

WEIGHTED HARDY-TYPE INEQUALITIES ON THE CONE OF QUASI-CONCAVE FUNCTIONS

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Abstract. The paper is devoted to the study of weighted Hardy-type inequalities on the cone of quasi-concave functions, which is equivalent to the study of the boundedness of the Hardy operator between the Lorentz Γ -spaces. For described inequalities we obtain necessary and sufficient conditions to hold for parameters $q \geq 1$, $p > 0$ and sufficient conditions for the rest of the range of parameters.

1. Introduction

In the remarkable paper [21] G.G. Lorentz introduced and characterized the basic properties of new function spaces $\Lambda_p(\phi)$ defined in terms of rearrangement

$$f^*(t) := \inf\{s > 0 : \text{mes}\{x : |f(x)| > s\} \leq t\}$$

of a function f on the semiaxis in decreasing order. More exactly, we say that $f \in \Lambda_p(\phi)$, if

$$\|f\|_{\Lambda_p(\phi)} := \left(\int_0^\infty [f^*]^p \phi \right)^{1/p} < \infty.$$

Since then the Lorentz spaces have become an important tool in various branches of analysis and its applications. In particular, the study of mapping properties of operators of classical analysis in Lorentz spaces has started. First of all, it concerns Interpolation Theory [4], where the Lorentz spaces play a crucial role in the extension of the classical Marcinkiewicz theorem. In fact, they appear naturally in the real interpolation method as intermediate spaces for L^p -spaces (see e.g. [4] and [5]). In spite of the fact that the original definition of $\Lambda_p(\phi)$ contains an arbitrary weight function ϕ , for a long period only the special case of power functions ϕ was used. A breakthrough happened in 1990, when M. Ariño and B. Muckenhoupt [1] characterized $\Lambda_p(\phi) \hookrightarrow \Lambda_p(\phi)$ property of the Hardy-Littlewood maximal operator M in the case $1 < p < \infty$ using the equivalence $(Mf)^* \approx f^{**}$, where

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) ds$$

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(see e.g. [2] for some simple proofs and historical facts concerning this estimate). It took about 15 years to find a precise characterization of $\Lambda_p(\phi) \leftrightarrow \Lambda_q(\psi)$ mapping properties of the maximal operator for all $0 < p, q < \infty$ and arbitrary weight functions ϕ and ψ (see [3], [6], [7], [12], [13], [17], [29], [31], [32] [33])). On the way E.T. Sawyer [29] introduced Lorentz Γ -spaces $\Gamma_p(v)$ such that $f \in \Gamma_p(v)$, if

$$\|f\|_{\Gamma_p(v)} := \left(\int_0^\infty [f^{**}]^p v \right)^{1/p} < \infty.$$

Similar to Λ -analysis, mapping properties of classical operators in the Lorentz Γ -spaces became a challenging task of numerous investigations (see, for example, research papers [6], [8], [14], [15], [16], [26], [30], [32] and monographs [19], [20]).

It has also become convenient to use functions with two different conditions of monotonicity ([15], [22], [30]). In particular, functions $u(t)$ are used, such that $u(t)$ is non-decreasing and $\frac{u(t)}{t}$ is non-increasing. It is known [5] that such functions are equivalent to concave functions and called *quasi-concave*. Some new information and historical remarks concerning such functions can be found in the paper [25].

The main motivation of this paper is to find the integral criteria of $\Gamma_p(v) \leftrightarrow \Gamma_q(w)$ boundedness for the Hardy–Littlewood maximal operator. This problem was first studied in [32] in the diagonal case $1 < p = q < \infty, v = w$. The next result was obtained in the paper [14] with answers in terms of implicit sequences as well as in the paper [15]. Later on, G.Sinnamon [30] gave integral criteria for the case $1 < p, q < \infty$ using a reduction principle for the inequalities on the cone of quasi-concave functions. In this paper we extend Sinnamon’s result to cover also the case $0 < p < 1$ and give an alternative approach for the other cases. Moreover, we obtain results not only for quasi-concave functions, but for a more generalized class of functions.

Let $\mathbb{R}_+ := [0, \infty)$ and \mathfrak{M}^+ be the class of all measurable functions $f: \mathbb{R}_+ \rightarrow [0, +\infty]$. Let ψ be a continuous strictly increasing function on $[0, \infty)$ such that $\psi(0) = 0$ and $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Such functions are called *admissible*. A function $u(t)$, such that $u(t)$ is non-decreasing and $\frac{u(t)}{\psi(t)}$ is non-increasing, is called ψ -*quasi-concave*. For an admissible function ψ let Ω_ψ be the subset of measurable functions $f \in \mathfrak{M}^+$ such that $f(t)$ is non-increasing and $\psi(t)f(t)$ is non-decreasing.

Let $u, v, w \in \mathfrak{M}^+$ be weights, $0 < p < \infty$. In Section 2 we study the inequality of the form

$$\left(\int_{[0, \infty)} (Af)^q w \right)^{\frac{1}{q}} \leq C \left(\int_{[0, \infty)} f^p v \right)^{\frac{1}{p}}, \quad f \in \Omega_\psi, \tag{1}$$

where

$$Af(t) = \left(\int_{[0, t]} f^p u \right)^{\frac{1}{p}}. \tag{2}$$

In Theorem 1 we obtain the necessary and sufficient conditions for this inequality to hold for parameters $q \geq 1, p > 0$. The sufficient conditions for the rest of parameters (more exactly, $0 < p \leq q < 1, 0 < q < p \leq 1, 0 < q < 1 < p < \infty$) are derived in Theorem 2. For a natural analog of (1) for $p = \infty$ or $q = \infty$ see [6], Section 2.

We also study the inequality (1), where the operator from (2) is replaced with the complementary operator

$$Bf(t) = \left(\int_{[t, \infty)} f^p u \right)^{\frac{1}{p}} \tag{3}$$

which is important in the interpolation theory ([4], [5]) and other areas. In this case we obtain necessary and sufficient conditions (Theorem 4) for the same range of p and q as in Theorem 1 using an extension of Sinnamon’s reduction theorem (see Theorem 3 and Lemma 1), and also make Remark 1 concerning reduction of the inequality (1) with the operator (2) to the one with the operator (3).

Throughout the paper expressions of the type $0 \cdot \infty$ are taken to equal 0. The relation $A \ll B$ means that $A \leq cB$ with a constant c depending only on the parameter of summation. We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$ and we use the symbol \mathbb{Z} for the set of all integers. We set $p' := p/(p - 1)$ for $0 < p < \infty$, $p \neq 1$ and use the notation $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ for $0 < q < p < \infty$. $L^p(u)$ denotes the set of all measurable functions f on $(0, \infty)$ such that $\|f\|_{p,u} := \left(\int_{(0, \infty)} |f|^p u \right)^{\frac{1}{p}} < \infty$. The constant C may be different at different occurrences.

2. Main results

First of all, we note that the inequality (1) for both operators (2) and (3) can be reduced to the case $\rho = 1$ if we substitute f^p by f because $f \in \Omega_\psi \Leftrightarrow f^p \in \Omega_{\psi^p}$.

Let u, v and w be weights.

We say that a measure $v(x)dx$ is *non-degenerate* with respect to some admissible function ϕ , if for every $t \in (0, \infty)$

$$\int_{[0, \infty)} \frac{v(s)ds}{\phi(s) + \phi(t)} < \infty, \quad \int_{[0,1]} \frac{v(s)ds}{\phi(s)} = \int_{[1, \infty)} v(s)ds = \infty.$$

THEOREM 1. *Let $q \geq 1, p > 0$ and the measure $v(x)dx$ be non-degenerate with respect to the function $\psi^p(x)$. Put*

$$\begin{aligned} V(t) &:= \int_{[0,t]} v, \quad \mathbf{U}(t) := \int_{[0,t]} u, \\ \mathbf{V}(t) &:= V(t) + \psi(t)^p \int_{[t, \infty)} \psi(s)^{-p} v(s)ds, \\ \mathbb{V}(t) &:= \mathbf{V}(t)^{-p'-1} V(t) \int_{[t, \infty)} \psi(s)^{-p} v(s)ds \end{aligned}$$

and

$$U(t, y) := \int_{[y,t]} \frac{u(s)}{\psi(s)} ds.$$

Then the inequality

$$\left(\int_{[0,\infty)} \left(\int_{[0,t]} f(s)u(s)ds \right)^q w(t)dt \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p v(t)dt \right)^{\frac{1}{p}} \tag{4}$$

holds for all functions $f \in \Omega_\Psi$ if and only if

(i) $A_1 < \infty$, if $q = 1, 0 < p \leq 1$, where

$$A_1 := \sup_{t \geq 0} \left(\int_{[0,t]} \left(\int_{[y,\infty)} U(z,y)w(z)dz \right) d\Psi(y) \right) \mathbf{V}(t)^{-\frac{1}{p}};$$

(ii) $A_2 < \infty$, if $q = 1, 1 < p < \infty$, where

$$A_2 := \left(\int_{[0,\infty)} \left(\int_{[0,t]} \left(\int_{[z,\infty)} U(s,z)w(s)ds \right) d\Psi(z) \right)^{p'} \nabla(t)d(\Psi^p(t)) \right)^{\frac{1}{p'}};$$

(iii) $A_3 < \infty$, if $q > 1, 0 < p \leq 1$, where

$$A_3 := \sup_{t \geq 0} \left(\int_{[0,\infty)} \left(\int_0^{\min(s,t)} U(s,y)d\Psi(y) \right)^q w(s)ds \right)^{\frac{1}{q}} \mathbf{V}(t)^{-\frac{1}{p}};$$

(iv) $A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty$, if $1 < p \leq q < \infty$, where

$$A_{4,1} := \sup_{t \geq 0} \left(\int_{[t,\infty)} \nabla(s)d(\Psi^p(s)) \right)^{\frac{1}{p'}} \left(\int_{[0,t]} \mathbf{U}^q(s)w(s)ds \right)^{\frac{1}{q}},$$

$$A_{4,2} := \sup_{t \geq 0} \left(\int_{[0,t]} U(t,s)^{p'} \nabla(s)d(\Psi^{pp'}(s)) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} w(s)ds \right)^{\frac{1}{q}},$$

$$A_{4,3} := \sup_{t \geq 0} \left(\int_{[0,t]} \nabla(s)d(\Psi^{pp'}(s)) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} U(y,t)^q w(y)dy \right)^{\frac{1}{q}},$$

$$A_{4,4} := \sup_{t \geq 0} \left(\int_{[0,t]} \nabla(s)\mathbf{U}(s)^{p'} d(\Psi^p(s)) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} w(s)ds \right)^{\frac{1}{q}};$$

(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

$$A_{5,1} := \left(\int_{[0,\infty)} \left(\int_{[0,x]} \mathbf{U}^q(s)w(s)ds \right)^{\frac{q}{r}} \left(\int_{[x,\infty)} \nabla(s)d(\Psi^p(s)) \right)^{\frac{r}{q}} \nabla(x)d(\Psi^p(x)) \right)^{\frac{1}{r}},$$

$$A_{5,2} := \left(\int_{[0,\infty)} \left(\int_{[0,t]} U(t,x)^{p'} \nabla(x)d(\Psi^{pp'}(x)) \right)^{\frac{p'}{r}} \left(\int_{[t,\infty)} w(x)dx \right)^{\frac{r}{p}} w(t)dt \right)^{\frac{1}{r}},$$

$$A_{5,3} := \left(\int_{[0,\infty)} \left(\int_{[0,t]} \nabla(s)d(\Psi^{pp'}(s)) \right)^{\frac{r}{q}} \left(\int_{[t,\infty)} U(y,t)^q w(y)dy \right)^{\frac{r}{q}} \nabla(t)d(\Psi^{pp'}(t)) \right)^{\frac{1}{r}},$$

$$A_{5,4} := \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} w(t) dt \right)^{\frac{1}{q}} \left(\int_{[0,x]} \nabla(t) \mathbf{U}(t)^{p'} d(\psi^p(t)) \right)^{\frac{1}{q'}} \nabla(x) U(x)^{p'} d(\psi^p(x)) \right)^{\frac{1}{r}}.$$

Proof. Let $f(t) \in \Omega_\psi$. Then by Lemma 2.8 in [8] there exists a non-increasing function $g(t)$ such that

$$f(t) \approx \frac{1}{\psi(t)} \int_{[0,t]} g(s) d\psi(s).$$

We denote the least possible constant C in (4) by $H_A(p, q)$, that is

$$H_A(p, q) := \sup_{f \in \Omega_\psi} \frac{\left(\int_{[0,\infty)} \left(\int_{[0,t]} f(s) u(s) ds \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f(t)^p v(t) dt \right)^{\frac{1}{p}}}.$$

Then we have

$$H_A(p, q) \approx \sup_{g \downarrow} \frac{\left(\int_{[0,\infty)} \left(\int_{[0,t]} \left(\frac{1}{\psi(s)} \int_{[0,s]} g d\psi \right) u(s) ds \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}}. \tag{5}$$

We write

$$\int_{[0,t]} \left(\frac{1}{\psi(s)} \int_{[0,s]} g d\psi \right) u(s) ds = \int_{[0,t]} U(t, y) g(y) d\psi(y).$$

Thus,

$$H_A(p, q) \approx \sup_{g \downarrow} \frac{\left(\int_{[0,\infty)} \left(\int_{[0,t]} U(t, y) g(y) d\psi(y) \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}}.$$

For $q = 1$ we obtain

$$H_A(p, 1) \approx \sup_{g \downarrow} \frac{\int_{[0,\infty)} \left(\int_{[y,\infty)} U(t, y) w(t) dt \right) g(y) d\psi(y)}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}},$$

and [8], Theorem 4.2 yields

$$H_A(p, 1) \approx \sup_{t \geq 0} \frac{\int_{[0,t]} \left(\int_{[y,\infty)} U(z, y) w(z) dz \right) d\psi(y)}{\left(V(t) + \psi(t)^p \int_{[t,\infty)} \psi(s)^{-p} v(s) ds \right)^{\frac{1}{p}}}$$

for $0 < p \leq 1$ and

$$H_A(p, 1) \approx \left(\int_{[0,\infty)} \psi(t)^{p'} \left[\sup_{y \geq t} \frac{1}{\psi(y)} \left(\int_{[0,y]} \left(\int_{[z,\infty)} U(t, z) w(t) dt \right) d\psi(z) \right) \right]^{p'} \times \nabla(t) d(\psi^p(t)) \right)^{\frac{1}{p'}}$$

for $1 < p < \infty$. Note that $\int_{[z,\infty)} U(t, z)w(t)dt$ is non-increasing in z on $(0, \infty)$. Hence,

$$\begin{aligned} & \sup_{y \geq t} \frac{1}{\psi(y)} \left(\int_{[0,y]} \left(\int_{[z,\infty)} U(t, z)w(t)dt \right) d\psi(z) \right) \\ &= \frac{1}{\psi(t)} \left(\int_{[0,t]} \left(\int_{[z,\infty)} U(t, z)w(t)dt \right) d\psi(z) \right), \end{aligned}$$

and we obtain

$$H_A(p, 1) \approx \left(\int_{[0,\infty)} \left(\int_{[0,t]} \left(\int_{[z,\infty)} U(t, z)w(t)dt \right) d\psi(z) \right)^{p'} \nabla(t)d(\psi^p(t)) \right)^{\frac{1}{p'}}$$

for $1 < p < \infty$. In the case $1 < q < \infty$ we have

$$\begin{aligned} H_A(p, q) &\approx \sup_{g \downarrow} \sup_{h \geq 0} \frac{\int_{[0,\infty)} \left(\int_{[0,t]} U(t, y)g(y)d\psi(y) \right) h(t)w(t)dt}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t)dt \right)^{\frac{1}{p}} \left(\int_{[0,\infty)} h^{q'} w \right)^{\frac{1}{q'}}} \\ &= \sup_{h \geq 0} \frac{1}{\left(\int_{[0,\infty)} h^{q'} w \right)^{\frac{1}{q'}}} \sup_{g \downarrow} \frac{\int_{[0,\infty)} \left(\int_{[y,\infty)} U(t, y)h(t)w(t)dt \right) g(y)d\psi(y)}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t)dt \right)^{\frac{1}{p}}}. \end{aligned}$$

If $0 < p \leq 1$, then, according to [8], Theorem 4.2 (i),

$$\begin{aligned} H_A(p, q) &\approx \sup_{h \geq 0} \frac{\|\mathbf{V}^{-\frac{1}{p}}(t) \int_{[0,t]} \left(\int_{[y,\infty)} U(s, y)h(s)w(s)ds \right) d\psi(y)\|_{\infty}}{\left(\int_{[0,\infty)} h^{q'} w \right)^{\frac{1}{q'}}} \\ &= \sup_{h \geq 0} \frac{\|\mathbf{V}^{-\frac{1}{p}}(t) \int_{[0,\infty)} \left(\int_0^{\min(s,t)} U(s, y)d\psi(y) \right) h(s)w(s)ds\|_{\infty}}{\left(\int_{[0,\infty)} h^{q'} w \right)^{\frac{1}{q'}}}. \end{aligned}$$

By using [18], Chapter XI, §1.5, Theorem 4 we get

$$H_A(p, q) \approx \sup_{t > 0} \mathbf{V}^{-\frac{1}{p}}(t) \left(\int_{[0,\infty)} \left(\int_0^{\min(s,t)} U(s, y)d\psi(y) \right)^q w(s)ds \right)^{\frac{1}{q}}.$$

If $1 < p < \infty$, then by using [8], Theorem 4.2 (ii) we find

$$H_A(p, q) \approx \sup_{h \geq 0} \frac{\left(\int_{[0,\infty)} \psi(t)^{p'} \left[\sup_{y \geq t} \frac{1}{\psi(y)} \int_{[0,y]} \Phi(s)d\psi(s) \right]^{p'} \nabla(t)d(\psi^p(t)) \right)^{\frac{1}{p'}}}{\left(\int_{[0,\infty)} h^{q'} w \right)^{\frac{1}{q'}}}, \tag{6}$$

where

$$\Phi(z) := \int_{[z, \infty)} U(t, z)h(t)w(t)dt.$$

It is easy to see that $\Phi(z) \downarrow$. Hence,

$$\sup_{y \geq t} \frac{1}{\Psi(y)} \int_{[0, y]} \Phi(s)d\psi(s) = \frac{1}{\Psi(t)} \int_{[0, t]} \Phi(s)d\psi(s),$$

and (6) is equivalent to

$$H_A(p, q) \approx \sup_{h \geq 0} \frac{\left(\int_{[0, \infty)} \left(\int_{[0, s]} \left(\int_{[z, \infty)} U(t, z)h(t)w(t)dt \right) d\psi(z) \right)^{p'} \nabla(s)d(\psi^p(t)) \right)^{\frac{1}{p'}}}{\left(\int_{[0, \infty)} h^q w \right)^{\frac{1}{q}}}.$$

Since

$$\begin{aligned} \int_{[0, s]} \left(\int_{[z, \infty)} U(t, z)h(t)w(t)dt \right) d\psi(z) &= \int_{[0, s]} \left(\int_{[z, s]} U(t, z)h(t)w(t)dt \right) d\psi(z) \\ &+ \int_{[0, s]} \left(\int_{[s, \infty)} U(t, z)h(t)w(t)dt \right) d\psi(z) =: I_1 + I_2, \end{aligned}$$

we have

$$\begin{aligned} I_1 &= \int_{[0, s]} \left(\int_{[0, t]} U(t, z)d\psi(z) \right) h(t)w(t)dt \\ &= \int_{[0, s]} h(t)w(t) \left(\int_{[0, t]} \left(\int_{[z, t]} \frac{u(\tau)}{\psi(\tau)} d\tau \right) d\psi(z) \right) dt = \int_{[0, s]} \mathbf{U}(t)h(t)w(t)dt \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{[0, s]} \left(\int_{[s, \infty)} [U(t, s) + U(s, z)]h(t)w(t)dt \right) d\psi(z) \\ &= \psi(s) \int_{[s, \infty)} U(t, s)h(t)w(t)dt + \mathbf{U}(s) \int_{[s, \infty)} h(t)w(t)dt. \end{aligned}$$

Thus, the characterization of (6) is equivalent to the following inequalities restricted to the set of non-negative functions:

$$\begin{aligned} \left(\int_{[0, \infty)} \left(\int_{[0, s]} \mathbf{U}hw \right)^{p'} \nabla(s)d(\psi^p(s)) \right)^{\frac{1}{p'}} &\leq B_1 \left(\int_{[0, \infty)} h^q w \right)^{\frac{1}{q}}, \\ \left(\int_{[0, \infty)} \left(\int_{[s, \infty)} U(t, s)h(t)w(t)dt \right)^{p'} \nabla(s)d(\psi^{pp'}(s)) \right)^{\frac{1}{p'}} &\leq B_2 \left(\int_{[0, \infty)} h^q w \right)^{\frac{1}{q}}, \\ \left(\int_{[0, \infty)} \left(\mathbf{U}(s) \int_{[s, \infty)} hw \right)^{p'} \nabla(s)d(\psi^p(s)) \right)^{\frac{1}{p'}} &\leq B_3 \left(\int_{[0, \infty)} h^q w \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$H_A(p, q) \approx B_1 + B_2 + B_3.$$

Applying well-known criteria for the weighted Hardy and Hardy-type inequalities [23], [24], [27], [28], we obtain

$$B_1 + B_2 + B_3 \approx A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4}$$

for $1 < p \leq q < \infty$ and

$$B_1 + B_2 + B_3 \approx A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4}$$

for $1 < q < p < \infty$.

For the range of parameters $0 < p \leq q < 1$, $0 < q < p \leq 1$, $0 < q < 1 < p < \infty$ integral criteria are unknown for validity of (4). Below we find the sufficient conditions for the inequality (4) to hold for $f \in \Omega_\psi$. We note that

$$\left(\int_{[0,\infty)} g^p v \right)^{\frac{1}{p}} \leq \left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}} \tag{7}$$

for any non-increasing function $g(t)$. Hence, the validity of the inequality

$$\left(\int_{[0,\infty)} \left(\int_{[0,t]} \left(\frac{1}{\psi(s)} \int_{[0,s]} g d\psi \right) u(s) ds \right)^q w(t) dt \right)^{\frac{1}{q}} \leq C_1 \left(\int_{[0,\infty)} g^p v \right)^{\frac{1}{p}} \tag{8}$$

for any non-increasing function g is sufficient for the validity of the inequality (4) for all $f \in \Omega_\psi$ and $H_A(p, q) \leq C_1$. However, if v and ψ are such that the reverse to (7) inequality holds (see, for instance [6], Theorem 4.1), then $H_A(p, q) \approx C_1$. It is easy to see that (8) is equivalent to the inequality

$$\left(\int_{[0,\infty)} \left(\int_{[0,t]} U(t, y) g(y) d\psi(y) \right)^q w(t) dt \right)^{\frac{1}{q}} \leq C_1 \left(\int_{[0,\infty)} g^p v \right)^{\frac{1}{p}}, \tag{9}$$

where $U(t, y)$ is defined in Theorem 1.

We now use [9], Theorem 5.7 (see also [10] and [11]) to estimate the constant C_1 in (9) for the range of parameters p and q mentioned above and obtain the following result:

THEOREM 2. *Let $p > 0, 0 < q < 1$. For the inequality (4) to hold for all functions $f \in \Omega_\psi$ it is sufficient that $C_1 < \infty$, where C_1 is defined as follows:*

(i) *if $0 < p \leq q < 1$, then*

$$C_1 = \sup_{x \geq 0} \left(\int_{[0,\infty)} \mathbf{U}^q(\min(x, y)) w(y) dy \right)^{\frac{1}{q}} V(x)^{-\frac{1}{p}},$$

(ii) *if $0 < q < p \leq 1$, then*

$$C_1 \approx \left(\sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} (\mathbf{U}(x_k) + \psi(x_k) U(y, x_k))^q w(y) dy \right)^{\frac{r}{q}} V(x_k)^{-\frac{r}{p}} \right)^{\frac{1}{r}},$$

(iii) if $0 < q < 1 < p < \infty$, then

$$C_1 \approx C_{1,1} + C_{1,2} + C_{1,3},$$

where

$$C_{1,1} := \left(\int_{[0,\infty)} \left(\int_{[0,x]} \mathbf{U}(y)^q w(y) dy \right)^{\frac{r}{p}} \mathbf{U}(x)^q w(x) V(x)^{-\frac{r}{p}} dx \right)^{\frac{1}{r}},$$

$$C_{1,2} := \left(\sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w \right)^{\frac{r}{q}} \right. \\ \left. \times \left(\int_{x_{k-1}}^{x_k} (\mathbf{U}(y) + \psi(y)U(x_k, y))^{p'} V(y)^{-p'} v(y) dy \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}},$$

$$C_{1,3} := \left(\sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} \left(\int_{[x_k, y]} \frac{u(s)}{\psi(s)} ds \right)^q w(y) dy \right)^{\frac{r}{q}} \right. \\ \left. \times \left(\int_{x_{k-1}}^{x_k} \mathbf{U}(y)^{p'} V(y)^{-p'} v(y) dy \right)^{\frac{r}{p'}} \right)^{\frac{1}{r}}.$$

Note that the case (i) is valid for the bigger range $0 < p \leq 1, p \leq q < \infty$. However, the sufficient condition for $0 < p \leq 1, q \geq 1$ is of no interest to us here, as we have obtained the necessary and sufficient conditions for this range of parameters in Theorem 1.

REMARK 1. To characterize the inequality (1) for the operator (3) we can reduce it to one for the operator (2) and then use the result of Theorem 1. Indeed, let

$$H_B(p, q) := \sup_{f \in \Omega_\psi} \frac{\left(\int_{[0,\infty)} \left(\int_{[t,\infty)} f u \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p v \right)^{\frac{1}{p}}}.$$

We have

$$\int_{[t,\infty)} f(s)u(s)ds = \int_{[0, \frac{1}{t}]} \frac{f(\frac{1}{s})}{s} \cdot \frac{u(\frac{1}{s})}{s} ds =: \int_{[0, \frac{1}{t}]} \tilde{f}(s)\tilde{u}(s)ds.$$

Observe that $f \in \Omega_\psi \Leftrightarrow \tilde{f} \in \Omega_{\tilde{\psi}}$ for $\tilde{f}(s) = \frac{1}{s}f\left(\frac{1}{s}\right)$. Then

$$\int_{[0,\infty)} \left(\int_{[t,\infty)} f u \right)^q w(t) dt = \int_{[0,\infty)} \left(\int_{[0, \frac{1}{t}]} \tilde{f} \tilde{u} \right)^q w(t) dt \\ = \int_{[0,\infty)} \left(\int_{[0, \frac{1}{t}]} \tilde{f} \tilde{u} \right)^q w\left(\frac{1}{t}\right) \frac{dt}{t^2} =: \int_{[0,\infty)} \left(\int_{[0, t]} \tilde{f} \tilde{u} \right)^q \tilde{w}(t) dt,$$

$$\int_{[0,\infty)} f^p(s)v(s)ds = \int_{[0,\infty)} \left[\frac{1}{s} f\left(\frac{1}{s}\right) \right]^p \cdot s^{p-2}v\left(\frac{1}{s}\right) ds =: \int_{[0,\infty)} \tilde{f}^p \tilde{v},$$

where

$$\begin{aligned} \tilde{u}(s) &= \frac{1}{s} u\left(\frac{1}{s}\right), \\ \tilde{w}(s) &= \frac{1}{s^2} w\left(\frac{1}{s}\right), \\ \tilde{v}(s) &= s^{p-2} v\left(\frac{1}{s}\right). \end{aligned}$$

Thus,

$$H_B(p, q)[u, v, w] = H_A(p, q)[\tilde{u}, \tilde{v}, \tilde{w}]$$

and applying Theorem 1 for $H_A(p, q)[\tilde{u}, \tilde{v}, \tilde{w}]$ we obtain criteria for $H_B(p, q)[u, v, w]$.

However, it is easy to see, that the functionals in Theorem 1 with weights $\tilde{u}, \tilde{v}, \tilde{w}$ are not always convenient. By this reason we find alternative characterization for the inequality (1) with the operator (3) by using a different approach, which is based on Sinnamon’s result ([30], Theorem 2.6) for the class of quasi-concave functions $\Omega_{0,1} = \left\{ f \in \mathfrak{M}^+ : f(t) \uparrow, \frac{f(t)}{t} \downarrow \right\}$.

For admissible function $\psi(t)$ we introduce the class of functions $\Omega_{\alpha,\beta}^\psi$, consisting of the functions $f(t) \in \mathfrak{M}^+$, such that $\psi(t)^\alpha f(t)$ is non-decreasing and $\psi(t)^{-\beta} f(t)$ is non-increasing. In particular, the class $\Omega_{0,1}^\psi$ consists of functions $f(t)$, such that $f(t)$ is non-decreasing and $\frac{f(t)}{\psi(t)}$ is non-increasing.

We also define the operators

$$H_{\psi,\alpha} h(x) := \psi(x)^{-\alpha} \int_{[0,x]} \psi(t)^\alpha h(t) dt$$

and

$$H^{\psi,\beta} h(x) := \psi(x)^\beta \int_{[x,\infty)} \psi(t)^{-\beta} h(t) dt.$$

For $\alpha + \beta > 0$ we also use the operator

$$H_{\psi,\alpha}^{\psi,\beta} h(x) := H_{\psi,\alpha} h(x) + H^{\psi,\beta} h(x),$$

which can be rewritten as

$$H_{\psi,\alpha}^{\psi,\beta} h(x) = \int_{[0,\infty)} \min \left(\left(\frac{\psi(t)}{\psi(x)} \right)^\alpha, \left(\frac{\psi(x)}{\psi(t)} \right)^\beta \right) h(t) dt.$$

It is easy to see that $\psi(x)^\alpha H_{\psi,\alpha}^{\psi,\beta} h(x)$ is non-decreasing and $\psi(x)^{-\beta} H_{\psi,\alpha}^{\psi,\beta} h(x)$ is non-increasing for any function $h(t) \in \mathfrak{M}^+$. Thus, $H_{\psi,\alpha}^{\psi,\beta} \mathfrak{M}^+ \subseteq \Omega_{\alpha,\beta}^\psi$.

It is also easy to check that $\int_{[0,\infty)} \left(H_{\psi,\alpha}^{\psi,\beta} h_1 \right) h_2 = \int_{[0,\infty)} h_1 \left(H_{\psi,\beta}^{\psi,\alpha} h_2 \right)$.

The following theorem characterizes the embedding $L_v^p \rightarrow L_u^q$ on $\Omega_{0,1}^\psi$ for $0 < p, q < \infty$.

THEOREM 3. (i) *If $0 < q < p < \infty$, $u, v \in \mathfrak{M}^+$, then*

$$\sup_{f \in \Omega_{0,1}^\Psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left(\int_{[0,\infty)} \left(H_{\Psi,p}^{\Psi,0} \right)^{-\frac{r}{p}} \left(H_{\Psi,q}^{\Psi,0} u \right)^{\frac{r}{p}} u \right)^{\frac{1}{r}}.$$

(ii) *If $0 < p \leq q < \infty$, $u, v \in \mathfrak{M}^+$, then*

$$\sup_{f \in \Omega_{0,1}^\Psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{t>0} \left(H_{\Psi,p}^{\Psi,0} v \right)^{-\frac{1}{p}} \left(H_{\Psi,q}^{\Psi,0} u \right)^{\frac{1}{q}}.$$

Proof. The part (i) of the theorem is proved in three steps. First we obtain the norm of the the embedding $L_v^1 \rightarrow L_u^q$ for the functions from $H_{\Psi,0}^{\Psi,1} \mathfrak{M}^+ \subset \Omega_{0,1}^\Psi$. Then we extend the result of step 1 to be valid on $\Omega_{0,1}^\Psi$. Finally, we obtain the norm of the embedding $L_v^p \rightarrow L_u^q$ on $\Omega_{0,1}^\Psi$.

Step 1. For $0 < q < 1$ we need to show the estimate

$$G := \sup_{f \in H_{\Psi,0}^{\Psi,1} \mathfrak{M}^+} \frac{\|f\|_{q,u}}{\|f\|_{1,v}} \approx \left(\int_{[0,\infty)} \left(H_{\Psi,1}^{\Psi,0} v \right)^{\frac{q}{q-1}} \left(H_{\Psi,q}^{\Psi,0} u \right)^{\frac{q}{1-q}} u \right)^{\frac{1-q}{q}}. \tag{10}$$

Since every $f \in H_{\Psi,0}^{\Psi,1} \mathfrak{M}^+$ has a representation $f = H_{\Psi,0}^{\Psi,1} h$ for some $h \in \mathfrak{M}^+$, we see, that G is the least possible constant C in the inequality

$$\left(\int_{[0,\infty)} \left(H_{\Psi,0}^{\Psi,1} h \right)^q u \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} \left(H_{\Psi,0}^{\Psi,1} h \right) v, \quad h \in \mathfrak{M}^+.$$

Since $\int_{[0,\infty)} \left(H_{\Psi,0}^{\Psi,1} h \right) v = \int_{[0,\infty)} h \left(H_{\Psi,1}^{\Psi,0} v \right)$, the last inequality is equivalent to

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} h(t) dt + \Psi(x) \int_{[x,\infty)} \frac{h(t)}{\Psi(t)} dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h(t) H_{\Psi,1}^{\Psi,0} v(t) dt.$$

By the Minkowsky inequality, $C \approx C_1 + C_2$, where C_1 and C_2 are the least constants for the following inequalities for $h \in \mathfrak{M}^+$:

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} h(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C_1 \int_{[0,\infty)} h(t) H_{\Psi,1}^{\Psi,0} v(t) dt,$$

$$\left(\int_{[0,\infty)} \left(\Psi(x) \int_{[x,\infty)} \frac{h(t)}{\Psi(t)} dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C_2 \int_{[0,\infty)} h(t) H_{\Psi,1}^{\Psi,0} v(t) dt.$$

It is easy to check that that $H_{\Psi,1}^{\Psi,0} v$ is a non-increasing function. Using [31], Theorem 3.3 with $V = H_{\Psi,1}^{\Psi,0} v$ and $U = u$, we obtain

$$C_1 \approx \left(\int_{[0,\infty)} \left(H_{\Psi,1}^{\Psi,0} v \right)^{\frac{q}{q-1}} \left(H_{\Psi,1}^{\Psi,0} u \right)^{\frac{q}{1-q}} u \right)^{\frac{1-q}{q}}.$$

For estimating C_2 we replace $\frac{h(t)}{\psi(t)}$ by $h(t)$, and the second part of [31], Theorem 3.3 with non-decreasing function $V(t) = \psi(t)H_{\psi,1}^{\psi,0}v(t)$ and $U(t) = \psi(t)^qu(t)$ yields

$$C_2 \approx \left(\int_{[0,\infty)} \left(H_{\psi,1}^{\psi,0} v \right)^{\frac{q}{q-1}} \left(H_{\psi,q} u \right)^{\frac{q}{1-q}} u \right)^{\frac{1-q}{q}}.$$

Since $C \approx C_1 + C_2$, we obtain the needed estimate (10).

Step 2. Let f be a function from $\Omega_{0,1}^\psi$. It was shown in [4], Proposition 2.5.10 that for a quasi-concave function ϕ there exists the least concave majorant $\tilde{\phi}$ and that it satisfies the estimate $1/2\tilde{\phi} \leq \phi \leq \tilde{\phi}$. The same estimate holds for the case of ψ -quasi-concave functions. Thus, there exists the smallest concave function \tilde{f} defined as the pointwise infimum of all dominating concave functions, which dominates f . In order to prove that $H_{\psi,0}^{\psi,1}\mathfrak{M}^+$ is dense in $\Omega_{0,1}^\psi$, we show that \tilde{f} is a pointwise limit of an increasing sequence of functions in $H_{\psi,0}^{\psi,1}\mathfrak{M}^+$.

Because of monotonicity of f and $\frac{f(x)}{\psi(x)}$ the limits $a = \lim_{x \rightarrow 0} f(x)$ and $b = \lim_{x \rightarrow \infty} \frac{f(x)}{\psi(x)}$ exist. Then $\tilde{f}(x) = a + b\psi(x) + g(x)$ with a concave function g such that $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow \infty} \frac{g(x)}{\psi(x)} = 0$. We take $h_n(t) := a\chi_{(0,1/n)}(t)$. Then $H_{\psi,0}^{\psi,1}h_n(x)$ is a non-decreasing sequence converging pointwise to a as $n \rightarrow \infty$. If we take $h_n(t) := b\psi(t)\chi_{(n,n+1)}(t)$, then $H_{\psi,0}^{\psi,1}h_n(x)$ converges to $b\psi(x)$ as $n \rightarrow \infty$. Now we show that $g(x)$ is also a pointwise limit of functions in $H_{\psi,0}^{\psi,1}\mathfrak{M}^+$. We have $g(x) = \int_{[0,x]} g'(t)dt$. Now, setting

$$h_n(t) := \frac{\psi(t)}{t \ln \left(\frac{n+1}{n} \right)} \left(\frac{g'(t)}{\psi'(t)} - \frac{g' \left(\frac{(n+1)t}{n} \right)}{\psi' \left(\frac{(n+1)t}{n} \right)} \right) dt,$$

we get

$$\int_{[y,\infty)} \frac{h_n(t)}{\psi(t)} dt = \left(\int_y^{(n+1)y/n} \frac{g'(t)}{t\psi'(t)} dt \right) / \left(\int_y^{(n+1)y/n} \frac{dt}{t} \right).$$

The sequence $\int_{[y,\infty)} \frac{h_n(t)}{\psi(t)} dt$ converges to $\frac{g'(y)}{\psi'(y)}$ for almost every y , when $n \rightarrow \infty$. This implies that

$$H_{\psi,0}^{\psi,1}h_n(x) = \int_{[0,x]} \left(\int_{[y,\infty)} \frac{h_n(t)}{\psi(t)} dt \right) d\psi(y)$$

converges to $\int_{[0,x]} g'(y)dy = g(x)$, when $n \rightarrow \infty$.

We use the observation above to extend the result of the first step to the cone $\Omega_{0,1}^\psi$. On the one hand, $H_{\psi,0}^{\psi,1}\mathfrak{M}^+ \subseteq \Omega_{0,1}^\psi$. On the other hand, if f_n is a sequence from $H_{\psi,0}^{\psi,1}\mathfrak{M}^+$ which converges pointwise to the least concave majorant \tilde{f} of f , then, by the Monotone Convergence Theorem,

$$\|f\|_{q,u} \leq \|\tilde{f}\|_{q,u} = \lim_{n \rightarrow \infty} \|f_n\|_{q,u} \approx \lim_{n \rightarrow \infty} \|f_n\|_{1,v} = \|\tilde{f}\|_{1,v} \leq 2\|f\|_{1,v}.$$

Thus, for $0 < q < 1$ we have

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{1,v}} \approx \left(\int_{[0,\infty)} \left(H_{\psi,1}^{\psi,0} v \right)^{\frac{q}{q-1}} \left(H_{\psi,q}^{\psi,0} u \right)^{\frac{q}{1-q}} u \right)^{\frac{1-q}{q}}.$$

Step 3. To extend this result to the case $0 < q < p < \infty$ we make the following observation:

$$f \in \Omega_{0,1}^\psi \iff g \in \Omega_{0,1}^\phi,$$

where $f(x)^p = g(x^p)$ and $\psi(x)^p = \phi(x^p)$. Then for $0 < p, q < \infty$ we have

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \left(\sup_{g \in \Omega_{0,1}^\phi} \frac{\|g\|_{q/p,U}}{\|g\|_{1,V}} \right)^{\frac{1}{p}},$$

where

$$U(x^p)d(x^p) = u(x)dx, \quad V(x^p)d(x^p) = v(x)dx.$$

Thus, for $0 < q < p < \infty$ we obtain

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left(\int_{[0,\infty)} \left(H_{\phi,1}^{\phi,0} V \right)^{-\frac{r}{p}} \left(H_{\phi,q/p}^{\phi,0} U \right)^{\frac{r}{p}} U \right)^{\frac{1}{r}}.$$

Making the substitution $t \rightarrow t^p$, we have

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left(\int_{[0,\infty)} \left(H_{\phi,1}^{\phi,0} V(t^p) \right)^{-\frac{r}{p}} \left(H_{\phi,q/p}^{\phi,0} U(t^p) \right)^{\frac{r}{p}} u(t) dt \right)^{\frac{1}{r}}.$$

Next we make the substitution $x \rightarrow x^p$ to get

$$H_{\phi,1}^{\phi,0} V(t^p) = \int_{[0,\infty)} \min \left(\frac{\phi(x)}{\phi(t^p)}, 1 \right) V(x) dx = \int_{[0,\infty)} \left(\frac{\phi(x^p)}{\phi(t^p)}, 1 \right) v(x) dx = H_{\psi,p}^{\psi,0} v(t).$$

Similarly, we have

$$H_{\phi,q/p}^{\phi,0} U(t^p) = H_{\psi,q}^{\psi,0} u(t).$$

Hence, we obtain the needed estimate

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left(\int_{[0,\infty)} \left(H_{\psi,p}^{\psi,0} v \right)^{-\frac{r}{p}} \left(H_{\psi,q}^{\psi,0} u \right)^{\frac{r}{p}} u \right)^{\frac{1}{r}}.$$

(ii) We can represent any function $f(t) \in \Omega_{0,1}^\psi$ as the integral $f(t) = \int_{[0,t]} g d\psi(s)$, where function $g(t)$ is non-increasing. Therefore, we have

$$\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \sup_{g \downarrow} \frac{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^q u(t) \psi(t)^q dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^p v(t) \psi(t)^p dt \right)^{\frac{1}{p}}},$$

and by using [8], Theorem 5.1 (i) we obtain the result for $0 < p \leq q < \infty$.

For an alternate characterization of $H_B(p, q)$ we need the following

LEMMA 1. *Let u, v be non-negative measurable functions and function ψ be admissible. Assume that the measure $v(t)dt$ is non-degenerate with respect to the function $\psi(t)^p$.*

(i) *If $0 < p \leq 1$, then*

$$\sup_{f \in \Omega_\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} \approx \sup_{x \geq 0} \left(\int_{[0,x]} u(t)dt + \psi(x) \int_{[x,\infty)} \frac{u(t)}{\psi(t)} dt \right) \mathbf{V}(x)^{-\frac{1}{p}},$$

where

$$\mathbf{V}(x) := \int_{[0,x]} v(t)dt + \psi(x)^p \int_{[x,\infty)} \frac{v(t)}{\psi(t)^p} dt.$$

(ii) *If $p > 1$, then*

$$\begin{aligned} \sup_{f \in \Omega_\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} &\approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} u(t)dt \right)^{p'} \mathbf{V}_1(x) dx \right)^{\frac{1}{p'}} \\ &+ \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{u(t)}{\psi(t)} dt \right)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{p'}}, \end{aligned}$$

where

$$\mathbf{V}_1(x) dx \approx d \left(\mathbf{V}(x)^{-\frac{p'}{p}} \right), \quad \mathbf{V}_2(x) dx \approx d \left(-\psi(x)^{p'} \mathbf{V}(x)^{-\frac{p'}{p}} \right).$$

Proof. (i) Theorem 3 yields

$$\begin{aligned} \sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} &\approx \left(\int_{[0,\infty)} \left(\frac{1}{\psi(x)^p} \int_{[0,x]} \psi(t)^p v(t) dt + \int_{[x,\infty)} v(t) dt \right)^{-\frac{p'}{p}} \right. \\ &\left. \times \left(\frac{1}{\psi(x)} \int_{[0,x]} \psi(t) u(t) dt + \int_{[x,\infty)} u(t) dt \right)^{\frac{p'}{p}} u(x) dx \right)^{\frac{1}{p'}}. \end{aligned}$$

Put $\mathbf{V}^*(x) := \frac{1}{\psi(x)^p} \int_{[0,x]} \psi(t)^p v(t) dt + \int_{[x,\infty)} v(t) dt$, then we have

$$\begin{aligned} \sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} &\approx \left(\int_{[0,\infty)} \frac{1}{\psi(x)^{\frac{p'}{p}}} \mathbf{V}^*(x)^{-\frac{p'}{p}} \left(\int_{[0,x]} \psi(t) u(t) dt \right)^{\frac{p'}{p}} u(x) dx \right)^{\frac{1}{p'}} \\ &+ \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} u \right)^{\frac{p'}{p}} \mathbf{V}^*(x)^{-\frac{p'}{p}} u(x) dx \right)^{\frac{1}{p'}} \end{aligned}$$

$$\begin{aligned}
 &= \left(\int_{[0,\infty)} \frac{1}{\psi(x)^{p'}} \mathbf{V}^*(x)^{-\frac{p'}{p}} d \left(\int_{[0,x]} \psi(t)u(t)dt \right)^{p'} \right)^{\frac{1}{p'}} \\
 &\quad + \left(\int_{[0,\infty)} \mathbf{V}^*(x)^{-\frac{p'}{p}} d \left(- \int_{[x,\infty)} u(t)dt \right)^{p'} \right)^{\frac{1}{p'}} \\
 &\approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} \psi(t)u(t)dt \right)^{p'} \mathbf{V}_1^*(x)dx \right)^{\frac{1}{p'}} + \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} u(t)dt \right)^{p'} \mathbf{V}_2^*(x)dx \right)^{\frac{1}{p'}} ,
 \end{aligned}$$

where

$$\mathbf{V}_1^*(x)dx \approx d \left(\mathbf{V}^*(x)^{-\frac{p'}{p}} / \psi(x)^{p'} \right), \quad \mathbf{V}_2^*(x)dx \approx d \left(-\mathbf{V}^*(x)^{-\frac{p'}{p}} \right).$$

For the class Ω_ψ we have

$$\sup_{f \in \Omega_{p, \psi}} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} = \sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{1, \frac{u(t)}{\psi(t)}}}{\|f\|_{p, \frac{v(t)}{\psi(t)^p}}}.$$

We make the substitutions $u(t) \rightarrow \frac{u(t)}{\psi(t)}$ and $v(t) \rightarrow \frac{v(t)}{\psi(t)^p}$ to obtain $V(t)/\psi(t)^p, V_1(t)$ and $V_2(t)$ instead of $V^*(t), V_1^*$ and $V_2^*(t)$, respectively, and thus (i) is proved.

(ii) The corresponding result for the case $0 < p \leq 1$ follows directly from Theorem 3. We have

$$\begin{aligned}
 \sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} &\approx \sup_{t>0} \left(\frac{1}{\psi(x)^p} \int_{[0,x]} \psi(t)^p v(t)dt + \int_{[x,\infty)} v(t)dt \right)^{-\frac{1}{p}} \\
 &\quad \times \left(\frac{1}{\psi(x)} \int_{[0,x]} \psi(t)u(t)dt + \int_{[x,\infty)} u(t)dt \right).
 \end{aligned}$$

Therefore,

$$\sup_{f \in \Omega_\psi} \frac{\|f\|_{1,u}}{\|f\|_{p,v}} = \sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{1, \frac{u(t)}{\psi(t)}}}{\|f\|_{p, \frac{v(t)}{\psi(t)^p}}} \approx \left(\frac{1}{\psi(x)} \int_{[0,x]} \psi(t)u(t)dt + \int_{[x,\infty)} u(t)dt \right) \mathbf{V}(x)^{-\frac{1}{p}}.$$

Now we are ready to prove an alternate characterization of $H_B(p, q)$.

THEOREM 4. *Let $q \geq 1, p > 0$ and the measure $v(x)dx$ be non-degenerate with respect to the function $\psi^p(x)$. Put*

$$\begin{aligned}
 W(t) &:= \int_{[0,t]} w(s)ds, \\
 \mathbf{V}(x) &:= \int_{[0,x]} v(t)dt + \psi(x)^p \int_{[x,\infty)} \frac{v(t)}{\psi(t)^p} dt,
 \end{aligned}$$

$$\mathbf{V}_1(x)dx \approx d\left(\mathbf{V}(x)^{-\frac{p'}{p}}\right), \quad \mathbf{V}_2(x)dx \approx d\left(-\psi(x)^{p'}\mathbf{V}(x)^{-\frac{p'}{p}}\right).$$

$$\mathbf{U}(x, s) := \int_{[s, x]} u(t)dt, \quad U_1(s) := \int_{[s, \infty)} \frac{u(t)}{\psi(t)} dt.$$

Then the inequality

$$\left(\int_{[0, \infty)} \left(\int_{[t, \infty)} f(s)u(s)ds\right)^q w(t)dt\right)^{\frac{1}{q}} \leq C \left(\int_{[0, \infty)} f^{p'}v(t)dt\right)^{\frac{1}{p}} \tag{11}$$

holds for all functions $f \in \Omega_\psi$ if and only if

(i) $A_1 < \infty$, if $q = 1, 0 < p \leq 1$, where

$$A_1 := \sup_{x \geq 0} \left(\int_{[0, x]} u(t)W(t)dt + \psi(x) \int_{[x, \infty)} \frac{u(t)}{\psi(t)} W(t)dt\right) \mathbf{V}(x)^{-\frac{1}{p}};$$

(ii) $A_2 < \infty$, if $q = 1, 1 < p < \infty$, where

$$A_2 := \left(\int_{[0, \infty)} \left(\int_{[0, x]} u(t)W(t)dt\right)^{p'} \mathbf{V}_1(x)dx + \int_{[0, \infty)} \left(\int_{[x, \infty)} \frac{u(t)}{\psi(t)} W(t)dt\right)^{p'} \mathbf{V}_2(x)dx\right)^{\frac{1}{p'}};$$

(iii) $A_3 < \infty$, if $q > 1, 0 < p \leq 1$, where

$$A_3 := \sup_{t \geq 0} \left[\left(\int_{[0, t]} \mathbf{U}(t, s)^q w(s)ds\right)^{\frac{1}{q}} + \psi(t) \left(\int_{[0, \infty)} U_1^q(\max(s, t))w(s)ds\right)^{\frac{1}{q}}\right] \mathbf{V}(t)^{-\frac{1}{p}};$$

(iv) $A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty$, if $1 < p \leq q < \infty$, where

$$A_{4,1} = \sup_{t \geq 0} \left(\int_{[t, \infty)} \mathbf{V}_1(x) \mathbf{U}(x, t)^{p'} dx\right)^{\frac{1}{p'}} \left(\int_{[0, t]} w(x)dx\right)^{\frac{1}{q}},$$

$$A_{4,2} = \sup_{t \geq 0} \left(\int_{[t, \infty)} \mathbf{V}_1(x)dx\right)^{\frac{1}{p'}} \left(\int_{[0, t]} \mathbf{U}(t, x)^q w(x)dx\right)^{\frac{1}{q}},$$

$$A_{4,3} = \sup_{t \geq 0} \left(\int_{[t, \infty)} U_1(x)^{p'} \mathbf{V}_2(x)dx\right)^{\frac{1}{p'}} \left(\int_{[0, t]} w(x)dx\right)^{\frac{1}{q}},$$

$$A_{4,4} = \sup_{t \geq 0} \left(\int_{[0, t]} \mathbf{V}_2(x)dx\right)^{\frac{1}{p'}} \left(\int_{[t, \infty)} U_1^q(x)w(x)dx\right)^{\frac{1}{q}};$$

(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

$$\begin{aligned}
 A_{5,1} &= \left(\int_{[0,\infty)} \left(\int_{[t,\infty)} \mathbf{V}_1(x) \mathbf{U}(x,t)^{p'} dx \right)^{\frac{r}{p'}} \left(\int_{[0,t]} w(x) dx \right)^{\frac{r}{p}} w(t) dt \right)^{\frac{1}{r}}, \\
 A_{5,2} &= \left(\int_{[0,\infty)} \left(\int_{[t,\infty)} \mathbf{V}_1(x) dx \right)^{\frac{r}{q'}} \left(\int_{[0,t]} \mathbf{U}(t,x)^q w(x) dx \right)^{\frac{r}{q}} \mathbf{V}_1(t) dt \right)^{\frac{1}{r}}, \\
 A_{5,3} &= \left(\int_{[0,\infty)} \left(\int_{[0,x]} w(t) dt \right)^{\frac{r}{q}} \left(\int_{[x,\infty)} U_1(t)^{p'} \mathbf{V}_2(t) dt \right)^{\frac{r}{q'}} U_1(x)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{r}}, \\
 A_{5,4} &= \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} U_1(s)^q w(s) ds \right)^{\frac{r}{q}} \left(\int_{[0,x]} \mathbf{V}_2(s) ds \right)^{\frac{r}{q'}} \mathbf{V}_2(x) dx \right)^{\frac{1}{r}}.
 \end{aligned}$$

Proof. Let $q = 1$, then

$$H_B(p, 1) = \sup_{f \in \Omega_t} \frac{\int_{[0,\infty)} \left(\int_{[t,\infty)} fu \right) w(t) dt}{\left(\int_{[0,\infty)} f^p v \right)^{\frac{1}{p}}} = \sup_{f \in \Omega_t} \frac{\int_{[0,\infty)} f(t) u(t) W(t) dt}{\left(\int_{[0,\infty)} f^p v \right)^{\frac{1}{p}}},$$

where $W(t) = \int_{[0,t]} w(s) ds$. We use Lemma 1 and obtain

$$\begin{aligned}
 H_B(p, 1) &\approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} u(t) W(t) dt \right)^{p'} \mathbf{V}_1(x) dx \right. \\
 &\quad \left. + \int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{u(t)}{t} W(t) dt \right)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{p'}}
 \end{aligned}$$

for $1 < p < \infty$ and

$$H_B(p, 1) \approx \sup_{x \geq 0} \left(\int_{[0,x]} u(t) W(t) dt + x \int_{[x,\infty)} \frac{u(t)}{t} W(t) dt \right) \mathbf{V}(x)^{-\frac{1}{p}}$$

for $0 < p \leq 1$. Let now $q > 1$. We have

$$\begin{aligned}
 H_B(p, q) &= \sup_{f \in \Omega_t} \frac{\left(\int_{[0,\infty)} \left(\int_{[t,\infty)} fu \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p v \right)^{\frac{1}{p}}} = \sup_{f \in \Omega_t} \sup_{g \geq 0} \frac{\int_{[0,\infty)} \left(\int_{[t,\infty)} fu \right) g(t) dt}{\|g\|_{q', w^{1-q'}} \|f\|_{p, v}} \\
 &= \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1-q'}}} \sup_{f \in \Omega_t} \frac{\int_{[0,\infty)} f(t) u(t) \left(\int_{[0,t]} g \right) dt}{\|f\|_{p, v}}.
 \end{aligned}$$

By using Lemma 1 for $p > 1$ we find

$$H_B(p, q) \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1-q'}}} \left(\int_{[0, \infty)} \left(\int_{[0, x]} u(t) \left(\int_{[0, t]} g \right) dt \right)^{p'} \mathbf{V}_1(x) dx + \int_{[0, \infty)} \left(\int_{[x, \infty)} \frac{u(t)}{t} \left(\int_{[0, t]} g \right) dt \right)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{p'}}$$

We can write

$$\int_{[x, \infty)} \frac{u(t)}{t} \left(\int_{[0, t]} g(s) ds \right) dt = \left(\int_{[0, x]} g(s) ds \right) \left(\int_{[x, \infty)} \frac{u(t)}{t} dt \right) + \int_{[x, \infty)} \left(\int_{[s, \infty)} \frac{u(t)}{t} dt \right) g(s) ds,$$

and, thus, the initial inequality (11) is equivalent to the following three inequalities on the set of non-negative functions:

$$\left(\int_{[0, \infty)} \left(\int_{[0, x]} \mathbf{U}^*(x, s) g(s) ds \right)^{p'} \mathbf{V}_1(x) dx \right)^{\frac{1}{p'}} \leq C_1 \|g\|_{q', w^{1-q'}},$$

$$\left(\int_{[0, \infty)} \left(\int_{[0, x]} g(s) ds \right)^{p'} \mathbf{U}_1(x)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{p'}} \leq C_2 \|g\|_{q', w^{1-q'}}$$

and

$$\left(\int_{[0, \infty)} \left(\int_{[x, \infty)} \mathbf{U}_1(s) g(s) ds \right)^{p'} \mathbf{V}_2(x) dx \right)^{\frac{1}{p'}} \leq C_3 \|g\|_{q', w^{1-q'}},$$

where

$$\mathbf{U}^*(x, s) = \int_{[s, x]} u(t) dt, \quad \mathbf{U}_1(s) = \int_{[s, \infty)} \frac{u(t)}{t} dt.$$

Criteria from [23], [24] yield the result of the theorem for $1 < p \leq q < \infty$ and $1 < q < p < \infty$. In case $0 < p \leq 1$ we have

$$H_B(p, q) \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1-q'}}} \times \sup_{t \geq 0} \left[\left(\int_{[0, t]} u(x) \left(\int_{[0, x]} g \right) dx + t \int_{[t, \infty)} \frac{u(x)}{x} \left(\int_{[0, x]} g \right) dx \right) \mathbf{V}(t)^{-\frac{1}{p}} \right]$$

$$= \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1-q'}}} \sup_{t \geq 0} \left[\left(\int_{[0, t]} \mathbf{U}^*(t, s) g(s) ds + t \mathbf{U}_1(t) \int_{[0, t]} g(s) ds + t \int_{[t, \infty)} \mathbf{U}_1(s) g(s) ds \right) \mathbf{V}(t)^{-\frac{1}{p}} \right]$$

$$= \sup_{g \geq 0} \frac{\|\mathbf{V}(t)^{-\frac{1}{p}} \left(\int_{[0,t]} \mathbf{U}^*(t,s)g(s)ds + t \int_{[0,\infty)} \mathbf{U}_1(\max(s,t))g(s)ds \right)\|_{\infty}}{\|g\|_{q',w^{1-q'}}}.$$

Then [18], Chapter XI, §1.5, Theorem 4 yields

$$H_B(p, q) \approx \sup_{t \geq 0} \mathbf{V}(t)^{-\frac{1}{p}} \left[\left(\int_{[0,t]} \mathbf{U}^*(t,s)^q w(s)ds \right)^{\frac{1}{q}} + t \left(\int_{[0,\infty)} \mathbf{U}_1^q(\max(s,t))w(s)ds \right)^{\frac{1}{q}} \right].$$

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