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SOME INEQUALITIES ON GENERAL L_p–CENTROID BODIES

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Abstract. In this article, we define the general L_p-centroid bodies, which extend the notion of L_p-centroid bodies by Lutwak and Zhang. Further, we generalize the two monotone inequalities by Wang, Lu and Leng, and establish the Brunn-Minkowski type inequalities of dual quermassintegrals for this new notion. In particular, the extremal values of dual quermassintegrals of the polars of general L_p-centroid bodies are also provided.

1. Introduction and main results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathscr{K}_o^n$. The unit ball in $\mathbb{R}^n$ and its surface will be denoted by B and S^{n-1}, respectively. $V(K)$ denotes the n-dimensional volume of a body K. We denote $\omega_n = V(B)$ for the volume of the unit ball B.

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see [6, 20])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \ \ x \in \mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y.

For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the Firey L_p-combination, $\lambda\cdot K +_p \mu\cdot L \in \mathscr{K}_o^n$, of K and L is defined by (see [5])

$$h(\lambda\cdot K +_p \mu\cdot L,\cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p, \tag{1.1}$$

where "$\cdot$" in $\lambda\cdot K$ denotes Firey L_p-scalar multiplication. Obviously, Firey L_p-scalar multiplication and usual scalar multiplication are related by $\lambda\cdot K = \lambda^{\frac{1}{p}}K$.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see[6, 20])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda\cdot u \in K\}, \ \ u \in S^{n-1}. \tag{1.2}$$

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If ρ_K is positive and continuous, call K a star body, and write $\mathscr{S}_o^n$ for the set of star bodies in $\mathbb{R}^n$. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathscr{S}_o^n$, of K and L is defined by (see [13])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}, \tag{1.3}$$

where $\lambda \star K$ denotes L_p-harmonic radial scalar multiplication, and we easily see $\lambda \star K = \lambda^{-\frac{1}{p}} K$.

If $K \in \mathscr{K}_o^n$, the polar body, K^*, of K is defined by (see [6, 20])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}. \tag{1.4}$$

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ was defined in [13] by

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}. \tag{1.5}$$

Centroid bodies are a classical notion from geometry which have attracted increased attention in recent years (see [2, 7, 14-19, 21]). In particular, Lutwak and Zhang [16] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real number $p \geqslant 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) &= \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx \\ &= \frac{1}{c_{n,p}(n+p)V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v) \end{aligned} \tag{1.6}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}, \text{and } \omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2}). \tag{1.7}$$

We recall that for $\tau \in [-1,1]$, Ludwig [11] introduced a function $\varphi_\tau : \mathbb{R} \to [0,\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t. \tag{1.8}$$

Now, we define a corresponding notion of general L_p-centroid bodies based on L_p-centroid bodies and definition (1.8). For $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p^\tau K}(u) &= \frac{1}{c_{n,p}(\tau) V(K)} \int_K \varphi_\tau(u \cdot x)^p dx \\ &= \frac{1}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho_K^{n+p}(v) dS(v), \end{aligned} \tag{1.9}$$

where

$$c_{n,p}(\tau)=\frac{1}{2}c_{n,p}[(1+\tau)^p+(1-\tau)^p].$$

The normalization is chosen such that $\Gamma_p^\tau B=B$ for every $\tau\in[-1,1]$, and $\Gamma_p^0K=\Gamma_pK$. Let $\varphi_+(u\cdot x)=\max\{u\cdot x,0\}$ $(\tau=1)$ in (1.9), then a special case of definition $\Gamma_p^\tau K$ is Γ_p^+K. Besides, we also define

$$\Gamma_p^-K=\Gamma_p^+(-K). \tag{1.10}$$

From the definitions of $\Gamma_p^\pm K$ and $\Gamma_p^\tau K$, it is easy to verity that

$$\Gamma_p^\tau K=f_1(\tau)\cdot\Gamma_p^+K+_p f_2(\tau)\cdot\Gamma_p^-K, \tag{1.11}$$

where

$$f_1(\tau)=\frac{(1+\tau)^p}{(1+\tau)^p+(1-\tau)^p},\quad f_2(\tau)=\frac{(1-\tau)^p}{(1+\tau)^p+(1-\tau)^p}. \tag{1.12}$$

From (1.12), it immediately follows that

$$f_1(\tau)+f_2(\tau)=1; \tag{1.13}$$

$$f_1(-\tau)=f_2(\tau),\quad f_2(-\tau)=f_1(\tau). \tag{1.14}$$

By (1.11) and definitions $\Gamma_p^\pm K$, we easily get

$$\Gamma_pK=\frac{1}{2}\cdot\Gamma_p^+K+_p\frac{1}{2}\cdot\Gamma_p^-K; \tag{1.15}$$

$$\Gamma_p^{-\tau}K=-\Gamma_p^\tau K. \tag{1.16}$$

The following are our main results: First, we show two results below which generalize the analogs of [21].

THEOREM 1.1. *For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, if $\widetilde{V}_{-p}(K,Q)\leqslant\widetilde{V}_{-p}(L,Q)$ for any $Q\in\mathscr{S}_o^n$, then*

$$\frac{V(\Gamma_p^{\tau,*}K)^{\frac{p}{n}}}{V(K)}\geqslant\frac{V(\Gamma_p^{\tau,*}L)^{\frac{p}{n}}}{V(L)}, \tag{1.17}$$

with equality if and only if $K=L$.

THEOREM 1.2. *For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, if $\widetilde{V}_{-p}(K,Q)\leqslant\widetilde{V}_{-p}(L,Q)$ for any $Q\in\mathscr{S}_o^n$, then*

$$\frac{V(\Gamma_p^{\tau}K)^{-\frac{p}{n}}}{V(K)}\geqslant\frac{V(\Gamma_p^{\tau}L)^{-\frac{p}{n}}}{V(L)}, \tag{1.18}$$

with equality if and only if $K=L$.

Moreover, we establish the following Brunn-Minkowski type inequalities of dual quermassintegrals for general L_p-centroid bodies.

THEOREM 1.3. *If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then for $i\geqslant n+p$*

$$\widetilde{W}_i(\Gamma_p^{\tau,*}(K\hat{+}_pL))^{-\frac{p}{n-i}}\leqslant\widetilde{W}_i(\Gamma_p^{\tau,*}K)^{-\frac{p}{n-i}}+\widetilde{W}_i(\Gamma_p^{\tau,*}L)^{-\frac{p}{n-i}};\tag{1.19}$$

for $i<n$ inequality (1.19) is reversed, with equality in every inequality if and only if K and L are dilates. Here $\widetilde{W}_i(K)$ is the dual quermassintegrals of $K\in\mathscr{S}_o^n$ defined by (see [5])

$$\widetilde{W}_i(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}dS(u)\tag{1.20}$$

for any real i. Clearly, $\widetilde{W}_0(K)=V(K)$.

Finally, the following theorem provides the extremal values of dual quermassintegrals for general L_p-centroid bodies.

THEOREM 1.4. *If $K\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then for $n\leqslant j\leqslant n+p\leqslant i$*

$$\frac{\widetilde{W}_i(\Gamma_p^{*}K)}{\widetilde{W}_j(\Gamma_p^{*}K)}\leqslant\frac{\widetilde{W}_i(\Gamma_p^{\tau,*}K)}{\widetilde{W}_j(\Gamma_p^{\tau,*}K)}\leqslant\frac{\widetilde{W}_i(\Gamma_p^{\pm,*}K)}{\widetilde{W}_j(\Gamma_p^{\pm,*}K)};\tag{1.21}$$

for $j\leqslant n\leqslant i\leqslant n+p$ inequality (1.21) is reversed, the left equality of every inequality holds if and only if $\Gamma_p^{\tau}K$ is origin-symmetric and the right equality of every inequality holds if and only if $\Gamma_p^{\pm}K$ is origin-symmetric.

2. Preliminaries

In this section, we collect some basic well-known facts that we will use in the proofs of our results.

According to the definitions of the polar body, the support function and radial function, it follows for $K\in\mathscr{K}_o^n$ that

$$h_{K^*}=\frac{1}{\rho_K},\quad \rho_{K^*}=\frac{1}{h_K}.\tag{2.1}$$

From (1.1), (1.3) and (2.1), we easily see that if $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), then

$$(\lambda\cdot K+_p\mu\cdot L)^*=\lambda\star K^*+_{-p}\mu\star L^*.\tag{2.2}$$

For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda\circ K\hat{+}_p\mu\circ L\in\mathscr{S}_o^n$, of K and L is defined by (see [4])

$$\frac{\rho(\lambda\circ K\hat{+}_p\mu\circ L,\cdot)^{n+p}}{V(\lambda\circ K\hat{+}_p\mu\circ L)}=\lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)}+\mu\frac{\rho(L,\cdot)^{n+p}}{V(L)}.\tag{2.3}$$

Here $\lambda\circ K$ is L_p-harmonic Blaschke scalar multiplication and $\lambda\circ K=\lambda^{\frac{1}{p}}K$.

For $K,L\in\mathscr{K}_o^n$, $\varepsilon>0$ and $p\geqslant 1$, the L_p-mixed volume $V_p(K,L)$ was defined in [12] by

$$\frac{n}{p}V_p(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K+_p\varepsilon\cdot L)-V(K)}{\varepsilon}.$$

Lutwak [12] proved that there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(u)dS_p(K,u). \tag{2.4}$$

Here $S_p(K,\cdot)$ is called the L_p-surface area measure of $K\in\mathscr{K}_o^n$. It turns out that the measure $S_p(K,\cdot)$ is absolutely continuous with respect to the classical surface area measure $S(K,\cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)}=h(K,\cdot)^{1-p}. \tag{2.5}$$

From formulas (2.4) and (2.5), it follows immediately that for each $K\in\mathscr{K}_o^n$

$$V_p(K,K)=V(K)=\frac{1}{n}\int_{S^{n-1}}h_K(v)dS(K,v). \tag{2.6}$$

The Minkowski inequality of L_p-mixed volume (see [13]) states that for $K,L\in\mathscr{K}_o^n$ and $p\geqslant 1$,

$$V_p(K,L)\geqslant V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.7}$$

with equality for $p=1$ if and only if K and L are homothetic, for $p>1$ if and only if K and L are dilates.

The definition of L_p-dual mixed volume (see (1.5)) and the polar coordinate formula for volume lead to the following integral representation of L_p-dual mixed volume (see [13]):

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(u)\rho_L^{-p}(u)dS(u), \tag{2.8}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}. From formula (2.8), we easily see that for $K\in\mathscr{S}_o^n$ and $p\geqslant 1$,

$$\widetilde{V}_{-p}(K,K)=V(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)dS(u). \tag{2.9}$$

For $K,L\in\mathscr{S}_o^n$ and $p\geqslant 1$, the L_p-dual Minkowski inequality (see [13]) is

$$\widetilde{V}_{-p}(K,L)\geqslant V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}, \tag{2.10}$$

with equality if and only if K and L are dilates.

For $K\in\mathscr{K}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, the general L_p-projection body, $\Pi_p^\tau K\in\mathscr{K}_o^n$, of K whose support function is given by (see [8])

$$h_{\Pi_p^\tau K}^p(u)=\alpha_{n,p}(\tau)\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p dS_p(K,v),$$

where

$$\alpha_{n,p}(\tau)=\frac{\alpha_{n,p}}{(1+\tau)^p+(1-\tau)^p}, \quad \text{and} \quad \alpha_{n,p}=\frac{1}{n\omega_n c_{n-2,p}}. \tag{2.11}$$

Haberl and Schuster [8] proved that if $K\in\mathscr{K}_o^n$, $L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then

$$V_p(K,M_p^\tau L)=\widetilde{V}_{-p}(L,\Pi_p^{\tau,*}K). \tag{2.12}$$

Here $M_p^\tau L$ denotes the L_p-moment body of $L\in\mathscr{S}_o^n$ which is defined by (see [8])

$$h^p_{M_p^\tau L}(u)=\alpha_{n,p}(\tau)\int_{S^{n-1}}\varphi_\tau(u\cdot v)\rho(L,v)^{n+p}dS(v) \tag{2.13}$$

for all $u\in S^{n-1}$, $\varphi_\tau(u\cdot v)$ and $\alpha_{n,p}(\tau)$ satisfy (1.8) and (2.11), respectively. From definitions (1.9) and (2.13), we easily get for $K\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$,

$$M_p^\tau K=(\frac{V(K)}{\omega_n})^{\frac{1}{p}}\Gamma_p^\tau K. \tag{2.14}$$

Combining (2.12) and (2.14), an immediate result is that if $K\in\mathscr{K}_o^n$, $L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then

$$V_p(K,\Gamma_p^\tau L)=\frac{\omega_n}{V(L)}\widetilde{V}_{-p}(L,\Pi_p^{\tau,*}K). \tag{2.15}$$

3. The proofs of main results

In this section, we prove Theorems 1.1-1.4. The proof of Theorem 1.1 needs the following lemmas:

LEMMA 3.1. *If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then*

$$\frac{\widetilde{V}_{-p}(K,\Gamma_p^{\tau,*}L)}{V(K)}=\frac{\widetilde{V}_{-p}(L,\Gamma_p^{\tau,*}K)}{V(L)}. \tag{3.1}$$

Proof. From (1.9) and (2.1), we have

$$\rho^{-p}_{\Gamma_p^{\tau,*}K}(u)=\frac{1}{c_{n,p}(\tau)(n+p)V(K)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_K^{n+p}(v)dS(v).$$

Together with (2.8), we get

$$\begin{aligned}\widetilde{V}_{-p}(L,\Gamma_p^{\tau,*}K)&=\frac{1}{n}\int_{S^{n-1}}\rho_L^{n+p}(u)\rho^{-p}_{\Gamma_p^{\tau,*}K}(u)dS(u)\\&=\frac{1}{n(n+p)c_{n,p}(\tau)V(K)}\int_{S^{n-1}}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_L^{n+p}(u)\rho_K^{n+p}(v)dS(v)dS(u)\\&=\frac{V(L)}{nV(K)}\int_{S^{n-1}}\rho_K^{n+p}(v)\rho^{-p}_{\Gamma_p^{\tau,*}L}(v)dS(v)=\frac{V(L)}{V(K)}\widetilde{V}_{-p}(K,\Gamma_p^{\tau,*}L).\end{aligned}$$

This yields the desired result. □

LEMMA 3.2. [22] *If* $K,L \in \mathscr{S}_o^n$ *and* $p \geqslant 1$, *then for any* $Q \in \mathscr{S}_o^n$,

$$\widetilde{V}_{-p}(K,Q) = \widetilde{V}_{-p}(L,Q)$$

if and only if $K = L$.

Proof of Theorem 1.1. From $\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q)$ for any $Q \in \mathscr{S}_o^n$, taking $\Gamma_p^{\tau,*}M$ for Q for any $M \in \mathscr{S}_o^n$, we have

$$\widetilde{V}_{-p}(K,\Gamma_p^{\tau,*}M) \leqslant \widetilde{V}_{-p}(L,\Gamma_p^{\tau,*}M), \tag{3.2}$$

with equality if and only if $K = L$ obtained from Lemma 3.2. Combining inequality (3.2) and equality (3.1), we get

$$\frac{V(K)\widetilde{V}_{-p}(M,\Gamma_p^{\tau,*}K)}{V(M)} \leqslant \frac{V(L)\widetilde{V}_{-p}(M,\Gamma_p^{\tau,*}L)}{V(M)}. \tag{3.3}$$

Taking $\Gamma_p^{\tau,*}L$ for M in (3.3) and using (2.9) and (2.10), we get that

$$V(L)V(\Gamma_p^{\tau,*}L) \geqslant V(K)\widetilde{V}_{-p}(\Gamma_p^{\tau,*}L,\Gamma_p^{\tau,*}K) \geqslant V(\Gamma_p^{\tau,*}L)^{\frac{n+p}{n}}V(\Gamma_p^{\tau,*}K)^{-\frac{p}{n}}V(K), \tag{3.4}$$

with equality in the second inequality of (3.4) if and only if K and L are dilates. Thus it follows from (3.4) that (1.17) holds.

From Lemma 3.1, we know that inequalities (3.2) and (3.3) are equivalent. Thus equality holds in first inequality of (3.4) if and only if $K = L$. Together with the equality condition of the second inequality of (3.4), we get that equality holds in (1.17) if and only if $K = L$. $\square$

Proof of Theorem 1.2. Since $\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q)$ for any $Q \in \mathscr{S}_o^n$, taking $\Pi_p^{\tau,*}M$ for Q for any $M \in \mathscr{K}_o^n$, we have

$$\widetilde{V}_{-p}(K,\Pi_p^{\tau,*}M) \leqslant \widetilde{V}_{-p}(L,\Pi_p^{\tau,*}M), \tag{3.5}$$

with equality if and only if $K = L$ obtained from Lemma 3.2. By (2.15), we get

$$V(K)V_p(M,\Gamma_p^{\tau}K) \leqslant V_p(M,\Gamma_p^{\tau}L)V(L). \tag{3.6}$$

Taking $\Gamma_p^{\tau}L$ for M in (3.6) and using (2.6) and (2.7), we obtain

$$V(L)V(\Gamma_p^{\tau}L) \geqslant V(\Gamma_p^{\tau}L)^{\frac{n-p}{n}}V(\Gamma_p^{\tau}K)^{\frac{p}{n}}V(K), \tag{3.7}$$

with equality if and only if K and L are dilates. Thus from inequality (3.7) this gets the desired result.

According to the equality conditions of (3.5) and (3.7), we see that equality holds in (1.18) if and only if $K = L$. $\square$

Proof of Theorem 1.3. From (1.9) and (2.3), and using (2.1), we easily get for any $u \in S^{n-1}$

$$\rho^{-p}_{\Gamma^{\tau,*}_p(K \hat{+}_p L)}(u) = \rho^{-p}_{\Gamma^{\tau,*}_p K}(u) + \rho^{-p}_{\Gamma^{\tau,*}_p L}(u). \tag{3.8}$$

Since $i \geqslant n+p$, it follows that $-\frac{n-i}{p} \geqslant 1$. Thus from (1.20), (3.8) and the Minkowski's integral inequality (see [9]), we get

$$\begin{aligned}\widetilde{W}_i(\Gamma^{\tau,*}_p(K \hat{+}_p L))^{-\frac{p}{n-i}} &= \left(\frac{1}{n}\int_{S^{n-1}} (\rho^{-p}_{\Gamma^{\tau,*}_p(K \hat{+}_p L)}(u))^{-\frac{n-i}{p}} dS(u)\right)^{-\frac{p}{n-i}} \\ &\leqslant \widetilde{W}_i(\Gamma^{\tau,*}_p K)^{-\frac{p}{n-i}} + \widetilde{W}_i(\Gamma^{\tau,*}_p L)^{-\frac{p}{n-i}}.\end{aligned}$$

This yields inequality (1.19). From $i < n \Rightarrow -\frac{n-i}{p} < 0$, similar to the proof of (1.19), we use the inverse Minkowski's integral inequality (see [9]) to get the reversed inequality of (1.19).

According to the equality conditions of Minkowski's integral inequalities, we see that equality holds in every inequality of Theorem 1.3 if and only if K and L are dilates. □

A special case of the reversed inequality of (1.19) is as follows:

COROLLARY 3.1. *If $K, L \in \mathscr{S}^n_o$, $p \geqslant 1$ and $\tau \in [-1,1]$, then*

$$V(\Gamma^{\tau,*}_p(K \hat{+}_p L))^{-\frac{p}{n}} \geqslant V(\Gamma^{\tau,*}_p K)^{-\frac{p}{n}} + V(\Gamma^{\tau,*}_p L)^{-\frac{p}{n}},$$

with equality if and only if K and L are dilates.

An extension of Beckenbach's inequality (see [1], sec. 24) was obtained by Dresher (see [3]) through the means of moment-space techniques.

LEMMA 3.3. (The Beckenbach-Dresher Inequality) *If $p \geqslant 1 \geqslant r \geqslant 0$, $f, g \geqslant 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leqslant \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}; \tag{3.9}$$

for $r \leqslant 0 \leqslant p \leqslant 1$ inequality (3.9) is reversed (see [10]), with equality in every inequality if and only if the functions f and g are positively proportional. Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

Proof of Theorem 1.4. We first prove the left inequality of (1.21), From (1.11), (1.13), (1.14), (1.15) and (2.1), we have

$$\rho^{-p}(\Gamma^{\tau,*}_p K, \cdot) + \rho^{-p}(\Gamma^{-\tau,*}_p K, \cdot) = \rho^{-p}(\Gamma^{+,*}_p K, \cdot) + \rho^{-p}(\Gamma^{-,*}_p K, \cdot). \tag{3.10}$$

Together with (1.15), we obtain

$$\rho^{-p}(\Gamma^{*}_p K, \cdot) = \frac{1}{2}\rho^{-p}(\Gamma^{\tau,*}_p K, \cdot) + \frac{1}{2}\rho^{-p}(\Gamma^{-\tau,*}_p K, \cdot). \tag{3.11}$$

Combining (1.20) and (3.11), we get

$$\widetilde{W}_{n-r}(\Gamma_p^* K) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}\rho^{-p}(\Gamma_p^{\tau,*}K,u) + \frac{1}{2}\rho^{-p}(\Gamma_p^{-\tau,*}K,u)\right)^{-\frac{r}{p}} dS(u). \tag{3.12}$$

Similarly,

$$\widetilde{W}_{n-s}(\Gamma_p^* K) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}\rho^{-p}(\Gamma_p^{\tau,*}K,u) + \frac{1}{2}\rho^{-p}(\Gamma_p^{-\tau,*}K,u)\right)^{-\frac{s}{p}} dS(u). \tag{3.13}$$

By the Beckenbach-Dresher inequality together with (3.12) and (3.13), we have

$$\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^* K)}{\widetilde{W}_{n-s}(\Gamma_p^* K)}\right)^{\frac{p}{s-r}} \leqslant \frac{1}{2}\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^{\tau,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{\tau,*} K)}\right)^{\frac{p}{s-r}} + \frac{1}{2}\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^{-\tau,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{-\tau,*} K)}\right)^{\frac{p}{s-r}}. \tag{3.14}$$

From Lemma 3.3, we know that $-\frac{r}{p} \geqslant 1 \geqslant -\frac{s}{p} \geqslant 0 \Rightarrow r \leqslant -p \leqslant s \leqslant 0$. Together with (1.16), and notice that $\widetilde{W}_i(-K) = \widetilde{W}_i(K)$ for any $K \in \mathscr{S}_o^n$, we get

$$\frac{\widetilde{W}_{n-r}(\Gamma_p^* K)}{\widetilde{W}_{n-s}(\Gamma_p^* K)} \leqslant \frac{\widetilde{W}_{n-r}(\Gamma_p^{\tau,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{\tau,*} K)}. \tag{3.15}$$

Let $r = n-i$ and $s = n-j$ in (3.15), from $r \leqslant -p \leqslant s \leqslant 0 \Rightarrow n \leqslant j \leqslant n+p \leqslant i$, this yields

$$\frac{\widetilde{W}_i(\Gamma_p^* K)}{\widetilde{W}_j(\Gamma_p^* K)} \leqslant \frac{\widetilde{W}_i(\Gamma_p^{\tau,*} K)}{\widetilde{W}_j(\Gamma_p^{\tau,*} K)}. \tag{3.16}$$

According to the equality condition of Lemma 3.3, we see that equality holds in inequality (3.16) if and only if $\rho(\Gamma_p^{\tau,*}K,u)$ and $\rho(\Gamma_p^{-\tau,*}K,u)$ are positively proportional, namely, $\Gamma_p^{\tau,*}K$ and $\Gamma_p^{-\tau,*}K$ are dilates. Since $\Gamma_p^{-\tau,*}K = -\Gamma_p^{\tau,*}K$, we have $\Gamma_p^{\tau,*}K = \Gamma_p^{-\tau,*}K$, i.e., $\Gamma_p^{\tau}K = \Gamma_p^{-\tau}K$. From (1.16), this gets that $\Gamma_p^{\tau}K$ is origin-symmetric. Thus with equality in inequality (3.16) if and only if $\Gamma_p^{\tau}K$ is origin-symmetric.

Now, we prove the right inequality of (1.21). From (1.3), (1.11), (1.20) and (2.2), we have

$$\widetilde{W}_{n-r}(\Gamma_p^{\tau,*} K) = \frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)\rho^{-p}_{\Gamma_p^{+,*}K}(u) + f_2(\tau)\rho^{-p}_{\Gamma_p^{-,*}K}(u)\right)^{-\frac{r}{p}} dS(u). \tag{3.17}$$

Similarly,

$$\widetilde{W}_{n-s}(\Gamma_p^{\tau,*} K) = \frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)\rho^{-p}_{\Gamma_p^{+,*}K}(u) + f_2(\tau)\rho^{-p}_{\Gamma_p^{-,*}K}(u)\right)^{-\frac{s}{p}} dS(u). \tag{3.18}$$

Similar to the proof of inequality (3.14), from (3.17), (3.18) and the Beckenbach-Dresher inequality, we get

$$\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^{\tau,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{\tau,*} K)}\right)^{\frac{p}{s-r}} \leqslant f_1(\tau)\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^{+,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{+,*} K)}\right)^{\frac{p}{s-r}} + f_2(\tau)\left(\frac{\widetilde{W}_{n-r}(\Gamma_p^{-,*} K)}{\widetilde{W}_{n-s}(\Gamma_p^{-,*} K)}\right)^{\frac{p}{s-r}}. \tag{3.19}$$

From (1.10) and (1.13), analogue to the deducing process of (3.16), this gets for inequality (3.19),

$$\frac{\widetilde{W}_i(\Gamma_p^{\tau,*}K)}{\widetilde{W}_j(\Gamma_p^{\tau,*}K)} \leqslant \frac{\widetilde{W}_i(\Gamma_p^{\pm,*}K)}{\widetilde{W}_j(\Gamma_p^{\pm,*}K)}. \tag{3.20}$$

Based on the equality condition of (3.9), we see that equality holds in inequality (3.20) if and only if $\rho(\Gamma_p^{+,*}K,u)$ and $\rho(\Gamma_p^{-,*}K,u)$ are positively proportional, i.e., $\Gamma_p^{+,*}K$ and $\Gamma_p^{-,*}K$ are dilates. Since $\Gamma_p^{-,*}K = -\Gamma_p^{+,*}K$, it follows that $\Gamma_p^{+,*}K = \Gamma_p^{-,*}K$, namely, $\Gamma_p^{+}K = \Gamma_p^{-}K$. From $\Gamma_p^{-}K = -\Gamma_p^{+}K$ or $\Gamma_p^{+}K = -\Gamma_p^{-}K$, we have $\Gamma_p^{+}K = -\Gamma_p^{+}K$ or $\Gamma_p^{-}K = -\Gamma_p^{-}K$. This means that $\Gamma_p^{\pm}K$ is origin-symmetric, thus with equality in inequality (3.20) if and only if $\Gamma_p^{\pm}K$ is origin-symmetric.

Applying the above method, it follows from the reversed Beckenbach-Dresher inequality that the reversed inequality of (1.21) holds. □

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