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REVERSES OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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Abstract. We show that if $x_1,\cdots,x_n$ are vectors in a normed linear space $(X,\|\cdot\|)$ and $s_1,\cdots,s_n$ belong to the interval $[0,\infty)$, then

$$f_n(s_1,\cdots,s_n)=\sum_{j=1}^{n}\|s_jx_j\|-\left\|\sum_{j=1}^{n}s_jx_j\right\|$$

is a non-negative valued continuous function such that $f_n(s_1,\cdots,s_n)\leqslant f_n(t_1,\cdots,t_n)$ for all $s_1,\cdots,s_n$ and $t_1,\cdots,t_n$ in $[0,\infty)$ with $s_j\leqslant t_j$ $(1\leqslant j\leqslant n)$. By using it, we prove several versions of reverse triangle inequality in inner product spaces and discuss equality attainedness of norm inequalities in strictly convex Banach spaces.

1. Introduction

The generalized triangle inequality, namely

$$\left\|\sum_{j=1}^{n}x_j\right\|\leqslant\sum_{j=1}^{n}\|x_j\|,$$

where $(X,\|\cdot\|)$ is a normed linear space over the real or complex field $\mathbb{K}$ and $x_j,\ j\in\{1,2,\cdots,n\}$ are vectors in X plays a fundamental role in establishing various analytic and geometric properties of such spaces. This inequality has been studied by several authors (see e.g. [2, 9, 15]). We are interested to know under which conditions the generalized triangle inequality on X is reversed, i.e., inequalities of the following type

$$\sum_{j=1}^{n}\|x_j\|\leqslant\left\|\sum_{j=1}^{n}x_j\right\|+C$$

with $C\geqslant 0$, which we call (additive) reverse of the triangle inequality.

Kato, Saito and Tamura [8] proved the following reverse of the triangle inequality.

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THEOREM 1.1. ([8, Theorem 1]) *Let* $(X,\|\cdot\|)$ *be a Banach space, and* $x_j \in X\setminus\{0\}$, $j\in\{1,\cdots,n\}$. *Then*

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \leqslant \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|. \tag{1.1}$$

Moreover they proved equality attainedness of (1.1) in a strictly convex Banach space.

THEOREM 1.2. ([8, Theorem 3]) *Let* $(X,\|\cdot\|)$ *be a strictly convex Banach space, and* $x_j\in X\setminus\{0\}$, $j\in\{1,\cdots,n\}$. *Let* $\|x_{j_0}\| = \min_{1\leqslant j\leqslant n}\|x_j\|$ *and* $\|x_{j_1}\| = \max_{1\leqslant j\leqslant n}\|x_j\|$. *Then the equality*

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| = \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

holds if and only if either

$$\|x_i\| = \|x_j\| \quad (\textit{for all} \quad i,j\in\{1,\cdots,n\})$$

or

$$\frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|} \textit{ for all } j\in J_1^c \textit{ and } \sum_{j=1}^{n}x_j = \left\|\sum_{j=1}^{n}x_j\right\|\frac{x_{j_0}}{\|x_{j_0}\|},$$

where J_1^c *is a complement of* $J_1 = \{j\in\{1,\cdots,n\} : \|x_j\| = \|x_{j_1}\|\}$ *in* $\{1,\cdots,n\}$.

After that, several authors improved and generalized these inequalities (cf. [5, 6, 11, 12, 13, 14]).

On the other hand, Dragomir [3] established the reverse of triangle inequality in real or complex inner product spaces as follows.

THEOREM 1.3. [3, Theorem 7] *Let* $(H,\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $e, x_j\in H$, $j\in\{1,\cdots,n\}$ *with* $\|e\|=1$. *If* $k_j\geqslant 0$, $j\in\{1,\cdots,n\}$, *are such that*

$$\|x_j\| - \mathrm{Re}\langle e,x_j\rangle \leqslant k_j, \ \textit{for each} \ \ j\in\{1,\cdots,n\},$$

then the following inequality holds

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \leqslant \sum_{j=1}^{n}k_j. \tag{1.2}$$

The equality holds in (1.2) *if and only if*

$$\sum_{j=1}^{n}\|x_j\| \geqslant \sum_{j=1}^{n}k_j$$

and

$$\sum_{j=1}^{n}x_j = \left(\sum_{j=1}^{n}\|x_j\| - \sum_{j=1}^{n}k_j\right)e.$$

Ansari and Moslehian modified the above result in [1], and related results can also be found in [4, 7].

The aim of this paper is to discuss equality attainedness of norm inequalities in a strictly convex Banach space. Moreover, we investigate the relation between the inequalities (1.1) and (1.2) in inner product spaces by using a continuous monotone function.

2. Equality attainedness in strictly convex Banach spaces

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$, and $x_j \in X,\ j \in \{1,2,\cdots,n\}$. Throughout this paper, for each $x_j \in X, j \in \{1,\cdots,n\}$, denote by f_n a function on $\prod_{j=1}^{n}[0,\infty)$ as

$$f_n(t_1,\cdots,t_n) = \sum_{j=1}^{n}\|t_jx_j\| - \left\|\sum_{j=1}^{n} t_jx_j\right\|, \qquad \left((t_1,\cdots,t_n) \in \prod_{j=1}^{n}[0,\infty)\right).$$

For $(s_1,\cdots,s_n), (t_1,\cdots,t_n) \in \prod_{j=1}^{n}[0,\infty)$, denote $(s_1,\cdots,s_n) \leqslant (t_1,\cdots,t_n)$ if $s_j \leqslant t_j$ for all $j \in \{1,\cdots,n\}$.

THEOREM 2.1. *Let $(X,\|\cdot\|)$ be a normed linear space, and $x_1,\cdots,x_n \in X$. Then f_n is a non-negative valued continuous function on $\prod_{j=1}^{n}[0,\infty)$ such that $f_n(s_1,\cdots,s_n) \leqslant f_n(t_1,\cdots,t_n)$ for all $(s_1,\cdots,s_n), (t_1,\cdots,t_n) \in \prod_{j=1}^{n}[0,\infty)$ with $(s_1,\cdots,s_n) \leqslant (t_1,\cdots,t_n)$.*

Proof. For any $(s_1,\cdots,s_n), (t_1,\cdots,t_n) \in \prod_{j=1}^{n}[0,\infty)$, applying the triangle inequality, we have

$$\begin{aligned}
&|f_n(t_1,\cdots,t_n) - f_n(s_1,\cdots,s_n)| \\
&= \left|\left\{\sum_{j=1}^{n}\|t_jx_j\| - \left\|\sum_{j=1}^{n} t_jx_j\right\|\right\} - \left\{\sum_{j=1}^{n}\|s_jx_j\| - \left\|\sum_{j=1}^{n} s_jx_j\right\|\right\}\right| \\
&= \left|\left\{\sum_{j=1}^{n}\|t_jx_j\| - \sum_{j=1}^{n}\|s_jx_j\|\right\} - \left\{\left\|\sum_{j=1}^{n} t_jx_j\right\| - \left\|\sum_{j=1}^{n} s_jx_j\right\|\right\}\right| \\
&\leqslant \left|\sum_{j=1}^{n}(t_j - s_j)\|x_j\|\right| + \left|\left\|\sum_{j=1}^{n} t_jx_j\right\| - \left\|\sum_{j=1}^{n} s_jx_j\right\|\right| \\
&\leqslant \sum_{j=1}^{n}\left|t_j - s_j\right| \|x_j\| + \left\|\sum_{j=1}^{n} t_jx_j - \sum_{j=1}^{n} s_jx_j\right\| \\
&\leqslant \sum_{j=1}^{n}\left|t_j - s_j\right| \|x_j\| + \left\|\sum_{j=1}^{n}(t_j - s_j)x_j\right\| \\
&\leqslant 2\sum_{j=1}^{n}\left|t_j - s_j\right| \|x_j\|.
\end{aligned}$$

Thus f_n is continuous.

Next we assume that $(s_1,\cdots,s_n) \leqslant (t_1,\cdots,t_n)$. Since, for each $j \in \{1,\cdots,n\}$, $t_j - s_j \geqslant 0$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n} t_j x_j\right\| &= \left\|\sum_{j=1}^{n} (t_j - s_j)x_j + \sum_{j=1}^{n} s_j x_j\right\| \\
&\leqslant \left\|\sum_{j=1}^{n} (t_j - s_j)x_j\right\| + \left\|\sum_{j=1}^{n} s_j x_j\right\| \\
&\leqslant \sum_{j=1}^{n} \left\|(t_j - s_j)x_j\right\| + \left\|\sum_{j=1}^{n} s_j x_j\right\| \\
&= \sum_{j=1}^{n} (t_j - s_j)\left\|x_j\right\| + \left\|\sum_{j=1}^{n} s_j x_j\right\| \\
&= \sum_{j=1}^{n} t_j\left\|x_j\right\| - \sum_{j=1}^{n} s_j\left\|x_j\right\| + \left\|\sum_{j=1}^{n} s_j x_j\right\| \\
&= \sum_{j=1}^{n} \left\|t_j x_j\right\| - \sum_{j=1}^{n} \left\|s_j x_j\right\| + \left\|\sum_{j=1}^{n} s_j x_j\right\|.
\end{aligned}$$

Therefore the inequality $f_n(s_1,\cdots,s_n) \leqslant f_n(t_1,\cdots,t_n)$ holds. This completes the proof. $\square$

This is a generalization of [8, Theorem 1]. Indeed if $x_j \neq 0$ for all $j \in \{1,\cdots,n\}$, then we see that

$$\frac{\min_{1\leqslant j\leqslant n}\|x_j\|}{\|x_j\|} \leqslant 1 \leqslant \frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_j\|} \qquad (j \in \{1,\cdots,n\}).$$

Thus, applying Theorem 2.1, we have

$$\begin{aligned}
f_n\left(\frac{\min_{1\leqslant j\leqslant n}\|x_j\|}{\|x_1\|},\cdots,\frac{\min_{1\leqslant j\leqslant n}\|x_j\|}{\|x_n\|}\right) &\leqslant f_n(1,\cdots,1) \\
&\leqslant f_n\left(\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_1\|},\cdots,\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_n\|}\right).
\end{aligned}$$

Since

$$f_n(1,\cdots,1) = \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|,$$

$$f_n\left(\frac{\min_{1\leqslant j\leqslant n}\|x_j\|}{\|x_1\|},\cdots,\frac{\min_{1\leqslant j\leqslant n}\|x_j\|}{\|x_n\|}\right) = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leqslant j\leqslant n}\|x_j\|$$

and

$$f_n\left(\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_1\|},\cdots,\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_n\|}\right) = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

we obtain [8, Theorem 1].

Next, we discuss when the equality $f_n(s_1,\cdots,s_n) = f_n(t_1,\cdots,t_n)$ holds in Theorem 2.1 in a strictly convex Banach space. To do it, we need the following lemma.

LEMMA 2.2. [8, Lemma 1] *Let $(X, \|\cdot\|)$ be a strictly convex Banach space. For each $x_j \in X \setminus \{0\}$, $j \in \{1,2,\cdots,n\}$, the following assertions are equivalent:*

(i) $\left\|\sum_{j=1}^n \alpha_j x_j\right\| = \sum_{j=1}^n \alpha_j \|x_j\|$ *with any positive numbers* $\alpha_1,\cdots,\alpha_n$*;*

(ii) $\left\|\sum_{j=1}^n \alpha_j x_j\right\| = \sum_{j=1}^n \alpha_j \|x_j\|$ *with some positive numbers* $\alpha_1,\cdots,\alpha_n$*;*

(iii) $\dfrac{x_i}{\|x_i\|} = \dfrac{x_j}{\|x_j\|} \quad (\forall i,j \in \{1,\cdots,n\})$.

THEOREM 2.3. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space and $(s_1,\cdots,s_n)$ and $(t_1,\cdots,t_n) \in \prod_{j=1}^n [0,\infty)$ satisfy $s_j \leqslant t_j$ for all $j \in \{1,\cdots,n\}$. Put $J = \{j \in \{1,\cdots,n\} : s_j < t_j\}$. Then the following assertions are equivalent:*
(i) *the equality*

$$f_n(s_1,\cdots,s_n) = f_n(t_1,\cdots,t_n) \tag{2.1}$$

holds;
(ii) *either $J = \varnothing$ or*

$$\sum_{j=1}^n \|s_j x_j\| \geqslant \sum_{j=1}^n \|t_j x_j\| - \left\|\sum_{j=1}^n t_j x_j\right\| \tag{2.2}$$

and

$$\sum_{j=1}^n s_j x_j = \left(\left\|\sum_{j=1}^n t_j x_j\right\| - \sum_{j=1}^n \|(t_j - s_j)x_j\|\right) \frac{x_i}{\|x_i\|} \text{ for some } i \in J; \text{and} \tag{2.3}$$

(iii) *either $J = \varnothing$ or*

$$\frac{x_i}{\|x_i\|} = \frac{x_j}{\|x_j\|} \quad \text{for all } i,j \in J \tag{2.4}$$

and

$$\sum_{j=1}^n s_j x_j = \left\|\sum_{j=1}^n s_j x_j\right\| \frac{x_i}{\|x_i\|} \quad \text{for some } i \in J. \tag{2.5}$$

Proof. (i)$\Rightarrow$(ii) We may assume that $J \neq \varnothing$. If (2.1) holds, then it is clear that (2.2) is valid and we need only to prove (2.3). If $\sum_{j=1}^n s_j x_j = 0$, then, by (2.1) and (2.2), we have

$$0 = \left\|\sum_{j=1}^n s_j x_j\right\| = \left\|\sum_{j=1}^n t_j x_j\right\| - \sum_{j=1}^n \|(t_j - s_j)x_j\|$$

and

$$\sum_{j=1}^{n} s_j x_j = 0 = \left(\left\| \sum_{j=1}^{n} t_j x_j \right\| - \sum_{j=1}^{n} \|(t_j - s_j)x_j\| \right) \frac{x_i}{\|x_i\|} \qquad (\forall i \in J).$$

Next, we consider the case $\sum_{j=1}^{n} s_j x_j \neq 0$. As in the proof of Theorem 2.1, we see that

$$\begin{aligned} \left\| \sum_{j=1}^{n} t_j x_j \right\| &= \left\| \sum_{j\in J} (t_j - s_j)x_j + \sum_{j=1}^{n} s_j x_j \right\| \\ &\leqslant \left\| \sum_{j\in J} (t_j - s_j)x_j \right\| + \left\| \sum_{j=1}^{n} s_j x_j \right\| \\ &\leqslant \sum_{j\in J} \left\| (t_j - s_j)x_j \right\| + \left\| \sum_{j=1}^{n} s_j x_j \right\| \\ &= \sum_{j=1}^{n} \left\| (t_j - s_j)x_j \right\| + \left\| \sum_{j=1}^{n} s_j x_j \right\| \\ &= \sum_{j=1}^{n} \left\| t_j x_j \right\| - \sum_{j=1}^{n} \left\| s_j x_j \right\| + \left\| \sum_{j=1}^{n} s_j x_j \right\|. \end{aligned} \tag{2.6}$$

Hence if (2.1) holds, then we have

$$\left\| \sum_{j\in J} (t_j - s_j)x_j + \sum_{j=1}^{n} s_j x_j \right\| = \sum_{j\in J} \left\| (t_j - s_j)x_j \right\| + \left\| \sum_{j=1}^{n} s_j x_j \right\|.$$

Applying Lemma 2.2, we see that, for each $i \in J$,

$$\frac{\sum_{j=1}^{n} s_j x_j}{\|\sum_{j=1}^{n} s_j x_j\|} = \frac{(t_i - s_i)x_i}{\|(t_i - s_i)x_i\|} = \frac{x_i}{\|x_i\|}.$$

Therefore, by (2.1), we have, for each $i \in J$,

$$\begin{aligned} \sum_{j=1}^{n} s_j x_j &= \left\| \sum_{j=1}^{n} s_j x_j \right\| \frac{x_i}{\|x_i\|} \\ &= \left(\left\| \sum_{j=1}^{n} t_j x_j \right\| - \sum_{j=1}^{n} \|(t_j - s_j)x_j\| \right) \frac{x_i}{\|x_i\|}. \end{aligned}$$

(ii)$\Rightarrow$(i) It is clear that if $J=\varnothing$, then (2.1) holds. Thus we assume that $J\neq\varnothing$. In this case, if (2.2) and (2.3) hold, then we have

$$\begin{aligned}\left\|\sum_{j=1}^n s_jx_j\right\| &= \left\|\left(\left\|\sum_{j=1}^n t_jx_j\right\| - \sum_{j=1}^n \|(t_j-s_j)x_j\|\right)\frac{x_i}{\|x_i\|}\right\| \\ &= \left|\left\|\sum_{j=1}^n t_jx_j\right\| - \sum_{j=1}^n \|(t_j-s_j)x_j\|\right| \\ &= \left\|\sum_{j=1}^n t_jx_j\right\| - \sum_{j=1}^n \|t_jx_j\| + \sum_{j=1}^n \|s_jx_j\|\end{aligned}$$

and the equality (2.1) holds.

(iii)$\Rightarrow$(i) Assume that (2.4) and (2.5) hold. Then, by (2.4), we see that, for each $i,j\in J$,

$$\frac{(t_i-s_i)x_i}{\|(t_i-s_i)x_i\|} = \frac{(t_j-s_j)x_j}{\|(t_j-s_j)x_j\|}.$$

By Lemma 2.2, we have

$$\left\|\sum_{j\in J}(t_j-s_j)x_j\right\| = \sum_{j\in J}\|(t_j-s_j)x_j\|.$$

Hence, if $\sum_{j=1}^n s_jx_j=0$, then

$$\begin{aligned}\left\|\sum_{j=1}^n t_jx_j\right\| &= \left\|\sum_{j=1}^n (t_j-s_j)x_j + \sum_{j=1}^n s_jx_j\right\| \\ &= \left\|\sum_{j\in J}^n (t_j-s_j)x_j\right\| = \sum_{j\in J}\|(t_j-s_j)x_j\| \\ &= \sum_{j=1}^n \|(t_j-s_j)x_j\| = \sum_{j=1}^n \|t_jx_j\| - \sum_{j=1}^n \|s_jx_j\| \\ &= \sum_{j=1}^n \|t_jx_j\| - \sum_{j=1}^n \|s_jx_j\| + \left\|\sum_{j=1}^n s_jx_j\right\|.\end{aligned}$$

Thus we have (2.1).

On the other hand, if $\sum_{j=1}^n s_jx_j\neq 0$, then, by (2.4) and (2.5), we see that, for each $i,j\in J$,

$$\frac{\sum_{j=1}^n s_ix_i}{\left\|\sum_{j=1}^n s_ix_i\right\|} = \frac{x_j}{\|x_j\|} = \frac{(t_j-s_j)x_j}{\|(t_j-s_j)x_j\|}.$$

Applying Lemma 2.2, we have

$$\left\|\sum_{j\in J}(t_j-s_j)x_j + \sum_{j=1}^n s_jx_j\right\| = \sum_{j\in J}\|(t_j-s_j)x_j\| + \left\|\sum_{j=1}^n s_jx_j\right\|$$

and

$$\left\|\sum_{j=1}^{n} t_j x_j\right\| = \sum_{j=1}^{n} \|t_j x_j\| - \sum_{j=1}^{n} \|s_j x_j\| + \left\|\sum_{j=1}^{n} s_j x_j\right\|.$$

(i)$\Rightarrow$ (iii) If (2.1) holds, then all the equalities in (2.6) hold. Thus, by Lemma 2.2, we have (2.4) and (2.5). This completes the proof. $\square$

Recall that, by Theorem 2.1, if $(t_1,\cdots,t_n)$ belongs to $\prod_{j=1}^{n}[1,\infty)$, then

$$f_n(1,\cdots,1) \leqslant f_n(t_1,\cdots,t_n),$$

that is, we have the reverse triangle inequality as follows:

$$\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leqslant \sum_{j=1}^{n} \|t_j x_j\| - \left\|\sum_{j=1}^{n} t_j x_j\right\|.$$

COROLLARY 2.4. *Let $(X,\|\cdot\|)$ be a strictly convex Banach space and $(t_1,\cdots,t_n) \in \prod_{j=1}^{n}[1,\infty)$. Put $I = \{j \in \{1,\cdots,n\} : 1 < t_j\}$. Then the following assertions are equivalent:*
(i) *the equality*

$$f_n(1,\cdots,1) = f_n(t_1,\cdots,t_n)$$

holds;
(ii) *either $I = \varnothing$ or*

$$\sum_{j=1}^{n} \|x_j\| \geqslant \sum_{j=1}^{n} \|t_j x_j\| - \left\|\sum_{j=1}^{n} t_j x_j\right\|$$

and

$$\sum_{j=1}^{n} x_j = \left(\left\|\sum_{j=1}^{n} t_j x_j\right\| - \sum_{j=1}^{n} \|(t_j-1)x_j\|\right) \frac{x_i}{\|x_i\|} \quad \textit{for some } i \in I;$$

and
(iii) *either $I = \varnothing$ or*

$$\frac{x_i}{\|x_i\|} = \frac{x_j}{\|x_j\|} \quad \textit{for all } i,j \in I$$

and

$$\sum_{j=1}^{n} x_j = \left\|\sum_{j=1}^{n} x_j\right\| \frac{x_i}{\|x_i\|} \quad \textit{for some } i \in I.$$

In Corollary 2.4, if we put

$$t_j = \frac{\max_{1\leqslant j\leqslant n} \|x_j\|}{x_j} \qquad (\forall j \in \{1,\cdots,n\}),$$

then we can obtain Theorem 1.2 [8, Theorem 3].

COROLLARY 2.5. (cf. [8, Theorem 3]) *Let* $(X,\|\cdot\|)$ *be a strictly convex Banach space and* $x_1,\cdots,x_n$ *belong to* $X\setminus\{0\}$. *If we put* $I=\{j\in\{1,\cdots,n\}:\ \|x_j\|<\max_{1\leqslant j\leqslant n}\|x_j\|\}$, *then the following assertions are equivalent:*
(i) *the equality*

$$\sum_{j=1}^n\|x_j\|-\left\|\sum_{j=1}^n x_j\right\|=\left(n-\left\|\sum_{j=1}^n\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

holds;
(ii) *either* $I=\varnothing$ $($*equivalently* $\|x_i\|=\|x_j\|$ *for all* $i,j\in\{1,\cdots,n\})$ *or*

$$\sum_{j=1}^n\|x_j\|\geqslant\left(n-\left\|\sum_{j=1}^n\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

and

$$\sum_{j=1}^n x_j=\left\{\left(n-\left\|\sum_{j=1}^n\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|+\sum_{j=1}^n\|x_j\|\right\}\frac{x_i}{\|x_i\|}\quad \textit{for some } i\in I;$$

and
(iii) *either* $I=\varnothing$ *or*

$$\frac{x_j}{\|x_j\|}=\frac{x_i}{\|x_i\|}\quad \textit{for all } i,j\in I$$

and

$$\sum_{j=1}^n x_j=\left\|\sum_{j=1}^n x_j\right\|\frac{x_i}{\|x_i\|}\quad \textit{for some } i\in I.$$

3. Inequalities in inner product spaces

Throughout this section, let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$. Note that all results in §2 are valid for $(H,\langle\cdot,\cdot\rangle)$. The following result makes it possible to connect Kato, Saito and Tamura's inequality to Dragomir's one.

LEMMA 3.1. *Let* $x_j\in H\setminus\{0\}$, $j\in\{1,\cdots,n\}$ *and* $(s_1,\cdots,s_n)\in\prod_{j=1}^n[0,\infty)$. *For each* $i\in\{1,\cdots,n\}$,

$$\lim_{t\to\infty}f_n(s_1,\cdots,\underset{i}{\hat{t}},\cdots,s_n)=\sum_{j=1}^n\left(\|s_jx_j\|-\mathrm{Re}\left\langle\frac{x_i}{\|x_i\|},s_jx_j\right\rangle\right).$$

Proof. Take $t > 0$ and put, for $i \in \{1, \cdots, n\}$, $X_i = \sum_{j \neq i} s_j x_j$. Then we see that

$$\begin{aligned}
f_n(s_1, \cdots, \underset{\substack{\wedge \\ i}}{t}, \cdots, s_n) &= \left(\sum_{j \neq i}^{n} \|s_j x_j\| + \|t x_i\| \right) - \left\| \sum_{j \neq i} s_j x_j + t x_i \right\| \\
&= \sum_{j \neq i}^{n} \|s_j x_j\| + (\|t x_i\| - \|X_i + t x_i\|) \\
&= \sum_{j \neq i}^{n} \|s_j x_j\| + \frac{(\|t x_i\| - \|X_i + t x_i\|)(\|t x_i\| + \|X_i + t x_i\|)}{\|t x_i\| + \|X_i + t x_i\|} \\
&= \sum_{j \neq i} \|s_j x_j\| - \frac{2\mathrm{Re}\,\langle x_i, X_i \rangle + \|\frac{1}{t} X_i\|^2}{\|x_i\| + \left\|\frac{1}{t} X_i + x_i\right\|} \\
&\longrightarrow \sum_{j \neq i} \|s_j x_j\| - \frac{\mathrm{Re}\,\langle x_i, X_i \rangle}{\|x_i\|} \qquad (as\ t \to \infty).
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{t \to \infty} f_n(s_1, \cdots, \underset{\substack{\wedge \\ i}}{t}, \cdots, s_n) &= \sum_{j \neq i} \|s_j x_j\| - \frac{\mathrm{Re}\,\langle x_i, X_i \rangle}{\|x_i\|} \\
&= \sum_{j \neq i} \|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, X_i \right\rangle \\
&= \sum_{j \neq i} \|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, (X_i + s_i x_i) - s_i x_i \right\rangle \\
&= \sum_{j \neq i} \|s_j x_j\| - \left(\mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} s_j x_j \right\rangle - s_i \|x_i\| \right) \\
&= \sum_{j=1}^{n} \|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} s_j x_j \right\rangle \\
&= \sum_{j=1}^{n} \left(\|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right). \quad \square
\end{aligned}$$

THEOREM 3.2. *Let* $x_j \in H \setminus \{0\}$, $j \in \{1, \cdots, n\}$ *and* $(s_1, \cdots, s_n) \in \prod_{j=1}^{n} [1, \infty)$. *Then, for each* $i \in \{1, \cdots, n\}$,

$$\sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leqslant \sum_{j=1}^{n} \left(\|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right). \tag{3.1}$$

The equality holds in (3.1) *if and only if*

$$\sum_{j=1}^{n} \|x_j\| \geqslant \sum_{j=1}^{n} \left(\|s_j x_j\| - \mathrm{Re} \left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right) \tag{3.2}$$

and

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \left(\operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right) \frac{x_i}{\|x_i\|}. \tag{3.3}$$

Proof. Since $1 \leqslant s_j$ $(j \in \{1,\cdots,n\})$, by Theorem 2.1, we have

$$f_n(1,\cdots,1) \leqslant f_n(s_1,\cdots,s_n) \leqslant \lim_{t\to\infty} f_n(s_1,\cdots,\underset{i}{\underset{\wedge}{t}},\cdots,s_n).$$

Thus, by Lemma 3.1, we deduce the desired inequality (3.1).

Assume that (3.2) and (3.3) hold. Since

$$\begin{aligned}
&\sum_{j=1}^{n} \left(\operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right) \\
&= \sum_{j=1}^{n} \left(\operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - (s_j-1)\|x_j\| \right) \\
&= \sum_{j=1}^{n} \left(\|x_j\| + \operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|s_j x_j\| \right) \\
&= \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} \left(\|s_j x_j\| - \operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right) \\
&\geqslant 0,
\end{aligned}$$

we have

$$\begin{aligned}
\left\| \sum_{j=1}^{n} x_j \right\| &= \left| \sum_{j=1}^{n} \left(\operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right) \right| \cdot \left\| \frac{x_i}{\|x_i\|} \right\| \\
&= \sum_{j=1}^{n} \left(\operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right) \\
&= \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} \left(\|s_j x_j\| - \operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right)
\end{aligned}$$

and the equality in (3.1) holds.

Conversely, if the equality holds in (3.1), then it is clear that (3.2) is valid and we need only to prove (3.3). Since f_n has monotonicity property, we see that

$$f_n(1,\cdots,1) \leqslant \lim_{t\to\infty} f_n(1,\cdots,\underset{i}{\underset{\wedge}{t}},\cdots,1) \leqslant \lim_{t\to\infty} f_n(s_1,\cdots,\underset{i}{\underset{\wedge}{t}},\cdots,s_n). \tag{3.4}$$

If the equality holds in (3.1), then both equalities in (3.4) hold, that is,

$$\begin{aligned}
\sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| &= \sum_{j=1}^{n} \left(\|x_j\| - \operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j \right\rangle \right) &(3.5)\\
&= \sum_{j=1}^{n} \left(\|s_j x_j\| - \operatorname{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right). &(3.6)
\end{aligned}$$

By the equality in (3.5), we obtain

$$\left\|\sum_{j=1}^{n} x_j\right\| = \sum_{j=1}^{n} \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j \right\rangle = \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} x_j \right\rangle. \tag{3.7}$$

We note that in Schwarz's type inequality $\mathrm{Re}\langle u, v\rangle \leqslant \|u\|\cdot\|v\|$ $(u, v \in H)$ the case of equality holds iff there exists a $\lambda \geqslant 0$ such that $u = \lambda v$. Consequently, the equality holds in (3.7) iff there exists a $\lambda \geqslant 0$ such that

$$\sum_{j=1}^{n} x_j = \lambda \frac{x_i}{\|x_i\|}.$$

Hence, by (3.7), we have

$$\lambda = \left\|\lambda \frac{x_i}{\|x_i\|}\right\| = \left\|\sum_{j=1}^{n} x_j\right\| = \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} x_j \right\rangle$$

and on the other hand, by (3.6), we get

$$\begin{aligned}
\mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} x_j \right\rangle &= \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} \left(\|s_j x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \right) \\
&= \sum_{j=1}^{n} \left(\mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - s_j\|x_j\| + \|x_j\| \right) \\
&= \sum_{j=1}^{n} \left(\mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right).
\end{aligned}$$

Therefore we have

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \left(\mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle - \|(s_j-1)x_j\| \right) \frac{x_i}{\|x_i\|}.$$

This completes the proof. □

Note that Theorem 3.2 is a generalization of [3, Theorem 7]. Indeed, we may state that if, for each fixed $i \in \{1, \cdots, n\}$, $k_j^i \geqslant 0$, $j \in \{1, \cdots, n\}$, satisfies the condition

$$\|s_j x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_j x_j \right\rangle \leqslant k_j^i,$$

then the following inequality holds;

$$\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leqslant \sum_{j=1}^{n} k_j^i. \tag{3.8}$$

The equality holds in (3.8) if and only if

$$\sum_{j=1}^{n} \|x_j\| \geqslant \sum_{j=1}^{n} k_j^i$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} k_j^i\right) \frac{x_i}{\|x_i\|}.$$

As a corollary, we obtain the following result similar to [3, Theorem 7].

COROLLARY 3.3. *Let* $x_j \in H\setminus\{0\}$, $j \in \{1,\cdots,n\}$. *If, for each fixed* $i \in \{1,\cdots,n\}$, $k_j^i \geqslant 0$, $j \in \{1,\cdots,n\}$, *are such that*

$$\|x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j \right\rangle \leqslant k_j^i,$$

then

$$\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leqslant \sum_{j=1}^{n} \left(\|x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j \right\rangle\right) \leqslant \sum_{j=1}^{n} k_j^i. \tag{3.9}$$

All equalities in (3.9) *hold if and only if*

$$\sum_{j=1}^{n} \|x_j\| \geqslant \sum_{j=1}^{n} k_j^i$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{n} k_j^i\right) \frac{x_i}{\|x_i\|}.$$

We are interested in the relation between Theorem 1.1 and Corollary 3.3. Recall that

$$f_n\left(\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_1\|}, \cdots, \frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_n\|}\right) = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leqslant j\leqslant n} \|x_j\|$$

and

$$\lim_{t\to\infty} f_n(1,\cdots,\underset{i}{\overset{t}{\wedge}},\cdots,1) = \sum_{j=1}^{n} \left(\|x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j \right\rangle\right).$$

Therefore, it might seem to be impossible to do it. Indeed, we have the following

EXAMPLE 3.4. Let $X = \mathbb{R}^2$. If we take $x_1 = \frac{1}{2}(\cos\theta, \sin\theta)$, $x_2 = \frac{1}{2}(0,1)$, $x_3 = (1,0)$, then $\|x_1\| = \|x_2\| = \frac{1}{2} < 1 = \|x_3\|$. Put

$$g(\theta) = f_3\left(\frac{\|x_3\|}{\|x_1\|}, \frac{\|x_3\|}{\|x_2\|}, \frac{\|x_3\|}{\|x_3\|}\right) - \lim_{t\to\infty} f_3(t,1,1).$$

Then we see that

$$g(\theta) = \frac{3}{2} + \cos\theta + \frac{1}{2}\sin\theta - \sqrt{3 + 2(\cos\theta + \sin\theta)}.$$

In the case of $\theta = \frac{\pi}{4}$, we have

$$g\left(\frac{\pi}{4}\right) = \frac{3}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} - \sqrt{3+2\sqrt{2}}$$
$$= (\sqrt{2}+1)\left(\frac{3}{2\sqrt{2}} - 1\right) > 0.$$

Hence we get

$$f_3\left(\frac{\|x_3\|}{\|x_1\|}, \frac{\|x_3\|}{\|x_2\|}, \frac{\|x_3\|}{\|x_3\|}\right) > \lim_{t\to\infty} f_3(t,1,1).$$

On the other hand, in the case of $\theta = \frac{3\pi}{4}$, we have

$$g\left(\frac{3\pi}{4}\right) = \frac{3}{2} - \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} - \sqrt{3}$$
$$= \frac{3}{2} - \sqrt{3} - \frac{1}{2\sqrt{2}} < 0.$$

Thus we get

$$f_3\left(\frac{\|x_3\|}{\|x_1\|}, \frac{\|x_3\|}{\|x_2\|}, \frac{\|x_3\|}{\|x_3\|}\right) < \lim_{t\to\infty} f_3(t,1,1).$$

It is possible to compare this with Theorem 1.1 and Corollary 1.3 in the special case.

THEOREM 3.5. *Let* $x_1,\cdots,x_n \in H\setminus\{0\}$ *and*

$$I = \left\{ i \in \{1,\cdots,n\} : \ \|x_i\| < \max_{1\leqslant j\leqslant n} \|x_j\| \right\}.$$

Then
(i) *if* $I = \varnothing$, *for each* $i \in \{1,\cdots,n\}$, *the following holds*

$$\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leqslant j\leqslant n} \|x_j\|$$
$$\leqslant \sum_{j=1}^{n} \left(\|x_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, x_j\right\rangle\right);$$

(ii) *if* $I = \{i_0\}$ *for some* $i_0 \in \{1,\cdots,n\}$, *then the following inequalities hold*

$$\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leqslant \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leqslant j\leqslant n} \|x_j\|$$
$$\leqslant \sum_{j=1}^{n} \left(\|x_j\| - \mathrm{Re}\left\langle \frac{x_{i_0}}{\|x_{i_0}\|}, x_j\right\rangle\right).$$

Proof. (ii) Since $\|x_i\| = \|x_j\|$ $(i, j \in I)$, we see that

$$f_n\left(1,\cdots,1,\frac{\|x_n\|}{\|x_{i_0}\|},1,\cdots,1\right) = f_n\left(\frac{\|x_n\|}{\|x_1\|},\cdots,\frac{\|x_n\|}{\|x_n\|}\right)$$
$$= \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

and

$$\lim_{t\to\infty} f_n(1,\cdots,1,\underset{i_0}{\hat{t}},1,\cdots,\cdots,1) = \sum_{j=1}^{n}\left(\|x_j\| - \operatorname{Re}\left\langle\frac{x_{i_0}}{\|x_{i_0}\|},x_j\right\rangle\right).$$

Thus we have

$$\left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\| = f_n\left(1,\cdots,1,\frac{\|x_n\|}{\|x_{i_0}\|},1,\cdots,1\right)$$
$$\leqslant \lim_{t\to\infty} f_n(1,\cdots,1,\underset{i_0}{\hat{t}},1,\cdots,\cdots,1)$$
$$= \sum_{j=1}^{n}\left(\|x_j\| - \operatorname{Re}\left\langle\frac{x_{i_0}}{\|x_{i_0}\|},x_j\right\rangle\right).$$

This completes the proof. $\square$

COROLLARY 3.6. *Let $x_1, x_2 \in H\setminus\{0\}$ with $\|x_1\| \leqslant \|x_2\|$. Then the following inequalities hold*

$$\|x_1\| + \|x_2\| - \|x_1 + x_2\| \leqslant \left(2 - \left\|\frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|}\right\|\right)\|x_2\|$$
$$\leqslant \|x_1\| + \|x_2\| - \operatorname{Re}\left\langle\frac{x_1}{\|x_1\|}, x_1 + x_2\right\rangle.$$

Finally, as an analogue of Theorem 1.1, we may state the following result as well:

THEOREM 3.7. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space, and $x_j \in H\setminus\{0\}$, $j \in \{1,\cdots,n\}$. Then, for each fixed $i \in \{1,\cdots,n\}$,*

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \leqslant \left(n - \operatorname{Re}\left\langle\frac{x_i}{\|x_i\|},\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\rangle\right)\max_{1\leqslant j\leqslant n}\|x_j\|. \tag{3.10}$$

The equality holds in (3.10) if and only if

$$\sum_{j=1}^{n}\|x_j\| \geqslant \left(n - \operatorname{Re}\left\langle\frac{x_i}{\|x_i\|},\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\rangle\right)\max_{1\leqslant j\leqslant n}\|x_j\|$$

and

$$\sum_{j=1}^{n}x_j = \left\{\sum_{j=1}^{n}\|x_j\| - \left(n - \operatorname{Re}\left\langle\frac{x_i}{\|x_i\|},\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\rangle\right)\max_{1\leqslant j\leqslant n}\|x_j\|\right\}\frac{x_i}{\|x_i\|}.$$

Proof. If we put $s_j = \frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_j\|}$ $(j \in \{1,\cdots,n\})$, then $s_j \geqslant 1$ $(j \in \{1,\cdots,n\})$. Thus, by Theorem 2.1 and Lemma 3.1, we have

$$\begin{aligned}
f_n(1,\cdots,1) &\leqslant \lim_{t\to\infty} f_n(s_1,\cdots,\underset{i}{\overset{t}{\wedge}},\cdots,s_n) \\
&= \sum_{j=1}^{n}\left(\|s_jx_j\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, s_jx_j\right\rangle\right) \\
&= \sum_{j=1}^{n}\left(\left\|\frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_j\|}x_j\right\| - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \frac{\max_{1\leqslant j\leqslant n}\|x_j\|}{\|x_j\|}x_j\right\rangle\right) \\
&= \sum_{j=1}^{n}\left(1 - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \frac{x_j}{\|x_j\|}\right\rangle\right)\max_{1\leqslant j\leqslant n}\|x_j\| \\
&= \left(n - \mathrm{Re}\left\langle \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\rangle\right)\max_{1\leqslant j\leqslant n}\|x_j\|. \qquad \square
\end{aligned}$$

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