{k-m}$, we get

$$(-1)^m x^{-(p+m)} J_{p+m}(x) = (-2)^m \sum_{k=m}^{\infty} c_k k(k-1)\cdots(k-m+1)(r^2-x^2)^{k-m}.$$

For $x = r$ we obtain that $c_m = \dfrac{r^{-(p+m)}J_{p+m}(r)}{2^m m!}$. Consequently,

$$x^{-p}J_p(x) = \sum_{k=0}^{\infty} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2 - x^2)^k, \qquad |x| < \infty. \tag{5.2}$$

To prove the inequalities (1.6), we have to estimate the remainder of the expansion (5.2), namely

$$\begin{aligned} &x^{-p}J_p(x) - \sum_{k=0}^{N} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2 - x^2)^k \\ &= (r^2 - x^2)^{N+1} \sum_{k=N+1}^{\infty} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2 - x^2)^{k-N-1}. \end{aligned}$$

Let $j_{n,1}$ denote the first positive zero of $J_n(x)$. It is well known that $j_{n,1} < j_{n+1,1}$ and that $J_n(x) > 0$ in $(0, j_{n,1})$. See Figure 1.

If r satisfies $0 < r < j_{p+1,1}$, all the coefficients in the tail series

$$R_N(x) = \sum_{k=N+1}^{\infty} \frac{r^{-(p+k)}J_{p+k}(r)}{2^k k!}(r^2 - x^2)^{k-N-1}$$

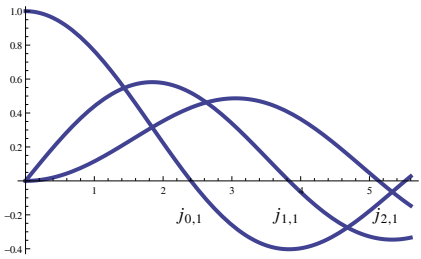

Figure 1: *Figure 1. The first zeros of $J_0(x)$, $J_1(x)$ and $J_2(x)$*

are positive for all N, hence $R_N(0) \geqslant R_N(x) \geqslant R_N(r)$. So, in the left hand side of (1.6), $\alpha_p = R_N(r) = \dfrac{r^{-(p+N+1)} J_{p+N+1}(r)}{2^{N+1}(N+1)!}$; and on the right hand side of (1.6),

$$\beta_p = R_N(0) = r^{-2(N+1)} \sum_{k=N+1}^{\infty} \frac{r^{k-p} J_{p+k}(r)}{2^k k!}.$$

β_p may be written as a finite sum. As $\lim_{x\to 0} x^{-p} J_p(x) = 1/2^p \Gamma(p+1)$, we get from (5.2) for $x = 0$ that

$$\frac{1}{2^p \Gamma(p+1)} = \sum_{k=0}^{\infty} \frac{r^{k-p} J_{p+k}(r)}{2^k k!}.$$

Thus,

$$\beta_p = \frac{1/2^p \Gamma(p+1) - \sum_{k=0}^{N} r^{k-p} J_{p+k}(r)/2^k k!}{r^{2(N+1)}}.$$

Inspired by a remark made by the referee, we suggest the following problem: Find similar inequalities for other special functions, based on the above methods.

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D. Aharonov
Dept. of Mathematics
Technion – Israel Institute of technology
Haifa 32000, Israel
e-mail: dova@tx.technion.ac.il

U. Elias
Dept. of Mathematics
Technion – Israel Institute of technology
Haifa 32000, Israel
e-mail: elias@tx.technion.ac.il