

ON THE EXTENSION OF THE ERDŐS–MORDELL TYPE INEQUALITIES

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(Communicated by J. Pečarić)

Abstract. We discuss the extension of inequality $R_A \geq \frac{c}{a}r_b + \frac{b}{a}r_c$ to the plane of triangle $\triangle ABC$. Based on the obtained extension, in regard to all three vertices of the triangle, we get the extension of Erdős-Mordell inequality, and some inequalities of Erdős-Mordell type.

1. Introduction

Let triangle $\triangle ABC$ be given in Euclidean plane. Denote by R_A, R_B and R_C the distances from the arbitrary point M in the interior of $\triangle ABC$ to the vertices A, B and C respectively, and denote by r_a, r_b and r_c the distances from the point M to the sides BC, CA and AB respectively (Figure 1).

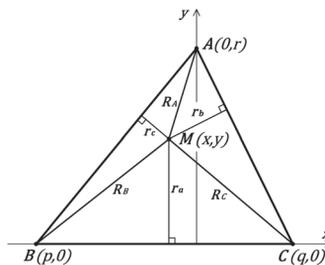


Figure 1: Erdős-Mordell inequality

Then Erdős-Mordell inequality is true:

$$R_A + R_B + R_C \geq 2(r_a + r_b + r_c) \quad (1)$$

where equality holds if and only if triangle ABC is equilateral and M is its center. This inequality was conjectured by P. Erdős as Amer. Math. Monthly Problem 3740 in 1935. [9], after his experimental conjecture in 1932. [13]. It was proved by L.J.

Mathematics subject classification (2010): 51M16, 51M04, 14H50.

Keywords and phrases: Erdős-Mordell inequality, inequality of Child, Erdős-Mordell curve.

Research is partially supported by the Ministry of Science and Education of the Republic of Serbia, Grant No. III 44006 and ON 174032.

Mordell in 1935. (in Hungarian, according to [13]), and as the solution of the Problem 3740 in 1937. [22].

Considering the Erdős-Mordell inequality (1) the goal of this research is to determine areas in the plane of the triangle, where the following three inequalities are valid:

$$R_A \geq \frac{c}{a}r_b + \frac{b}{a}r_c \quad (2)$$

$$R_B \geq \frac{c}{b}r_a + \frac{a}{b}r_c \quad (3)$$

$$R_C \geq \frac{b}{c}r_a + \frac{a}{c}r_b \quad (4)$$

where $a = |BC|$, $b = |CA|$, $c = |AB|$.

In this paper we determine a set of points E for which

$$R_A + R_B + R_C \geq \left(\frac{c}{b} + \frac{b}{c}\right)r_a + \left(\frac{c}{a} + \frac{a}{c}\right)r_b + \left(\frac{a}{b} + \frac{b}{a}\right)r_c \quad (5)$$

is valid. It is known that the triangular area of $\triangle ABC$ is contained in the set E [3], [4], [11], [13], [14], [26]. Here we show that the set E is greater than the triangle $\triangle ABC$, and we give a geometric interpretation of the set E .

The proofs of Erdős-Mordell inequality are often based on different proofs of inequality (2), as given in [4], [6], [7], [11], [12], [23], [26]. N. Derigades in [8] proved the inequality (5) valid in the whole plane of the triangle, where r_a , r_b and r_c , are signed distances. A similar result was given by B. Malešević [20], [21].

Note that V. Pambuccian [24] recently proved that the Erdős-Mordell inequality is equivalent to non-positive curvature. Overview of recent results on Erdős-Mordell inequalities and related inequalities is given in [1]–[3], [5], [8], [10], [13]–[21], [24], [25], [27]–[30].

2. The main results

In this section we analyze only the inequality (2). Let $\triangle ABC$ be a triangle with vertices $A(0, r)$, $B(p, 0)$, $C(q, 0)$, $p \neq q$, $r \neq 0$. Without diminishing generality, let $p < q$. We denote by $M(x, y)$ an arbitrary point in the plane of the triangle $\triangle ABC$. The distance from the point M to the point A , and the distance from the point M to the straight lines b and c are given by functions:

$$R_A = \sqrt{x^2 + (y - r)^2} \quad (6)$$

$$r_b = \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} \quad (7)$$

$$r_c = \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}} \quad (8)$$

respectively. Consider the inequality (2) related to the vertex A . The analytical notation of this inequality is:

$$\sqrt{x^2 + (y - r)^2} \geq \frac{\sqrt{r^2 + p^2}}{|q - p|} \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} + \frac{\sqrt{r^2 + q^2}}{|q - p|} \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}}, \tag{9}$$

i.e.

$$|q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{x^2 + (y - r)^2} \geq (r^2 + p^2) |-qy - rx + qr| + (r^2 + q^2) |py + rx - pr|. \tag{10}$$

Let $y = kx + r, k \in \mathbb{R}$, then the inequality (10) reads as follows:

$$|x| |q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1 + k^2} \geq |x| \left((r^2 + p^2) |-qk - r| + (r^2 + q^2) |pk + r| \right) \tag{11}$$

For $x = 0$, the previous inequality is reduced to an equality which solution is the point $A(0, r)$. For $x \neq 0$ we obtain inequality by a single variable k :

$$|q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1 + k^2} \geq (r^2 + p^2) |-qk - r| + (r^2 + q^2) |pk + r|. \tag{12}$$

Solution of the inequality (12) reduces to four cases per parameter k :

$$(\alpha_1) : \begin{cases} pk + r \geq 0 \\ -qk - r \geq 0, \end{cases} \tag{13}$$

$$(\alpha_2) : \begin{cases} pk + r < 0 \\ -qk - r \geq 0, \end{cases} \tag{14}$$

$$(\alpha_3) : \begin{cases} pk + r \geq 0 \\ -qk - r < 0, \end{cases} \tag{15}$$

$$(\alpha_4) : \begin{cases} pk + r < 0 \\ -qk - r < 0. \end{cases} \tag{16}$$

Note that the value k corresponds to the points $(x, y) \in \mathbb{R}^2$ located on the straight line $y = kx + r$. With its values, the mentioned parameter of the line $y = kx + r$ decomposes \mathbb{R}^2 on four corner areas. Inquiring the existence of parameter k (i.e. the pencil of lines $y = kx + r$ through the vertex A) depending on the signs of parameters p, q and r , we provide the following table of existing corner areas $(\alpha_1) - (\alpha_4)$:

	p	q	r	(α_1)	(α_2)	(α_3)	(α_4)
1.	> 0	> 0	> 0	+	+	+	-
2.	< 0	> 0	> 0	+	-	+	+
3.	< 0	< 0	> 0	-	+	+	+
4.	> 0	> 0	< 0	-	+	+	+
5.	< 0	> 0	< 0	+	+	-	+
6.	< 0	< 0	< 0	+	+	+	-
7.	$= 0$	> 0	> 0	+	-	+	-
8.	$= 0$	> 0	< 0	-	+	-	+
9.	< 0	$= 0$	> 0	-	-	+	+
10.	< 0	$= 0$	< 0	+	+	-	-

Table 1: The existence of the corner area depending on the parameters p , q and r

The corner areas (α_1) and (α_4) are always in the interior of $\sphericalangle BAC$ and its cross angle, while the areas (α_2) and (α_3) are in the interior of its supplementary angle (Figure 2).

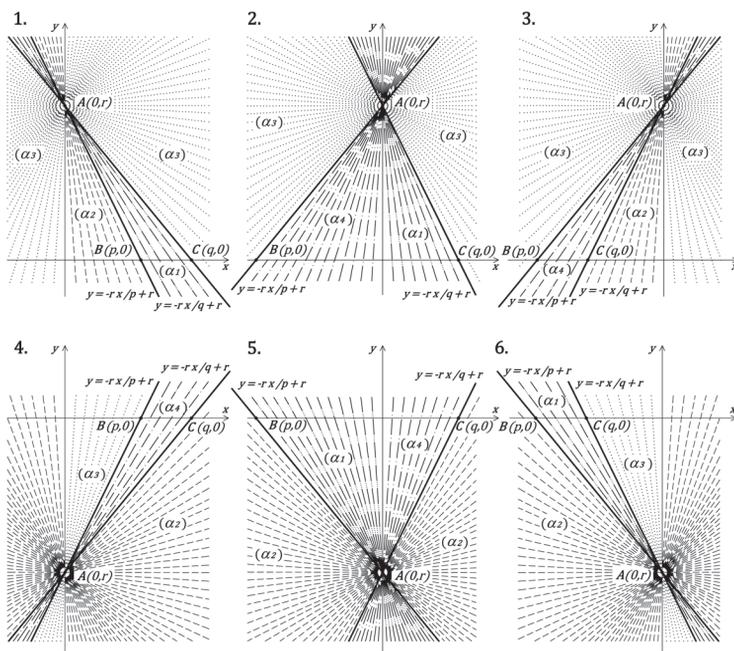


Figure 2: Existence of the corner area for the vertex A (Cases 1. to 6. in the Table 1)

Let us consider the equation:

$$(q - p) \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1 + k^2} = (r^2 + p^2) |-qk - r| + (r^2 + q^2) |pk + r|. \quad (17)$$

D) Let k fulfill (α_1) or (α_4) . Then the previous equation can be rewritten in a way that follows, with positive sign (+) in the case of area (α_1) and negative sign (-) in the case of area (α_4)

$$(q - p)\sqrt{r^2 + p^2}\sqrt{r^2 + q^2}\sqrt{1 + k^2} = \pm((-qk - r)(r^2 + p^2) + (pk + r)(r^2 + q^2)) \tag{18}$$

i.e.

$$(q - p)\sqrt{r^2 + p^2}\sqrt{r^2 + q^2}\sqrt{1 + k^2} = \pm(q - p)(r(q + p) + k(pq - r^2)) \tag{19}$$

abbreviated written as

$$\lambda\sqrt{1 + k^2} = \pm\beta k \pm \gamma = \begin{cases} \beta k + \gamma, & k \in (\alpha_1) \\ -\beta k - \gamma, & k \in (\alpha_4) \end{cases} \tag{20}$$

where at:

$$\lambda = (q - p)\sqrt{r^2 + p^2}\sqrt{r^2 + q^2} \quad \text{and} \quad \lambda > 0 \tag{21}$$

$$\beta = (pq - r^2)(q - p) \tag{22}$$

$$\gamma = r(q^2 - p^2). \tag{23}$$

As $p < q$, the equation (19) can be divided by $q - p \neq 0$ and then squared:

$$(r^2 + p^2)(r^2 + q^2)(1 + k^2) = (r(q + p) + k(pq - r^2))^2 \tag{24}$$

which transforms into

$$(r(p + q)k - (pq - r^2))^2 = 0. \tag{25}$$

Based on the above equation, we conclude that there exists the unique solution:

$$k_1 = \frac{pq - r^2}{r(p + q)} \tag{26}$$

only if, for $k = k_1$:

$$\pm\beta k \pm \gamma \geq 0 \tag{27}$$

is valid.

Hence, the straight line $y = k_1x + r$ is in the interior of $\triangle BAC$ and its cross angle, or it doesn't exist. The cases where values k_1 from the formula (26) does not meet the condition (27) are presented in the *Table 1* with:

- in the case 1: $k_1 > -r/q \iff p(q^2 + r^2) > 0$;
- in the case 3: $k_1 > -r/p \iff (-q)(p^2 + r^2) > 0$;
- in the case 4: $k_1 < -r/q \iff p(q^2 + r^2) > 0$;
- in the case 6: $k_1 < -r/p \iff (-q)(p^2 + r^2) > 0$.

LEMMA 1. For $k \in (\alpha_1) \cup (\alpha_4)$ inequality (12) is valid, where equality holds for $k = k_1$ if (27) is fulfilled.

Proof. (12) $\iff (r(p+q)k - (pq - r^2))^2 \geq 0$. \square

COROLLARY 1. Inequality (12) is valid for lines **b** and **c**.

II) Let k fulfill (α_2) or (α_3) . Then equation (17) can be rewritten in a way that follows, with negative sign (-) in the case of area (α_2) and positive sign (+) in the case of area (α_3)

$$(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2} = \pm((qk+r)(r^2+p^2) + (pk+r)(r^2+q^2)) \quad (28)$$

or abbreviated written as

$$\lambda\sqrt{1+k^2} = \pm\delta k \pm \varepsilon = \begin{cases} \delta k + \varepsilon, & k \in (\alpha_3) \\ -\delta k - \varepsilon, & k \in (\alpha_2) \end{cases} \quad (29)$$

with parameters:

$$\lambda = (q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2} \quad \text{and} \quad \lambda > 0$$

$$\delta = (r^2 + pq)(q + p) \quad (30)$$

$$\varepsilon = r(2r^2 + q^2 + p^2). \quad (31)$$

The equation (29) is considered under the following condition:

$$\pm\delta k \pm \varepsilon \geq 0. \quad (32)$$

By squaring the equation (29) we obtain

$$P(k) = \lambda^2(1+k^2) - (\pm\delta k \pm \varepsilon)^2 = (\lambda^2 - \delta^2)k^2 - 2\delta\varepsilon k + (\lambda^2 - \varepsilon^2) = 0. \quad (33)$$

For the square trinomial

$$P(k) = \widehat{\mathbf{A}}k^2 + \widehat{\mathbf{B}}k + \widehat{\mathbf{C}} \quad (34)$$

coefficients $\widehat{\mathbf{A}}$, $\widehat{\mathbf{B}}$, $\widehat{\mathbf{C}}$ are determined by:

$$\widehat{\mathbf{A}} = \lambda^2 - \delta^2 = (q-p)^2(r^2+p^2)(r^2+q^2) - (r^2+pq)^2(q+p)^2 \quad (35)$$

$$\widehat{\mathbf{B}} = -2\delta\varepsilon = -2r(r^2+pq)(q+p)(2r^2+q^2+p^2) \quad (36)$$

$$\widehat{\mathbf{C}} = \lambda^2 - \varepsilon^2 = (r^2+pq)((pq-r^2)(q-p)^2 - 2r^2(2r^2+q^2+p^2)). \quad (37)$$

Let us consider the equation:

$$\widehat{\mathbf{A}} = -4pqr^4 + (p^4+q^4-4pq^3-4p^3q-2p^2q^2)r^2 - 4p^3q^3 = 0. \quad (38)$$

It has real solutions for r in the following form:

$$\begin{cases} r_{1,2} = \frac{1}{4\sqrt{pq}} \left((q-p)^2 \pm \sqrt{(q-p)^4 - 16p^2q^2} \right) > 0 \\ r_{3,4} = -\frac{1}{4\sqrt{pq}} \left((q-p)^2 \pm \sqrt{(q-p)^4 - 16p^2q^2} \right) < 0 \end{cases} \quad (39)$$

iff

$$(p \geq 0 \wedge q \geq (3 + 2\sqrt{2})p) \vee (p < 0 \wedge q \leq (3 - 2\sqrt{2})p). \quad (40)$$

REMARK 1. When $p < 0$ and $q > 0$ then $\widehat{A} = 4|p|qr^4 + (q^2 - p^2)^2r^2 + 4|p|q(p^2 + q^2)r^2 + 4|p|^3q^3 > 0$ is valid. Note that the equation $\widehat{A} = 0$ is not considered for $p = 0$ or $q = 0$ (because we obtain the contradictions: $p = 0, q \neq 0: \widehat{A} = rq^4 = 0 \implies r = 0$; i.e. $p \neq 0, q = 0: \widehat{A} = rp^4 = 0 \implies r = 0$).

We distinguish the cases:

a) Let $r = r_j$ for some $j = 1, 2, 3, 4$, then $\widehat{A} = 0$. In this case, $\widehat{B} \neq 0$, because $r^2 + pq \neq 0$ and $q + p \neq 0$ (in the case of equilateral triangle, there will be valid $q + p = 0$ and then $r = \pm pi, i = \sqrt{-1}$). Therefore, by solving the linear equation $\widehat{B}k + \widehat{C} = 0$ we find that:

$$k_2 = -\frac{\widehat{C}}{\widehat{B}} = \frac{\lambda^2 - \varepsilon^2}{2\delta\varepsilon} = \frac{(q-p)^2(r^2+p^2)(r^2+q^2) - r^2(2r^2+q^2+p^2)^2}{2r(q+p)(2r^2+q^2+p^2)}. \quad (41)$$

For $p < 0$ and $q > 0$ the case **a)** is not considered (because $\widehat{A} > 0$). Let us examine when the value k_2 meet the condition (32). It is valid that:

$$\pm\delta k_2 \pm \varepsilon \geq 0 \iff \pm(\delta k_2 + \varepsilon) = \pm\left(\delta \frac{\lambda^2 - \varepsilon^2}{2\delta\varepsilon} + \varepsilon\right) = \pm\left(\frac{\lambda^2 + \varepsilon^2}{2\varepsilon}\right) \geq 0.$$

Based on $\varepsilon = r(2r^2 + q^2 + p^2)$ we conclude:

if $r > 0$ then $\delta k_2 + \varepsilon \geq 0$ is fulfilled, whereby k_2 fulfills condition (32) and $k_2 \in (\alpha_3)$;

if $r < 0$ then $-\delta k_2 - \varepsilon \geq 0$ is fulfilled, whereby k_2 fulfills condition (32) and $k_2 \in (\alpha_2)$.

In this case, the line $y = k_2x + r$ is in the exterior of $\sphericalangle BAC$ and its cross angle.

b) Let $r \neq r_j$ for each $j = 1, 2, 3, 4$, then $\widehat{A} \neq 0$ and in this case, by solving the quadratic equation (33), we find the values:

$$\begin{aligned} k_{2,3} &= \frac{-\delta\varepsilon \pm \sqrt{\lambda^2(\delta^2 + \varepsilon^2 - \lambda^2)}}{\delta^2 - \lambda^2} \\ &= \frac{r(p+q)(r^2+pq)(q^2+p^2+2r^2) \pm 2(r^2+p^2)(r^2+q^2)(q-p)\sqrt{r^2+pq}}{(q-p)^2(r^2+p^2)(r^2+q^2) - (r^2+pq)^2(q+p)^2}. \end{aligned} \quad (42)$$

If $r^2 + pq \geq 0$ then exists $k_{2,3} \in \mathbb{R}$. Incidence of $k_{2,3} \in \mathbb{R}$ to the area (α_3) , as to the area (α_2) is determined by the inequality (32). The expression $\delta k_{2,3} + \varepsilon$ exists for $\delta \neq \pm \lambda$, whereby the expression $\delta k_{2,3} + \varepsilon$ is either positive or negative (because $\delta k_{2,3} + \varepsilon = 0 \implies \delta = \pm \lambda$).

Based on the Corollary 1, the straight lines $y = k_s x + r$, ($s = 2, 3$) are in the exterior of $\triangle BAC$ and its cross angle (Figure 3).

Consider the limiting case for $k_{2,3}$ when $r \rightarrow r_j$. Note that $\widehat{A} = \lambda^2 - \delta^2 \xrightarrow{r \rightarrow r_j} 0$ is valid, whereat from

$$k_{2,3} = \frac{-\varepsilon}{(\delta - \lambda)(\delta + \lambda)} \cdot \left(\delta \mp |\lambda| \sqrt{1 + \frac{\delta^2 - \lambda^2}{\varepsilon^2}} \right)$$

follows

$$\lim_{r \rightarrow r_j} k_2 = \frac{-\varepsilon}{(\delta + \lambda)} \quad \wedge \quad \lim_{r \rightarrow r_j} k_3 = \infty.$$

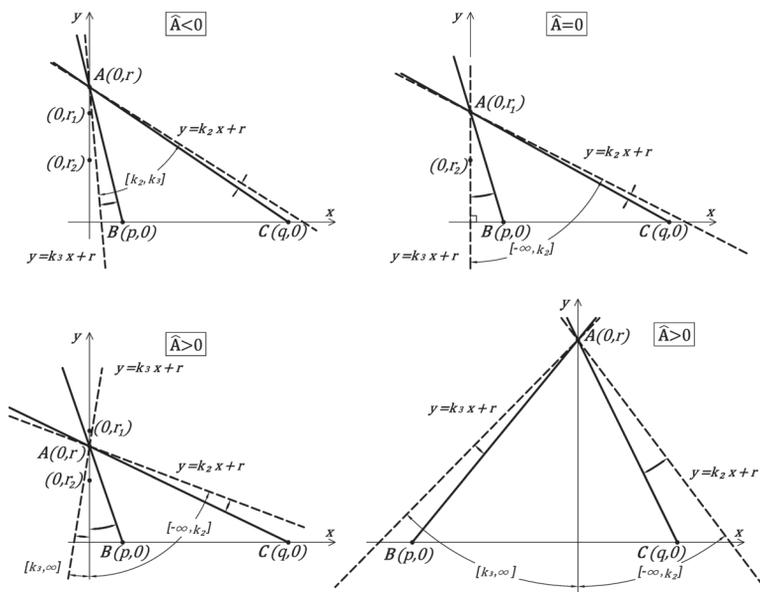


Figure 3: The existence of the lines $y = k_s x + r$, ($s = 2, 3$) depending on the parameter \widehat{A}

Related to the $\triangle BAC$ we distinguish the cases:

1. $\triangle BAC < \pi/2 \iff r^2 + pq > 0$ and if $\widehat{A} \neq 0$ then there are two real and different values of k_2 and k_3 . In this case, the following lemma is valid:

LEMMA 2. For $\triangle BAC < \pi/2$, $k \in (\alpha_2) \cup (\alpha_3)$ the inequality (12) is valid, just in the cases:

1. $\widehat{A} > 0 \wedge k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus ((\alpha_1) \cup (\alpha_4))$;
2. $\widehat{A} = 0 \wedge k \in [-\infty, k_2] \setminus ((\alpha_1) \cup (\alpha_4))$;
3. $\widehat{A} < 0 \wedge k \in [k_2, k_3] \setminus ((\alpha_1) \cup (\alpha_4))$;

where the equality holds for $k = k_2$ or $k = k_3$.

2. If $\sphericalangle BAC = \pi/2 \iff r^2 + pq = 0$ then $\widehat{A} = -qp(q-p)^4$, $\widehat{B} = 0$ and $\widehat{C} = 0$, according to the equation (42) that $k_{2,3} = 0$. Hence is valid:

LEMMA 3. For $\sphericalangle BAC = \pi/2$ and $k \in (\alpha_2) \cup (\alpha_3)$ the inequality (12) is valid. The equality is valid only for $k = 0$.

Proof. (12) $\iff \widehat{A}k^2 + \widehat{B}k + \widehat{C} \geq 0 \iff -qp(q-p)^4 k^2 \geq 0$. \square

3. $\sphericalangle BAC > \pi/2 \iff r^2 + pq < 0$. In this case, for: $r^2 < -pq$ and for the coefficient \widehat{A} :

$$\begin{aligned} \widehat{A} &> 4r^6 + (p^4 + q^4)r^2 + 4(p^2 + q^2)r^4 - 2r^6 + 4p^2q^2r^2 \\ &= 2r^6 + 4(p^2 + q^2)r^4 + (p^4 + q^4 + 4p^2q^2)r^2 > 0 \end{aligned}$$

is valid. Since $k_{2,3} \in \mathbb{C}$ and $\widehat{A} > 0$ the inequality (12) is valid, which proves the claim:

LEMMA 4. For $\sphericalangle BAC > \pi/2$ and $k \in (\alpha_2) \cup (\alpha_3)$ the inequality (12) is valid in the strict form.

Based on the previous considerations in **I**) and **II**), follows:

STATEMENT 1. The inequality (12) holds in following cases:

$$k \in (\alpha_1) \cup (\alpha_4)$$

or

$$k \in (\alpha_2) \cup (\alpha_3) \text{ for } \sphericalangle BAC \geq \pi/2$$

i.e.

$$k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus ((\alpha_1) \cup (\alpha_4)) \wedge \widehat{A} > 0$$

$$k \in [-\infty, k_2] \setminus ((\alpha_1) \cup (\alpha_4)) \wedge \widehat{A} = 0$$

$$k \in [k_2, k_3] \setminus ((\alpha_1) \cup (\alpha_4)) \wedge \widehat{A} < 0,$$

for $\sphericalangle BAC < \pi/2$.

3. Conclusion

For the vertex A , let us define

$$E_A = \left\{ (x, y) \mid R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c \right\},$$

and for the vertices B and C , let us define

$$E_B = \left\{ (x, y) \mid R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \right\},$$

$$E_C = \left\{ (x, y) \mid R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b \right\},$$

respectively. Based on the analysis of the inequalities (2), (3) and (4), the inequality (5) is valid in the intersection of the areas:

$$E = E_A \cap E_B \cap E_C. \tag{43}$$

Therefore follows

STATEMENT 2. *Erdős-Mordell inequality is valid in the area E .*

Let us define the set M by the intersection of the corner areas formed from E_A , E_B and E_C , containing the initial triangle. Then the set of points M is quadrilateral or hexagonal shape, and is contained the area E (Figure 4).

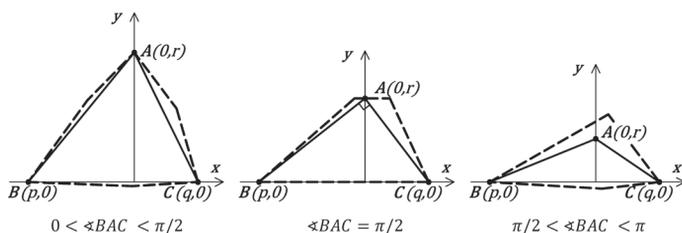


Figure 4: Extension of the triangle ABC to the area $M \subset E$

Let us define Erdős-Mordell curve in the plane of triangle, by the following equation:

$$R_A + R_B + R_C = 2(r_a + r_b + r_c), \tag{44}$$

where

$$R_A = \sqrt{x^2 + (y-r)^2}, \quad R_B = \sqrt{(x-p)^2 + y^2}, \quad R_C = \sqrt{(x-q)^2 + y^2},$$

$$r_a = \frac{|y(q-p)|}{\sqrt{(q-p)^2}} = |y|, \quad r_b = \frac{|-q(y-r) - rx|}{\sqrt{r^2 + q^2}}, \quad r_c = \frac{|-p(y-r) - rx|}{\sqrt{r^2 + p^2}}.$$

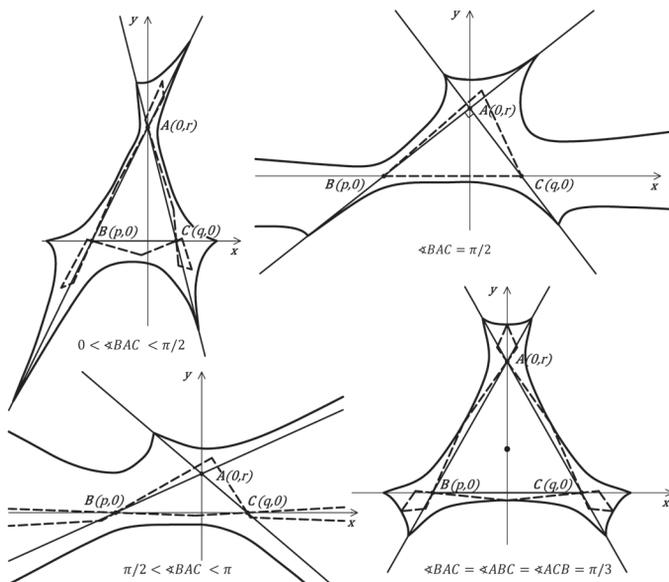


Figure 5: Erdős-Mordell curve and the area E

The curve (44) is a union of parts of algebraic curves of order eight (Figure 5).

Let us denote by E' the part of the plane \mathbb{R}^2 bounded by the Erdős-Mordell's curve and consisting the triangle $\triangle ABC$. Thus, according to the fact that inequality (5) is valid in the area of the triangle $\triangle ABC$, and based on continuity, it follows that inequality (5) is valid in the area E' . Remark that the area E' allows us to precise when, except for the inequality (5), some of the inequalities (2), (3) and/or (4) are true. For example, in the area $(E' \setminus E_A) \cap E_B \cap E_C$ the inequalities (5), (4), (3) are true and (2) is not true. At end of this section let us emphasize that the following statement is true.

STATEMENT 3. All geometric inequalities based on the inequalities (2), (3) and (4) can be extended from the triangle interior to the area E .

EXAMPLE 1. In the area E , the inequality of Child [7] is valid:

$$R_A \cdot R_B \cdot R_C \geq 8 \cdot r_a \cdot r_b \cdot r_c \tag{45}$$

because, based on inequality between arithmetic and geometric mean, follows:

$$a \cdot R_A \geq b \cdot r_c + c \cdot r_b \geq 2\sqrt{b \cdot c \cdot r_b \cdot r_c} \tag{46}$$

$$b \cdot R_B \geq c \cdot r_a + a \cdot r_c \geq 2\sqrt{c \cdot a \cdot r_c \cdot r_a} \tag{47}$$

$$c \cdot R_C \geq a \cdot r_b + b \cdot r_a \geq 2\sqrt{a \cdot b \cdot r_a \cdot r_b}. \tag{48}$$

Hence, by multiplying the left and right sides of inequalities (46) - (48), we get the inequality (45) in the area E . \square

At the end of this paper, let us set up an open problem (proposed by anonymous reviewer): prove or disprove that there exist a positive number ε such that the area of E' is bigger than $1+\varepsilon$ times the area of the triangle for every triangle. Thus, we set a conjecture: for the finite area of E' the value ε is determined in the case of equilateral triangle.

Acknowledgement. The authors would like to thank anonymous reviewer for his/her valuable comments and suggestions, which were helpful in improving the paper.

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(Received April 4, 2012)

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