

## PRODUCTS OF RADIAL DERIVATIVE AND MULTIPLICATION OPERATOR BETWEEN MIXED NORM SPACES AND ZYGmund–TYPE SPACES ON THE UNIT BALL

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*Abstract.* In this paper, we obtain some characterizations of the boundedness and compactness of the products of the radial derivative and multiplication operator  $\mathcal{RM}_u$  between mixed norm spaces  $H(p, q, \phi)$  and Zygmund-type spaces on the unit ball.

### 1. Introduction

Let  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  be points in the complex vector space  $\mathbb{C}^n$  and  $z\bar{w} := \langle z, w \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \dots + z_n\bar{w}_n$ , where  $\bar{w}_k$  is the complex conjugate of  $w_k$ . We also write

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{\sum_{j=1}^n |z_j|^2}.$$

Let  $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$  be the open unit ball in  $\mathbb{C}^n$ ,  $S = \partial\mathbb{B}$  its boundary, and  $H(\mathbb{B})$  denote the class of all holomorphic functions on  $\mathbb{B}$ . For  $f \in H(\mathbb{B})$ , let

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

stand for the radial derivative of  $f$  at  $z$  ([12, 36]).

The iterated radial derivative operator  $\mathcal{R}^m f$  is defined inductively by ([1, 2, 25]):

$$\mathcal{R}^m f = \mathcal{R}(\mathcal{R}^{m-1} f), \quad m \in \mathbb{N} - \{1\}.$$

A positive continuous function  $\phi$  on  $[0, 1)$  is called normal, if there is a  $t \in [0, 1)$  and  $a$  and  $b$ ,  $0 < a < b$  such that (see, for example, [13]).

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [t, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^a} = 0,$$

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$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [t, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^b} = \infty,$$

If we say that a function  $\phi: \mathbb{B} \rightarrow [0, \infty)$  is normal, we also assume that it is radial, that is,  $\phi(z) = \phi(|z|)$ ,  $z \in \mathbb{B}$ .

For  $p, q \in (0, \infty)$ , let

$$\|f\|_{p,q,\phi} = \left( \int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} dr \right)^{\frac{1}{p}},$$

where

$$M_q(f,r) = \left( \int_S |f(r\zeta)|^q d\sigma(\zeta) \right)^{\frac{1}{q}}, \quad 0 \leq r < 1,$$

and  $d\sigma$  is the normalized surface measure on  $S$ . The mixed norm space  $H(p, q, \phi)$  consists of all  $f \in H(\mathbb{B})$  such that  $\|f\|_{p,q,\phi} < \infty$ . For  $1 \leq p, q < \infty$ ,  $H(p, q, \phi)$ , equipped with the norm  $\|f\|_{p,q,\phi}$ , is a Banach space. While for the other values of  $p$  and  $q$ ,  $\|\cdot\|_{p,q,\phi}$  is a quasinorm on  $H(p, q, \phi)$ ,  $H(p, q, \phi)$  is a Fréchet space but not a Banach space. Note that if  $\phi(r) = (1-r)^{(\alpha+1)/p}$ , then  $H(p, q, \phi)$  is equivalent to the weighted Bergman space  $A_\alpha^p(\mathbb{B})$  defined for  $0 < p < \infty$  and  $\alpha > -1$ , as the space of all  $f \in H(\mathbb{B})$  such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{B}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

Let  $\mu$  be a normal function on  $[0, 1)$ . We say that an  $f \in H(\mathbb{B})$  belongs to the space  $\mathcal{L}_\mu = \mathcal{L}_\mu(\mathbb{B})$ , if

$$\sup \{ \mu(|z|) |\mathcal{R}^2 f(z)| : z \in \mathbb{B} \} < \infty.$$

It is easy to check that  $\mathcal{L}_\mu$  becomes a Banach space under the norm

$$\|f\|_{\mathcal{L}_\mu} = |f(0)| + \sup \{ \mu(|z|) |\mathcal{R}^2 f(z)| : z \in \mathbb{B} \}.$$

$\mathcal{L}_\mu$  will be called the Zygmund-type space. Let  $\mathcal{L}_{\mu,0} = \mathcal{L}_{\mu,0}(\mathbb{B})$  denote the class of holomorphic functions  $f \in \mathcal{L}_\mu$  such that

$$\lim_{|z| \rightarrow 1} \mu(|z|) |\mathcal{R}^2 f(z)| = 0,$$

$\mathcal{L}_{\mu,0}$  is called the little Zygmund-type space (see [7, 17, 24]). It is easy to see that  $\mathcal{L}_{\mu,0}$  is a closed subspace of  $\mathcal{L}_\mu$ . When  $\mu(r) = 1 - r^2$ , Zygmund-type space  $\mathcal{L}_\mu$  (little Zygmund-type space  $\mathcal{L}_{\mu,0}$ ) is the classical Zygmund space  $\mathcal{Z}$  (little Zygmund-type space  $\mathcal{Z}_0$ ). The weighted iterated radial-derivative composition operator is defined in [25, 27] as follows:

$$\mathcal{R}_{u,\phi}^m f(z) = (M_u C_\phi \mathcal{R}^m) f(z) = u(z) \mathcal{R}^m f(\phi(z)), \quad z \in \mathbb{B}.$$

Some characterizations of the boundedness and compactness of the operator  $\mathcal{R}_{u,\phi}^m$  between various spaces of holomorphic functions on the unit ball can be found in [25, 27]. Some related operators between mixed norm spaces and various spaces on the unit ball, are treated, for example, in [1, 8, 10, 14, 15, 18, 19, 20, 21, 22, 26, 28, 29, 30, 31, 32, 35, 38]. For related one-dimensional operators, see, for example [3, 4, 5, 9, 11, 16, 23, 34, 37], as well as the related references therein. When  $m = 1$  and  $\phi(z) = z$ , we can get the operator  $M_u\mathcal{R}$ . Inspired by this operator, we can define the operator  $\mathcal{R}M_u$  as follows:

$$\begin{aligned} \mathcal{R}M_u f(z) &= \mathcal{R}(u(z)f(z)) \\ &= u(z) \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) + \sum_{j=1}^n z_j \frac{\partial u}{\partial z_j}(z) f(z) \\ &= u(z)\mathcal{R}f(z) + \mathcal{R}u(z)f(z) \\ &= M_u\mathcal{R}f(z) + f(z)\mathcal{R}u(z), z \in \mathbb{B}. \end{aligned}$$

The purpose of this paper is to study the boundedness and compactness of the operator  $\mathcal{R}M_u$  between  $H(p, q, \phi)$  spaces and Zygmund-type spaces on the unit ball.

Throughout the paper, the letter  $C$  denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

### 2. Auxiliary results

Here we state several auxiliary results most of which will be used in the proofs of the main results.

The following lemma can be found in [15, 25].

LEMMA 1. Assume that  $m \in \mathbb{N}$ ,  $0 < p, q < \infty$ ,  $\phi$  is normal and  $f \in H(p, q, \phi)$ . Then there is a positive constant  $C$  independent of  $f$  such that

$$|\mathcal{R}^m f(z)| \leq \frac{C|z|}{\phi(|z|)(1 - |z|^2)^{m + \frac{n}{q}}} \|f\|_{p,q,\phi},$$

and

$$|f(z)| \leq C \frac{\|f\|_{p,q,\phi}}{\phi(|z|)(1 - |z|^2)^{\frac{n}{q}}}, \quad z \in \mathbb{B}.$$

The next folklore lemma can be found, e.g. in [1, 15].

LEMMA 2. Assume that  $0 < p, q < \infty$ , for  $\beta > t$ ,  $\omega \in \mathbb{B}$  and

$$f_\omega(z) = \frac{(1 - |\omega|^2)^\beta}{\phi(|\omega|)(1 - z\bar{\omega})^{\beta + \frac{n}{q}}}, \quad z \in \mathbb{B}.$$

Then  $f_\omega \in H(p, q, \phi)$  and there is a positive constant  $C$  independent of  $f$  such that

$$\sup_{\omega \in \mathbb{B}} \|f_\omega\|_{p,q,\phi} \leq C.$$

The next Schwartz-type lemma is proved in a standard way (see, e.g. [14, Lemma 3]).

LEMMA 3. Assume  $\phi$  and  $\mu$  are normal,  $0 < p, q < \infty$  and  $u \in H(\mathbb{B})$ , and let  $X$  and  $Y$  be one of the spaces  $H(p, q, \phi)$ ,  $\mathcal{L}_\mu$ ,  $\mathcal{L}_{\mu,0}$ . Then  $\mathcal{R}M_u : X \rightarrow Y$  is compact if and only if  $\mathcal{R}M_u : X \rightarrow Y$  is bounded and for any bounded sequence  $\{f_n\}$  in  $X$  which converges to zero uniformly on the compact subsets of  $\mathbb{B}$  as  $n \rightarrow \infty$ , we have  $\|\mathcal{R}M_u(f_n)\|_{\mathcal{L}_\mu} \rightarrow 0, n \rightarrow \infty$ .

To investigate the compactness of the operator  $\mathcal{R}M_u$ , which map a space into  $\mathcal{L}_{\mu,0}$ , we also need the next lemma (see, e.g. [6, 38]).

LEMMA 4. A closed set  $K$  in  $\mathcal{L}_{\mu,0}$  is compact if and only if it is bounded and satisfies

$$\limsup_{|z| \rightarrow 1} \sup_{f \in K} \mu(|z|) |\mathcal{R}^2 f(z)| = 0.$$

For the case when  $\mu$  is a normal weight we have the following point evaluation estimate, which was proved in Lemma 3.1 in [33].

LEMMA 5. Assume that  $f \in H(\mathbb{B})$  and  $\mu$  is a normal weight. Then

$$|f(z)| \leq C \|f\|_{\mathcal{L}_\mu} \left( 1 + \int_0^{|z|} \frac{ds}{\mu(s)} \right), \quad z \in \mathbb{B}.$$

The next two lemmas are proved in [24].

LEMMA 6. Assume that  $\mu$  is normal and  $f \in \mathcal{L}_\mu$ . Then,

$$|f(z)| \leq C \|f\|_{\mathcal{L}_\mu} \left( 1 + \int_0^{|z|} \int_0^t \frac{ds}{\mu(s)} dt \right), \quad z \in \mathbb{B}.$$

Moreover, if

$$\int_0^1 \int_0^t \frac{ds}{\mu(s)} dt < \infty,$$

then

$$|f(z)| \leq C \|f\|_{\mathcal{L}_\mu}, \quad z \in \mathbb{B}.$$

LEMMA 7. Assume  $\mu$  is normal and  $\int_0^1 \int_0^t \frac{ds}{\mu(s)} dt < \infty$  holds. Then, for every bounded sequence  $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{L}_\mu$  converging to 0 uniformly on the compact subsets of  $\mathbb{B}$ , we have that

$$\limsup_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0.$$

**3. The boundedness and compactness of  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu (\mathcal{L}_{\mu,0})$**

In this section we formulate and prove our main results. Assume that  $u \in H(\mathbb{B})$ ,  $\phi$  and  $\mu$  are normal.

**THEOREM 1.** *Assume that  $0 < p, q < \infty$ . Then  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is bounded if and only if*

$$\sup_{z \in \mathbb{B}} \frac{\mu(|z|) |\mathcal{R}^3 u(z)|}{\phi(|z|) (1 - |z|^2)^{\frac{n}{q}}} < \infty, \tag{1}$$

$$\sup_{z \in \mathbb{B}} \frac{\mu(|z|) |z \mathcal{R}^2 u(z)|}{\phi(|z|) (1 - |z|^2)^{1 + \frac{n}{q}}} < \infty, \tag{2}$$

$$\sup_{z \in \mathbb{B}} \frac{\mu(|z|) |z \mathcal{R} u(z)|}{\phi(|z|) (1 - |z|^2)^{2 + \frac{n}{q}}} < \infty, \tag{3}$$

and

$$\sup_{z \in \mathbb{B}} \frac{\mu(|z|) |zu(z)|}{\phi(|z|) (1 - |z|^2)^{3 + \frac{n}{q}}} < \infty. \tag{4}$$

*Proof.* First let us assume that conditions (1), (2), (3) and (4) hold. For any  $f \in H(p, q, \phi)$ , by Lemma 1, we have

$$\begin{aligned} & \mu(|z|) |\mathcal{R}^2 ((\mathcal{R}M_u)f)(z)| \\ &= \mu(|z|) |\mathcal{R}^2 (\mathcal{R}u(z)f(z) + u(z)\mathcal{R}f(z))| \\ &= \mu(|z|) |\mathcal{R}^3 u(z)f(z) + 3\mathcal{R}^2 u(z)\mathcal{R}f(z) + 3\mathcal{R}u(z)\mathcal{R}^2 f(z) + u(z)\mathcal{R}^3 f(z)| \\ &\leq C \|f\|_{p,q,\phi} \left( \frac{\mu(|z|) |\mathcal{R}^3 u(z)|}{\phi(|z|) (1 - |z|^2)^{\frac{n}{q}}} + \frac{\mu(|z|) |z \mathcal{R}^2 u(z)|}{\phi(|z|) (1 - |z|^2)^{1 + \frac{n}{q}}} + \frac{\mu(|z|) |z \mathcal{R} u(z)|}{\phi(|z|) (1 - |z|^2)^{2 + \frac{n}{q}}} \right. \\ & \quad \left. + \frac{\mu(|z|) |zu(z)|}{\phi(|z|) (1 - |z|^2)^{3 + \frac{n}{q}}} \right). \end{aligned}$$

And we have  $(\mathcal{R}M_u)f(0) = \mathcal{R}u(0)f(0) + u(0)\mathcal{R}f(0) = 0$ . Combing these two facts we get that the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is bounded.

Conversely, assume that the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is bounded. Then for any  $f \in H(p, q, \phi)$ , there is a positive constant  $C$  independent of  $f$  such that  $\|\mathcal{R}M_u f\|_{\mathcal{L}_\mu} \leq C \|f\|_{p,q,\phi}$ . For a fixed  $\omega \in \mathbb{B}$  and constants  $A, B, C, D$ , set

$$\begin{aligned} f_\omega(z) &= A \frac{(1 - |\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{1}{(1 - z\bar{\omega})^{t+1 + \frac{n}{q}}} \\ & \quad + B \frac{(1 - |\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{1}{(1 - z\bar{\omega})^{t+2 + \frac{n}{q}}} \end{aligned}$$

$$\begin{aligned}
 &+C \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{1}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+D \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{1}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}}, \quad z \in \mathbb{B},
 \end{aligned} \tag{5}$$

then

$$\begin{aligned}
 \mathcal{R}f_\omega(z) &= A \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \frac{1}{(1-z\bar{\omega})^{t+1+\frac{n}{q}}} \\
 &+B \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \frac{1}{(1-z\bar{\omega})^{t+2+\frac{n}{q}}} \\
 &+C \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \frac{1}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+D \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \frac{1}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &= AC_1 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+2+\frac{n}{q}}} \\
 &+BC_2 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+CC_3 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+DC_4 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}}, \tag{6} \\
 \mathcal{R}^2 f_\omega(z) &= AC_1 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \mathcal{R} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+2+\frac{n}{q}}} \\
 &+BC_2 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \mathcal{R} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+CC_3 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \mathcal{R} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+DC_4 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \mathcal{R} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}} \\
 &= AC_1 C_2 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+BC_2 C_3 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}}
 \end{aligned}$$

$$\begin{aligned}
 &+CC_3C_4 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}} \\
 &+DC_4C_5 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+6+\frac{n}{q}}} \\
 &+AC_1 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+2+\frac{n}{q}}} \\
 &+BC_2 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+CC_3 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+DC_4 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}}, \tag{7}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{R}^3 f_\omega(z) = &3AC_1C_2 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}} \\
 &+3BC_2C_3 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+3CC_3C_4 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}} \\
 &+3DC_4C_5 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{(z\bar{\omega})^2}{(1-z\bar{\omega})^{t+6+\frac{n}{q}}} \\
 &+AC_1C_2C_3 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{(z\bar{\omega})^3}{(1-z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+BC_2C_3C_4 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{(z\bar{\omega})^3}{(1-z\bar{\omega})^{t+5+\frac{n}{q}}} \\
 &+CC_3C_4C_5 \frac{(1-|\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{(z\bar{\omega})^3}{(1-z\bar{\omega})^{t+6+\frac{n}{q}}} \\
 &+DC_4C_5C_6 \frac{(1-|\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{(z\bar{\omega})^3}{(1-z\bar{\omega})^{t+7+\frac{n}{q}}} \\
 &+AC_1 \frac{(1-|\omega|^2)^{t+1}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+2+\frac{n}{q}}} \\
 &+BC_2 \frac{(1-|\omega|^2)^{t+2}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1-z\bar{\omega})^{t+3+\frac{n}{q}}}
 \end{aligned}$$

$$\begin{aligned}
 &+CC_3 \frac{(1 - |\omega|^2)^{t+3}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1 - z\bar{\omega})^{t+4+\frac{n}{q}}} \\
 &+DC_4 \frac{(1 - |\omega|^2)^{t+4}}{\phi(|\omega|)} \frac{z\bar{\omega}}{(1 - z\bar{\omega})^{t+5+\frac{n}{q}}}, \tag{8}
 \end{aligned}$$

where  $C_j = t + j + \frac{n}{q}$ ,  $j = 1, 2, 3, 4, 5, 6$ . Applying Lemma 2 we see that  $f_\omega \in H(p, q, \phi)$  for every  $\omega \in \mathbb{B}$  and  $\sup_{\omega \in \mathbb{B}} \|f_\omega\|_{p,q,\phi} \leq C$ . We choose the corresponding function in (5) with

$$A = -\frac{C_4}{C_1}, \quad B = 3\frac{C_4}{C_2}, \quad C = -3\frac{C_4}{C_3}, \quad D = 1$$

and denote it by  $f_\omega$ . By applying (5), (6), (7) and (8) we get

$$\mathcal{R}f_\omega(\omega) = \mathcal{R}^2f_\omega(\omega) = \mathcal{R}^3f_\omega(\omega) = 0, \quad f_\omega(\omega) = M \frac{1}{\phi(|\omega|)(1 - |\omega|^2)^{\frac{n}{q}}}, \tag{9}$$

where  $M = -\frac{C_4}{C_1} + 3\frac{C_4}{C_2} - 3\frac{C_4}{C_3} + 1$ , thus for any  $\omega \in \mathbb{B}$ , we get

$$\begin{aligned}
 \frac{\mu(|\omega|)|\mathcal{R}^3u(\omega)|}{\phi(|\omega|)(1 - |\omega|^2)^{\frac{n}{q}}} &\leq C\mu(|\omega|)|\mathcal{R}^3u(\omega)f_\omega(\omega) + 3\mathcal{R}^2u(\omega)\mathcal{R}f_\omega(\omega) \\
 &\quad + 3\mathcal{R}u(\omega)\mathcal{R}^2f_\omega(\omega) + u(\omega)\mathcal{R}^3f_\omega(\omega)| \\
 &\leq C\|\mathcal{R}M_u(f_\omega)\|_{\mathcal{X}_\mu} \leq C\|\mathcal{R}M_u\|_{H(p,q,\phi) \rightarrow \mathcal{X}_\mu}. \tag{10}
 \end{aligned}$$

Hence, we have

$$\sup_{|\omega|<1} \frac{\mu(|\omega|)|\mathcal{R}^3u(\omega)|}{\phi(|\omega|)(1 - |\omega|^2)^{\frac{n}{q}}} \leq C. \tag{11}$$

That is, (1) holds.

To prove (4), we choose the corresponding function in (5) with

$$A = \frac{C_3 + 2C_4}{C_1}, \quad B = \frac{-2C_3 - 3C_4}{C_2}, \quad C = 1, \quad D = 1$$

and denote it by  $g_\omega$ . By applying (5), (6), (7) and (8), we get

$$\begin{aligned}
 \mathcal{R}g_\omega(\omega) &= \mathcal{R}^2g_\omega(\omega) = 0, \quad g_\omega(\omega) = N \frac{1}{\phi(|\omega|)(1 - |\omega|^2)^{\frac{n}{q}}}, \\
 \mathcal{R}^3g_\omega(\omega) &= O \frac{|\omega|^6}{\phi(|\omega|)(1 - |\omega|^2)^{3+\frac{n}{q}}}. \tag{12}
 \end{aligned}$$

where  $N = \frac{C_3+2C_4}{C_1} - \frac{2C_3+3C_4}{C_2} + 2$ ,  $O = C_2C_3^2 + 2C_2C_3C_4 - 2C_3^2C_4 - 3C_3C_4^2 + C_3C_4C_5 +$

$C_4C_5C_6$ , thus for any  $\omega \in \mathbb{B}$ , by using (1), (12) and the triangle inequality we get

$$\begin{aligned}
 |O| \frac{\mu(|\omega|)|u(\omega)||\omega|^6}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} &= \mu(|\omega|)|u(\omega)\mathcal{R}^3g_\omega(\omega)| \\
 &\leq \mu(|\omega|) \left| \mathcal{R}^3u(\omega)g_\omega(\omega) + 3\mathcal{R}^2u(\omega)\mathcal{R}g_\omega(\omega) + 3\mathcal{R}u(\omega)\mathcal{R}^2g_\omega(\omega) + u(\omega)\mathcal{R}^3g_\omega(\omega) \right| \\
 &\quad + |N| \frac{\mu(|\omega|)|\mathcal{R}^3u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}} \\
 &\leq \|\mathcal{R}M_u(g_\omega)\|_{\mathcal{X}_\mu} + C \\
 &\leq C\|\mathcal{R}M_u\|_{H(p,q,\phi) \rightarrow \mathcal{X}_\mu} + C.
 \end{aligned} \tag{13}$$

Let  $r \in (0, 1)$ , from (13) we get

$$\begin{aligned}
 \sup_{r < |\omega| < 1} \frac{\mu(|\omega|)|\omega u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} &\leq \frac{C}{r^5} \sup_{r < |\omega| < 1} \frac{\mu(|\omega|)|u(\omega)||\omega|^6}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} \\
 &\leq C\|\mathcal{R}M_u\|_{H(p,q,\phi) \rightarrow \mathcal{X}_\mu} + C.
 \end{aligned} \tag{14}$$

Using the fact

$$\sup_{|\omega| \leq r} \frac{\mu(|\omega|)|\omega u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} \leq C \sup_{|\omega| \leq r} \mu(|\omega|)|u(\omega)| \leq C,$$

we can get that (4) holds.

To prove (3), we choose the corresponding function in (5) with

$$A = \frac{-C_3^2C_4 + C_3C_4 + C_4C_5C_6}{2C_1C_3}, \quad B = \frac{-3C_3C_4 - C_4C_5C_6 + C_2C_3^2}{2C_2C_3}, \quad C = 1, \quad D = 1$$

and denote it by  $h_\omega$ . By applying (5), (6), (7) and (8), we get

$$\begin{aligned}
 \mathcal{R}h_\omega(\omega) &= 0, \quad h_\omega(\omega) = Q \frac{1}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}}, \\
 \mathcal{R}^2h_\omega(\omega) &= R \frac{|\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}}, \\
 \mathcal{R}^3h_\omega(\omega) &= 3R \frac{|\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}},
 \end{aligned} \tag{15}$$

where

$$\begin{aligned}
 Q &= \frac{-C_3^2C_4 + C_3C_4 + C_4C_5C_6}{2C_1C_3} + \frac{-3C_3C_4 - C_4C_5C_6 + C_2C_3^2}{2C_2C_3} + 2, \\
 R &= \frac{-C_2C_3^2C_4 + C_2C_3C_4 + C_2C_4C_5C_6}{2C_3} + \frac{-3C_3C_4 - C_4C_5C_6 + C_2C_3^2}{2} + C_3C_4 + C_4C_5,
 \end{aligned}$$

thus for any  $\omega \in \mathbb{B}$ , by using (1), (4), (15) and the triangle inequality we get

$$\begin{aligned}
 & 3|R| \frac{\mu(|\omega|)|\mathcal{R}u(\omega)||\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \\
 & \leq \mu(|\omega|) |\mathcal{R}^3u(\omega)h_\omega(\omega) + 3\mathcal{R}^2u(\omega)\mathcal{R}h_\omega(\omega) + 3\mathcal{R}u(\omega)\mathcal{R}^2h_\omega(\omega) + u(\omega)\mathcal{R}^3h_\omega(\omega)| \\
 & \quad + |Q| \frac{\mu(|\omega|)|R^3u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}} + 3|R| \frac{\mu(|\omega|)|u(\omega)||\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \\
 & \leq \|M_u\mathcal{R}(h_\omega)\|_{\mathcal{X}_\mu} + |Q| \frac{\mu(|\omega|)|R^3u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}} + 3|R| \frac{\mu(|\omega|)|u(\omega)||\omega|}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} \\
 & \leq C\|\mathcal{R}M_u\|_{H(p,q,\phi)\rightarrow\mathcal{X}_\mu} + C.
 \end{aligned} \tag{16}$$

Let  $r \in (0, 1)$ , from (16) we get

$$\begin{aligned}
 \sup_{r<|\omega|<1} \frac{\mu(|\omega|)|\omega\mathcal{R}u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} & \leq \frac{C}{r^3} \sup_{r<|\omega|<1} \frac{\mu(|\omega|)|\mathcal{R}u(\omega)||\omega|^4}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \\
 & \leq C\|\mathcal{R}M_u\|_{H(p,q,\phi)\rightarrow\mathcal{X}_\mu} + C.
 \end{aligned} \tag{17}$$

Using the fact

$$\sup_{|\omega|\leq r} \frac{\mu(|\omega|)|\omega\mathcal{R}u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{2+\frac{n}{q}}} \leq C \sup_{|\omega|\leq r} \mu(|\omega|)|\mathcal{R}u(\omega)| \leq C,$$

we get that (3) holds.

To prove (2), we choose the corresponding function in (5) with

$$A = \frac{-C_3C_4^2 + C_4C_5^2}{C_1C_2}, \quad B = \frac{-C_4C_5C_6 + C_3^2C_4}{C_2C_3}, \quad C = 1, \quad D = 1$$

and denote it by  $l_\omega$ . By applying (5), (6), (7) and (8), we get

$$\begin{aligned}
 l_\omega(\omega) & = S \frac{1}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}}, \\
 \mathcal{R}l_\omega(\omega) & = \mathcal{R}^2l_\omega(\omega) = \mathcal{R}^3l_\omega(\omega) = T \frac{|\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}},
 \end{aligned} \tag{18}$$

where  $S = \frac{-C_3C_4^2 + C_4C_5^2}{C_1C_2} + \frac{-C_4C_5C_6 + C_3^2C_4}{C_2C_3} + 2$ ,  $T = \frac{-C_3C_4^2 + C_4C_5^2}{C_2} + \frac{-C_4C_5C_6 + C_3^2C_4}{C_3} + C_3 + C_4$ , thus for any  $\omega \in \mathbb{B}$ , by using (1), (3), (4), (18) and the triangle inequality we get

$$\begin{aligned}
 & 3|T| \frac{\mu(|\omega|)|\mathcal{R}^2u(\omega)||\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}} \\
 & \leq \mu(|\omega|) |\mathcal{R}^3u(\omega)l_\omega(\omega) + 3\mathcal{R}^2u(\omega)\mathcal{R}l_\omega(\omega) + 3\mathcal{R}u(\omega)\mathcal{R}^2l_\omega(\omega) + u(\omega)\mathcal{R}^3l_\omega(\omega)| \\
 & \quad + |S| \frac{\mu(|\omega|)|\mathcal{R}^3u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}} + 3|T| \frac{\mu(|\omega|)|\mathcal{R}u(\omega)||\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}} + |T| \frac{\mu(|\omega|)|u(\omega)||\omega|^2}{\phi(|\omega|)(1-|\omega|^2)^{1+\frac{n}{q}}}
 \end{aligned}$$

$$\begin{aligned} &\leq \| \mathcal{R}M_u(l_\omega) \|_{\mathcal{X}_\mu} + |S| \frac{\mu(|\omega|)|\mathcal{R}^3 u(\omega)|}{\phi(|\omega|)(1-|\omega|^2)^{\frac{n}{q}}} + 3|T| \frac{\mu(|\omega|)|\mathcal{R}u(\omega)||\omega|}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} \\ &\quad + |T| \frac{\mu(|\omega|)|u(\omega)||\omega|}{\phi(|\omega|)(1-|\omega|^2)^{3+\frac{n}{q}}} \\ &\leq C \| \mathcal{R}M_u \|_{H(p,q,\phi) \rightarrow \mathcal{X}_\mu} + C. \end{aligned} \tag{19}$$

From (19) we have that

$$\begin{aligned} &\sup_{\omega \in \mathbb{B}} \frac{\mu(|\omega|)|\mathcal{R}^2 u(\omega)||\omega|}{\phi(|\omega|)(1-|\omega|)^{1+\frac{n}{q}}} \\ &\leq C \sup_{\{\omega \in \mathbb{B}: |\omega| \leq \frac{1}{2}\}} \mu(|\omega|)|\mathcal{R}^2 u(\omega)| + 2 \sup_{\{\omega \in \mathbb{B}: \frac{1}{2} < |\omega| < 1\}} \frac{\mu(|\omega|)|\mathcal{R}^2 u(\omega)||\omega|^2}{\phi(|\omega|)(1-|\omega|)^{1+\frac{n}{q}}} \\ &\leq C. \end{aligned}$$

From this (2) follows, finishing the proof of the theorem.  $\square$

**THEOREM 2.** *Assume that  $0 < p, q < \infty$ . Then  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{X}_\mu$  is compact if and only if*

$$\lim_{|z| \rightarrow 1} \frac{\mu(|z|)|\mathcal{R}^3 u(z)|}{\phi(|z|)(1-|z|^2)^{\frac{n}{q}}} = 0, \tag{20}$$

$$\lim_{|z| \rightarrow 1} \frac{\mu(|z|)|z\mathcal{R}^2 u(z)|}{\phi(|z|)(1-|z|^2)^{1+\frac{n}{q}}} = 0, \tag{21}$$

$$\lim_{|z| \rightarrow 1} \frac{\mu(|z|)|z\mathcal{R}u(z)|}{\phi(|z|)(1-|z|^2)^{2+\frac{n}{q}}} = 0, \tag{22}$$

and

$$\lim_{|z| \rightarrow 1} \frac{\mu(|z|)|zu(z)|}{\phi(|z|)(1-|z|^2)^{3+\frac{n}{q}}} = 0. \tag{23}$$

*Proof.* First assume that the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{X}_\mu$  is compact. Let  $\{z_k\}$  be a sequence in  $\mathbb{B}$  such that  $|z_k| \rightarrow 1$  as  $k \rightarrow \infty$ . Set

$$f_k(z) = f_{z_k}(z), \quad g_k(z) = g_{z_k}(z), \quad h_k(z) = h_{z_k}(z), \quad l_k(z) = l_{z_k}(z), \quad k \in \mathbb{N}.$$

Then  $f_k, g_k, h_k, l_k \in H(p, q, \phi)$ ,  $\sup_{k \in \mathbb{N}} \|f_k\|_{p,q,\phi} \leq C$ ,  $\sup_{k \in \mathbb{N}} \|g_k\|_{p,q,\phi} \leq C$ ,  $\sup_{k \in \mathbb{N}} \|h_k\|_{p,q,\phi} \leq C$ ,  $\sup_{k \in \mathbb{N}} \|l_k\|_{p,q,\phi} \leq C$ , and  $f_k, g_k, h_k, l_k$  converge to 0 uniformly on the compact subsets of  $\mathbb{B}$ , using the compactness of  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{X}_\mu$  and Lemma 3, we get

$$\lim_{k \rightarrow \infty} \| \mathcal{R}M_u(f_k) \|_{\mathcal{X}_\mu} = \lim_{k \rightarrow \infty} \| \mathcal{R}M_u(g_k) \|_{\mathcal{X}_\mu} = \lim_{k \rightarrow \infty} \| \mathcal{R}M_u(h_k) \|_{\mathcal{X}_\mu} = \lim_{k \rightarrow \infty} \| \mathcal{R}M_u(l_k) \|_{\mathcal{X}_\mu} = 0.$$

By (9) we have

$$\mathcal{R}f_k(z_k) = \mathcal{R}^2 f_k(z_k) = \mathcal{R}^3 f_k(z_k) = 0, \quad f_k(z_k) = M \frac{1}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}},$$

so

$$\begin{aligned}
 & |M| \frac{\mu(|z_k|)|\mathcal{R}^3 u(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}} \\
 &= \mu(|z_k|) |\mathcal{R}^3 u(z_k) f_k(z_k) + 3\mathcal{R}^2 u(z_k) R f_k(z_k) + 3\mathcal{R} u(z_k) R^2 f_k(z_k) + u(z_k) \mathcal{R}^3 f_k(z_k)| \\
 &\leq \|\mathcal{R} M_u(f_k)\|_{\mathcal{X}_\mu}.
 \end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \frac{\mu(|z_k|)|\mathcal{R}^3 u(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}} = 0, \tag{24}$$

so (20) holds.

By (12), we have

$$\begin{aligned}
 \mathcal{R} g_k(z_k) &= \mathcal{R}^2 g_k(z_k) = 0, \quad g_k(z_k) = N \frac{1}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}}, \\
 \mathcal{R}^3 g_k(z_k) &= \frac{|z_k|^6}{\phi(|z_k|)(1-|z_k|^2)^{3+\frac{n}{q}}},
 \end{aligned}$$

so

$$\begin{aligned}
 |O| \frac{\mu(|z_k|)|u(z_k)||z_k|^6}{\phi(|z_k|)(1-|z_k|^2)^{3+\frac{n}{q}}} &\leq \|\mathcal{R} M_u(g_k)\|_{\mathcal{X}_\mu} + |N| \frac{\mu(|z_k|)|\mathcal{R}^3 u(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}} \rightarrow 0, \\
 &\text{as } k \rightarrow \infty,
 \end{aligned} \tag{25}$$

hence, (23) holds.

By (15), we have

$$\begin{aligned}
 \mathcal{R} h_k(z_k) &= 0, \quad h_k(z_k) = Q \frac{1}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}}, \\
 \mathcal{R}^2 h_k(z_k) &= R \frac{|z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}}, \\
 \mathcal{R}^3 h_k(z_k) &= 3R \frac{|z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}},
 \end{aligned}$$

therefore,

$$\begin{aligned}
 & 3|R| \frac{\mu(|z_k|)|\mathcal{R} u(z_k)||z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}} \\
 &\leq \|\mathcal{R} M_u(h_k)\|_{\mathcal{X}_\mu} + |Q| \frac{\mu(|z_k|)|\mathcal{R}^3 u(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}} + 3|R| \frac{\mu(|z_k|)|u(z_k)||z_k|^4}{\phi(|z_k|)(1-|z_k|^2)^{2+\frac{n}{q}}} \\
 &\leq \|\mathcal{R} M_u(h_k)\|_{\mathcal{X}_\mu} + |Q| \frac{\mu(|z_k|)|\mathcal{R}^3 u(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{\frac{n}{q}}} + 3|R| \frac{\mu(|z_k|)|u(z_k)||z_k|}{\phi(|z_k|)(1-|z_k|^2)^{3+\frac{n}{q}}} \rightarrow 0 \\
 &\text{as } k \rightarrow \infty,
 \end{aligned} \tag{26}$$

hence, (22) is true.

By (18), we have

$$l_k(z_k) = S \frac{1}{\phi(|\omega|)(1 - |z_k|^2)^{\frac{n}{q}}},$$

$$\mathcal{R}l_k(z_k) = \mathcal{R}^2l_k(z_k) = \mathcal{R}^3l_k(z_k) = T \frac{|z_k|^2}{\phi(|\omega|)(1 - |z_k|^2)^{1+\frac{n}{q}}}.$$

By Lemma 3, we have

$$\begin{aligned} & |T| \frac{\mu(|z_k|)|\mathcal{R}^2u(z_k)||z_k|^2}{\phi(|z_k|)(1 - |z_k|^2)^{1+\frac{n}{q}}} \\ & \leq \|\mathcal{R}M_u(l_k)\|_{\mathcal{L}_\mu} + |S| \frac{\mu(|z_k|)|\mathcal{R}^3u(z_k)|}{\phi(|z_k|)(1 - |z_k|^2)^{\frac{n}{q}}} + 3|T| \frac{\mu(|z_k|)|\mathcal{R}u(z_k)||z_k|^2}{\phi(|z_k|)(1 - |z_k|^2)^{1+\frac{n}{q}}} \\ & \quad + |T| \frac{\mu(|z_k|)|u(z_k)||z_k|^2}{\phi(|z_k|)(1 - |z_k|^2)^{1+\frac{n}{q}}} \\ & \leq \|\mathcal{R}M_u(l_k)\|_{\mathcal{L}_\mu} + |S| \frac{\mu(|z_k|)|\mathcal{R}^3u(z_k)|}{\phi(|z_k|)(1 - |z_k|^2)^{\frac{n}{q}}} + 3|T| \frac{\mu(|z_k|)|\mathcal{R}u(z_k)||z_k|}{\phi(|z_k|)(1 - |z_k|^2)^{2+\frac{n}{q}}} \\ & \quad + |T| \frac{\mu(|z_k|)|u(z_k)||z_k|}{\phi(|z_k|)(1 - |z_k|^2)^{3+\frac{n}{q}}} \\ & \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned} \tag{27}$$

so (21) holds.

To prove the other direction of the theorem, suppose (20), (21), (22) and (23) hold. We see that (1), (2), (3) and (4) hold. By Theorem 1, we get the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is bounded and  $\varepsilon > 0$ , there is a  $\delta \in (0, 1)$  such that for  $\delta < |z| < 1$

$$\frac{\mu(|z|)|\mathcal{R}^3u(z)|}{\phi(|z|)(1 - |z|^2)^{\frac{n}{q}}} < \varepsilon, \tag{28}$$

$$\frac{\mu(|z|)|z\mathcal{R}^2u(z)|}{\phi(|z|)(1 - |z|^2)^{1+\frac{n}{q}}} < \varepsilon, \tag{29}$$

$$\frac{\mu(|z|)|z\mathcal{R}u(z)|}{\phi(|z|)(1 - |z|^2)^{2+\frac{n}{q}}} < \varepsilon, \tag{30}$$

and

$$\frac{\mu(|z|)|zu(z)|}{\phi(|z|)(1 - |z|^2)^{3+\frac{n}{q}}} < \varepsilon. \tag{31}$$

Let  $a_k \in H(p, q, \phi)$ ,  $\sup_{k \in \mathbb{N}} \|a_k\|_{p,q,\phi} \leq L$ , and  $\{a_k\}$  converge to zero uniformly on the compact subsets of  $\mathbb{B}$ , by Lemma 1, (28), (29), (30) and (31), we have that for

sufficiently large  $k$

$$\begin{aligned}
 & \| \mathcal{R}M_u(a_k) \|_{\mathcal{L}_\mu} = | \mathcal{R}M_u(a_k)(0) | + \sup_{z \in \mathbb{B}} \mu(|z|) | \mathcal{R}^2(RM_u(a_k))(z) | \\
 &= \sup_{z \in \mathbb{B}} \mu(|z|) \mu(|z|) | \mathcal{R}^3 u(z) a_k(z) + 3 \mathcal{R}^2 u(z) \mathcal{R} a_k(z) + 3 \mathcal{R} u(z) \mathcal{R}^2 a_k(z) + u(z) \mathcal{R}^3 a_k(z) | \\
 &\leq \sup_{\{z \in \mathbb{B}: |z| \leq \delta\}} \mu(|z|) | \mathcal{R}^3 u(z) a_k(z) + 3 \mathcal{R}^2 u(z) \mathcal{R} a_k(z) + 3 \mathcal{R} u(z) \mathcal{R}^2 a_k(z) + u(z) \mathcal{R}^3 a_k(z) | \\
 &\quad + \sup_{\{z \in \mathbb{B}: |z| > \delta\}} \mu(|z|) | \mathcal{R}^3 u(z) a_k(z) + 3 \mathcal{R}^2 u(z) \mathcal{R} a_k(z) + 3 \mathcal{R} u(z) \mathcal{R}^2 a_k(z) + u(z) \mathcal{R}^3 a_k(z) | \\
 &\leq \varepsilon \sup_{\{z \in \mathbb{B}: |z| \leq \delta\}} \mu(|z|) ( | \mathcal{R}^3 u(z) | + | 3 \mathcal{R}^2 u(z) | + | 3 \mathcal{R} u(z) | + | u(z) | ) \\
 &\quad + CL \sup_{\{z \in \mathbb{B}: |z| > \delta\}} \left( \frac{\mu(|z|) | \mathcal{R}^3 u(z) |}{\phi(|z|) (1 - |z|^2)^{\frac{n}{q}}} + \frac{\mu(|z|) |z \mathcal{R}^2 u(z)|}{\phi(|z|) (1 - |z|^2)^{1 + \frac{n}{q}}} + \frac{\mu(|z|) |z \mathcal{R} u(z)|}{\phi(|z|) (1 - |z|^2)^{2 + \frac{n}{q}}} \right. \\
 &\quad \left. + \frac{\mu(|z|) |zu(z)|}{\phi(|z|) (1 - |z|^2)^{3 + \frac{n}{q}}} \right) \\
 &< (C + 4CL) \varepsilon,
 \end{aligned}$$

hence

$$\lim_{k \rightarrow \infty} \| \mathcal{R}M_u(a_k) \|_{\mathcal{L}_\mu} = 0,$$

so by Lemma 3  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is compact.  $\square$

**THEOREM 3.** Assume that  $0 < p, q < \infty$ . Then  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu,0}$  is compact if and only if  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is compact.

*Proof.* First assume that the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is compact, by Theorem 2, for any  $f \in H(p, q, \phi)$

$$\begin{aligned}
 & \mu(|z|) | \mathcal{R}^2((\mathcal{R}M_u) f)(z) | \\
 &= \mu(|z|) | \mathcal{R}^3 u(z) f(z) + 3 \mathcal{R}^2 u(z) \mathcal{R} f(z) + 3 \mathcal{R} u(z) \mathcal{R}^2 f(z) + u(z) \mathcal{R}^3 f(z) | \\
 &\leq C \|f\|_{p,q,\phi} \left( \frac{\mu(|z|) | \mathcal{R}^3 u(z) |}{\phi(|z|) (1 - |z|^2)^{\frac{n}{q}}} + \frac{3 \mu(|z|) |z \mathcal{R}^2 u(z)|}{\phi(|z|) (1 - |z|^2)^{1 + \frac{n}{q}}} + \frac{3 \mu(|z|) |z \mathcal{R} u(z)|}{\phi(|z|) (1 - |z|^2)^{2 + \frac{n}{q}}} \right. \\
 &\quad \left. + \frac{\mu(|z|) |zu(z)|}{\phi(|z|) (1 - |z|^2)^{3 + \frac{n}{q}}} \right) \\
 &\rightarrow 0, \quad |z| \rightarrow 1.
 \end{aligned} \tag{32}$$

we see that  $\mathcal{R}M_u(f) \in \mathcal{L}_{\mu,0}$ . Since the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_\mu$  is bounded, we obtain that the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu,0}$  is bounded. Hence the set

$$\mathcal{R}M_u \{ f \in H(p, q, \phi) : \|f\|_{p,q,\phi} \leq 1 \}$$

is bounded in  $\mathcal{L}_{\mu,0}$ . By Lemma 4, we wish to show

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{p,q,\phi} \leq 1} \mu(|z|) |\mathcal{R}^2(\mathcal{R}M_u f)(z)| = 0. \tag{33}$$

In fact, since the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu}$  is compact, by Theorem 2, (20), (21),(22) and (23) hold. By taking the supremum in (32) over the unit ball in  $H(p, q, \phi)$ , using (20), (21),(22) and (23) we see that (33) follows. Therefore, the operator  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu,0}$  is compact.

Conversely, the compactness of  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu,0}$  implies the compactness of  $\mathcal{R}M_u : H(p, q, \phi) \rightarrow \mathcal{L}_{\mu}$  is obvious. The proof is completed.  $\square$

**4. The boundedness and compactness of  $\mathcal{R}M_u : \mathcal{L}_{\mu} (\mathcal{L}_{\mu,0}) \rightarrow H(p, q, \phi)$**

In this section we investigate the boundedness and compactness of the operator  $\mathcal{R}M_u : \mathcal{L}_{\mu} (\mathcal{L}_{\mu,0}) \rightarrow H(p, q, \phi)$ .

Before we formulate and prove the next result, we must ensure that the following lemma is true:

LEMMA 8. Assume that  $0 < p < \infty, 1 < q < \infty, f, g \in L^q(\Omega, \mu)$ , then

$$\left( \int_{\Omega} |f + g|^q d\mu \right)^{\frac{p}{q}} \leq C \left( \left( \int_{\Omega} |f|^q d\mu \right)^{\frac{p}{q}} + \left( \int_{\Omega} |g|^q d\mu \right)^{\frac{p}{q}} \right).$$

*Proof.* Using  $C_r$ -inequality and Minkowski-inequality, we have

$$\begin{aligned} \left( \int_{\Omega} |f + g|^q d\mu \right)^{\frac{p}{q}} &= \|f + g\|_{L^q}^p \\ &\leq (\|f\|_{L^q} + \|g\|_{L^q})^p \leq C_p (\|f\|_{L^q}^p + \|g\|_{L^q}^p) \\ &= C_p \left( \left( \int_{\Omega} |f|^q d\mu \right)^{\frac{p}{q}} + \left( \int_{\Omega} |g|^q d\mu \right)^{\frac{p}{q}} \right), \end{aligned}$$

where  $C_p = \max\{1, 2^{p-1}\}$ .  $\square$

THEOREM 4. Assume that  $0 < p < \infty, 1 < q < \infty, u \in H(B), \phi$  and  $\mu$  are normal, and  $\mu$  satisfies condition  $\int_0^1 \frac{dt}{\mu(t)} < \infty$ . If  $\mathcal{R}u, u \in H(p, q, \phi)$ , then the operator  $\mathcal{R}M_u : \mathcal{L}_{\mu}(\mathcal{L}_{\mu,0}) \rightarrow H(p, q, \phi)$  is bounded.

*Proof.* Assume that  $\mathcal{R}u, u \in H(p, q, \phi)$ . For any  $f \in \mathcal{L}_{\mu}$ , by Lemma 5, Lemma 6 and Lemma 8, we have

$$\begin{aligned} &\|\mathcal{R}M_u(f)\|_{H(p,q,\phi)}^p \\ &= \int_0^1 M_q^p(\mathcal{R}M_u(f), r) \frac{\phi^p(r)}{1-r} dr \\ &\leq C \int_0^1 M_q^p(f\mathcal{R}u, r) \frac{\phi^p(r)}{1-r} dr + C \int_0^1 M_q^p(u\mathcal{R}f, r) \frac{\phi^p(r)}{1-r} dr \end{aligned}$$

$$\begin{aligned}
 &\leq C \sup_{z \in \mathbb{B}} |f(z)|^p \int_0^1 M_q^p(\mathcal{R}u, r) \frac{\phi^p(r)}{1-r} dr + C \sup_{z \in \mathbb{B}} |\mathcal{R}f(z)|^p \int_0^1 M_q^p(u, r) \frac{\phi^p(r)}{1-r} dr \\
 &\leq C \|f\|_{\mathcal{Z}_\mu}^p \int_0^1 M_q^p(\mathcal{R}u, r) \frac{\phi^p(r)}{1-r} dr + C \|\mathcal{R}f\|_{\mathcal{B}_\mu}^p \int_0^1 M_q^p(u, r) \frac{\phi^p(r)}{1-r} dr \\
 &\leq C \|f\|_{\mathcal{Z}_\mu}^p \int_0^1 M_q^p(\mathcal{R}u, r) \frac{\phi^p(r)}{1-r} dr + C \|f\|_{\mathcal{Z}_\mu}^p \int_0^1 M_q^p(u, r) \frac{\phi^p(r)}{1-r} dr \\
 &\leq C \|f\|_{\mathcal{Z}_\mu}^p \|\mathcal{R}u\|_{H(p,q,\phi)}^p + C \|f\|_{\mathcal{Z}_\mu}^p \|u\|_{H(p,q,\phi)}^p.
 \end{aligned} \tag{34}$$

from which it follows that the operator  $\mathcal{R}M_u : \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0}) \rightarrow H(p, q, \phi)$  is bounded.  $\square$

**THEOREM 5.** *Assume that  $0 < p < \infty, 1 < q < \infty, u \in H(B), \phi$  and  $\mu$  are normal, and  $\mu$  satisfies condition  $\int_0^1 \frac{dt}{\mu(t)} < \infty$ . If  $\mathcal{R}u, u \in H(p, q, \phi)$ , then the operator  $\mathcal{R}M_u : \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0}) \rightarrow H(p, q, \phi)$  is also compact.*

*Proof.* Assume that  $\mathcal{R}u, u \in H(p, q, \phi)$ . Let  $\{f_k\}_{k \in \mathbb{N}}$  be a bounded sequence in  $\mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$  converging to zero on the compact subsets of  $\mathbb{B}$  as  $k \rightarrow \infty$ . By Lemma 7, we have that

$$\limsup_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0.$$

By the definition of the Zygmund-type space  $\mathcal{Z}_\mu$ , we obtain  $\{\mathcal{R}f_k\}$  is a bounded sequence in  $\mathcal{B}_\mu$ . An application of Cauchy integral estimates implies that the sequence  $\{\mathcal{R}f_k\}$  converges to zero on the compact subsets of  $\mathbb{B}$  as  $k \rightarrow \infty$ . Applying the corresponding result for the  $\mu$ -Bloch space (see [33, Lemma 4.2]), we also have

$$\limsup_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |\mathcal{R}f_k(z)| = 0.$$

Hence,

$$\begin{aligned}
 &\|\mathcal{R}M_u(f_k)\|_{H(p,q,\phi)}^p \\
 &= \int_0^1 M_q^p(\mathcal{R}M_u(f_k), r) \frac{\phi^p(r)}{1-r} dr \\
 &\leq \int_0^1 M_q^p(\mathcal{R}u f_k, r) \frac{\phi^p(r)}{1-r} dr + \int_0^1 M_q^p(u \mathcal{R}f_k, r) \frac{\phi^p(r)}{1-r} dr \\
 &\leq C \sup_{z \in \mathbb{B}} |f_k(z)|^p \int_0^1 M_q^p(\mathcal{R}u, r) \frac{\phi^p(r)}{1-r} dr + C \sup_{z \in \mathbb{B}} |\mathcal{R}f_k(z)|^p \int_0^1 M_q^p(u, r) \frac{\phi^p(r)}{1-r} dr \\
 &= \sup_{z \in \mathbb{B}} |f_k(z)|^p \|\mathcal{R}u\|_{H(p,q,\phi)}^p + \sup_{z \in \mathbb{B}} |\mathcal{R}f_k(z)|^p \|u\|_{H(p,q,\phi)}^p \\
 &\rightarrow 0, \text{ as } k \rightarrow \infty,
 \end{aligned} \tag{35}$$

by Lemma 3, the operator  $\mathcal{R}M_u : \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0}) \rightarrow H(p, q, \phi)$  is compact.  $\square$

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## REFERENCES

- [1] Z. HU, *Extended Cesàro operators on mixed norm spaces*, Proc. Amer. Math. Soc. **131**, 7 (2003), 2171–2179 (electronic).
- [2] B. LI, C. OUYANG, *Higher radial derivative of functions of  $Q_p$  spaces and its applications*, J. Math. Anal. Appl. **327**, 2 (2007), 1257–1272.
- [3] S. LI, S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338**, 2 (2008), 1282–1295.
- [4] S. LI, S. STEVIĆ, *Products of Volterra type operator and composition operator from  $H^\infty$  and Bloch spaces to Zygmund spaces*, J. Math. Anal. Appl. **345**, 1 (2008), 40–52.
- [5] S. LI, S. STEVIĆ, *Composition followed by differentiation between  $H^\infty$  and  $\alpha$ -Bloch spaces* (English summary), Houston J. Math. **35**, 1 (2009), 327–340.
- [6] S. LI, S. STEVIĆ, *Integral-type operators from Bloch-type spaces to Zygmund-type spaces*, Appl. Math. Comput. **215**, 2 (2009), 464–473.
- [7] S. LI, S. STEVIĆ, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput. **217** (2010), 3144–3154.
- [8] Y. LIU, *Boundedness of the Bergman type operators on mixed norm spaces*, Proc. Amer. Math. Soc. **130**, 8 (2002), 2363–2367 (electronic).
- [9] Y. LIU, Y. YU, *Weighted differentiation composition operators from mixed-norm to Zygmund spaces*, Numer. Funct. Anal. Optim. **31**, 8 (2010), 936–954.
- [10] Y. LIU, Y. YU, *On compactness for iterated commutators*, Acta Math. Sci. Ser. B Engl. Ed. **31B**, 2 (2011), 491–500.
- [11] Y. LIU, Y. YU, *Composition followed by differentiation between  $H^\infty$  and Zygmund spaces*, Complex. Anal. Oper. Theory **6**, 1 (2012), 121–137.
- [12] W. RUDIN, *Function Theory in the Unit Ball of  $\mathbb{C}^n$* , Springer-Verlag, New York-Berlin, 1980.
- [13] A.L. SHIELDS, D.L. WILLIAMS, *Bounded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. **162** (1971), 287–302.
- [14] S. STEVIĆ, *Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc*, Siberian Math. J. **48**, 3 (2007), 559–569.
- [15] S. STEVIĆ, *Weighted composition operators between mixed norm spaces and  $H^\infty_\alpha$  spaces in the unit ball*, J. Inequal. Appl. **2007** (2007), Art. ID 28629, 9 pp. doi:10.1155/2007/28629.
- [16] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces*, Appl. Math. Comput. **211**, 1 (2009), 222–233.
- [17] S. STEVIĆ, *On an integral operator from the Zygmund space to the Bloch-type space on the unit ball*, Glasg. J. Math. **51** (2009), 275–287.
- [18] S. STEVIĆ, *On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces*, Nonlinear Anal. **71**, 12 (2009), 6323–6342.
- [19] S. STEVIĆ, *On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball*, J. Math. Anal. Appl. **354**, 2 (2009), 426–434.
- [20] S. STEVIĆ, *On an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball*, Appl. Math. Comput. **215**, 11 (2010), 3817–3823.
- [21] S. STEVIĆ, *On operator  $P_\phi^g$  from the logarithmic Bloch-type space to the mixed-norm space on the unit ball*, Appl. Math. Comput. **215**, 12 (2010), 4248–4255.
- [22] S. STEVIĆ, *Extended Cesàro operators between mixed-norm spaces and Bloch-type spaces in the unit ball*, Houston J. Math. **36**, 3 (2010), 843–858.
- [23] S. STEVIĆ, *Composition followed by differentiation from  $H^\infty$  and the Bloch space to  $n$ th weighted-type spaces on the unit disk*, Appl. Math. Comput. **216**, 12 (2010), 3450–3458.
- [24] S. STEVIĆ, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstr. Appl. Anal. **2010** (2010), Article ID 198608, 7 pp..
- [25] S. STEVIĆ, *Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball*, Abstr. Appl. Anal. **2010** (2010), Article ID 801264, 14 pp..
- [26] S. STEVIĆ, *On an integral operator between Bloch-type spaces on the unit ball*, Bull. Sci. Math. **134** (2010), 329–339.
- [27] S. STEVIĆ, *On a product-type operator from Bloch spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput. **217** (2011), 5930–5935.

- [28] S. STEVIĆ, *On some integral-type operators between a general space and Bloch-type spaces*, Appl. Math. Comput. **218**, 6 (2011), 2600–2618.
- [29] S. STEVIĆ, *Boundedness and compactness of an integral-type operator from Bloch-type spaces with normal weights to  $F(p, q, s)$  space*, Appl. Math. Comput. **218**, 9 (2012), 5414–5421.
- [30] S. STEVIĆ, *Weighted iterated radial operators between different weighted Bergman spaces on the unit ball*, Appl. Math. Comput. **218**, 17 (2012), 8288–8294.
- [31] S. STEVIĆ, A. K. SHARMA, *Iterated differentiation followed by composition from Bloch-type spaces to weighted BMOA spaces*, Appl. Math. Comput. **218**, 7 (2011), 3574–3580.
- [32] S. STEVIĆ, S. I. UEKI, *Integral-type operators acting between weighted-type spaces on the unit ball*, Appl. Math. Comput. **215**, 7 (2009), 2464–2471.
- [33] X TANG, *Extended Cesàro operators between Bloch-type spaces in the unit ball of  $\mathbb{C}^n$* , J. Math. Anal. Appl. **326**, 2 (2007), 1199–1211.
- [34] W. YANG, *Products of composition and differentiation operators from  $Q_K(p, q)$  spaces to Bloch-type spaces*, Abstr. Appl. Anal. **2009** (2009), Article ID 741920, 14 pp.
- [35] X. ZHANG, J. XIAO, Z. HU, *The multipliers between the mixed norm spaces in  $\mathbb{C}^n$* , J. Math. Anal. Appl. **311**, 2 (2005), 664–674.
- [36] K. ZHU, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Text in Mathematics **226**, Springer, New York, 2005.
- [37] X. ZHU, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces* (English summary), Integral Transforms Spec. Funct. **18**, 3/4 (2007), 223–231.
- [38] X. ZHU, *Integral-type operators from iterated logarithmic Bloch spaces to Zygmund-type spaces*, Appl. Math. Comput. **215**, 3 (2009), 1170–1175.

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