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COMPLETE MONOTONICITY PROPERTIES AND ASYMPTOTIC EXPANSIONS OF THE LOGARITHM OF THE GAMMA FUNCTION

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Abstract. We prove two conjectures of Chen concerning the complete monotonicity properties of some functions involving the gamma and polygamma functions. We prove asymptotic expansions of the logarithm of the gamma function in terms of the polygamma functions, and provide recurrence relations to calculate the coefficients of the asymptotic expansions. By using the results obtained, we derive recursive relations of the Bernoulli numbers.

1. Introduction

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad (x \in I; \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1,2,3,\ldots\}). \tag{1}$$

Dubourdieu [4, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a,\infty)$, then strict inequality holds true in (1). See also [6] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. See [8, p. 161].

The gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas are widely scattered. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\{\ln\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \ln\Gamma(x) = \int_1^x \psi(t)\,\mathrm{d}t$$

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is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{d^n}{dx^n}\{\psi(x)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

Recently, Chen [3] presented new asymptotic expansions of the logarithm of the gamma function in terms of the polygamma functions. More precisely, the author proved that, as $x \to \infty$,

$$\ln\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty} a_j \psi^{(2j-1)}\left(x+\frac{1}{2}\right) \tag{2}$$

and

$$\ln\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty} b_j \psi^{(j)}(x), \tag{3}$$

where the coefficients a_j and b_j are given by

$$a_j = \frac{4j}{2^{2j+1}(2j+1)!}, \qquad j \in \mathbb{N} \tag{4}$$

and

$$b_j = \frac{j}{2(j+2)!}, \qquad j \in \mathbb{N}, \tag{5}$$

respectively. Burić et al. [2] dealt with the same problem. Based on the asymptotic expansions (2) and (3), Chen [3] proposed the following conjectures.

CONJECTURE 1. *For all $m \in \mathbb{N}_0$, the functions $R_m(x)$ defined by*

$$R_m(x) = \ln\Gamma(x+1) - \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{m} \frac{4j}{2^{2j+1}(2j+1)!}\psi^{(2j-1)}\left(x+\frac{1}{2}\right)\right] \tag{6}$$

are completely monotonic on $(0,\infty)$.

CONJECTURE 2. *For all $m \in \mathbb{N}_0$, the functions $S_m(x)$ defined by*

$$S_m(x) = (-1)^m\left[\ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m} \frac{j}{2(j+2)!}\psi^{(j)}(x)\right] \tag{7}$$

are completely monotonic on $(0,\infty)$.

Our first aim in this paper is to prove these two conjectures. Our second aim in this paper is to give recursive relations for determining the coefficients a_j and b_j in (2) and (3), respectively. By using the results obtained, we derive recursive relations of the Bernoulli numbers.

2. Completely monotonic functions

In this section, we prove Conjectures 1 and 2.

THEOREM 1. *For all $m \in \mathbb{N}_0$, the functions $R_m(x)$, defined by (6), are completely monotonic on $(0,\infty)$.*

Proof. The noted Binet's first formula [7, p. 16] states that

$$\ln\Gamma(x) = \left(x-\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \qquad (x>0). \tag{8}$$

It is known (see [1, p. 260]) that

$$\psi^{(j)}(x) = (-1)^{j+1}\int_0^\infty \frac{t^j}{1-e^{-t}}e^{-xt}\,\mathrm{d}t, \qquad x>0 \quad \text{and} \quad j\in\mathbb{N}. \tag{9}$$

By using (8) and (9), we have

$$\begin{aligned} R_m(x) &= \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t - \sum_{j=1}^m \frac{4j}{2^{2j+1}(2j+1)!}\int_0^\infty \frac{t^{2j-1}}{1-e^{-t}}e^{-(x+1/2)t}\,\mathrm{d}t \\ &= \int_0^\infty \frac{p(t)}{2\sinh(\frac{t}{2})}e^{-xt}\,\mathrm{d}t, \end{aligned}$$

with

$$\begin{aligned} p(t) &= \frac{1}{t}\cosh\left(\frac{t}{2}\right) - \frac{2}{t^2}\sinh\left(\frac{t}{2}\right) - \sum_{j=1}^m \frac{4j}{2^{2j+1}(2j+1)!}t^{2j-1} \\ &= \sum_{j=m+1}^\infty \frac{4j}{2^{2j+1}(2j+1)!}t^{2j-1} > 0, \qquad t>0. \end{aligned}$$

We then obtain that

$$(-1)^n R_m^{(n)}(x) > 0 \quad \text{for} \quad x>0 \quad \text{and} \quad m,n\in\mathbb{N}_0.$$

The proof of Theorem 1 is complete. □

THEOREM 2. *For all $m \in \mathbb{N}_0$, the functions $S_m(x)$, defined by (7), are completely monotonic on $(0,\infty)$.*

Proof. By using (8) and (9), we have

$$\begin{aligned} S_m(x) &= (-1)^m\left[\int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t - \sum_{j=1}^m \frac{(-1)^{j-1}j}{2(j+2)!}\int_0^\infty \frac{t^j}{1-e^{-t}}e^{-xt}\,\mathrm{d}t\right] \\ &= \int_0^\infty (-1)^m\left(q(t) - \sum_{j=1}^m \frac{(-1)^{j-1}j}{2(j+2)!}t^j\right)\frac{e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \end{aligned} \tag{10}$$

where

$$q(t)=\frac{1}{2t}-\frac{1}{t^2}+\left(\frac{1}{2t}+\frac{1}{t^2}\right)e^{-t}=\sum_{j=1}^{m}(-1)^{j-1}\frac{j}{2(j+2)!}t^j,\qquad t>0. \tag{11}$$

We claim that for all $t>0$ and $m\in\mathbb{N}_0$,

$$(-1)^m\left(q(t)-\sum_{j=1}^{m}(-1)^{j-1}\frac{j}{2(j+2)!}t^j.\right)>0, \tag{12}$$

it suffices to show that for all $t>0$ and $k\in\mathbb{N}_0$,

$$\sum_{j=1}^{2k}(-1)^{j-1}\frac{j}{2(j+2)!}t^j<q(t)<\sum_{j=1}^{2k+1}(-1)^{j-1}\frac{j}{2(j+2)!}t^j. \tag{13}$$

We are now in a position to prove (13). It is easy to see that for $t>0$,

$$q(t)e^t=\sum_{j=1}^{\infty}\frac{j}{2\cdot(j+2)!}t^j>0$$

and

$$\left(q(t)-\frac{1}{12}t\right)e^t=-\sum_{j=2}^{\infty}\frac{j(j-1)(j+4)}{12\cdot(j+2)!}t^j<0.$$

Hence, we have

$$0<q(t)<\frac{1}{12}t,\qquad t>0. \tag{14}$$

Let $t>0$ be fixed. Denote

$$a_j=\frac{j}{2(j+2)!}t^j$$

and

$$S_k=\sum_{j=1}^{k}(-1)^{j-1}a_j,\qquad q(t)=S=\sum_{j=1}^{\infty}(-1)^{j-1}a_j.$$

Then we have

$$a_{j-1}\leqslant a_j\iff j+1-\frac{2}{j}\leqslant t.$$

So, the sequence (a_j) is unimodal. Let j^* be the minimal index for which elements of this sequence obtains the maximal value. We consider two cases to show (13).

Case 1. $j^*=2m$, and

$$a_1<a_2<\ldots<a_{2m-1}<a_{2m},\qquad a_{2m}\geqslant a_{2m+1}>a_{2m+2}>\ldots.$$

For $k \leqslant m$ we have

$$S_{2k} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2k-1} - a_{2k}) < 0 < q(t),$$

and

$$S_{2k-1} = a_1 + (-a_2 + a_3) + \ldots + (-a_{2k-2} + a_{2k-1}) > a_1 = \frac{t}{12} > q(t).$$

For $k > m$ we have

$$q(t) = S = S_{2k} + (a_{2k+1} - a_{2k+2}) + \ldots > S_{2k}$$

and

$$q(t) = S = S_{2k-1} - (a_{2k} - a_{2k+1}) - \ldots < S_{2k-1}.$$

Case 2. $j^* = 2m+1$, and

$$a_1 < a_2 < \ldots < a_{2m} < a_{2m+1}, \qquad a_{2m+1} \geqslant a_{2m+2} > a_{2m+3} > \ldots.$$

For $k \leqslant m$ we have

$$S_{2k} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2k-1} - a_{2k}) < 0 < q(t)$$

and

$$S_{2k+1} = a_1 + (-a_2 + a_3) + \ldots + (-a_{2k} + a_{2k+1}) > a_1 = \frac{t}{12} > q(t).$$

For $k > m$ we have

$$q(t) = S = S_{2k} + (a_{2k+1} - a_{2k+2}) + \ldots > S_{2k}$$

and

$$q(t) = S = S_{2k+1} - (a_{2k+2} - a_{2k+3}) - \ldots < S_{2k+1}.$$

This proves the claim.

We then obtain from (10) that

$$(-1)^n S_m^{(n)}(x) > 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad m, n \in \mathbb{N}_0.$$

The proof of Theorem 2 is complete. $\square$

REMARK 1. From $S_m(x) > 0$ (for $x > 0$ and $m \in \mathbb{N}_0$), we obtain the following upper and lower bounds for the gamma function in terms of the polygamma functions

$$\begin{aligned} \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{k=1}^{2m} \frac{k}{2(k+2)!}\psi^{(k)}(x)\right) &< \Gamma(x+1) \\ &< \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{k=1}^{2m+1} \frac{k}{2(k+2)!}\psi^{(k)}(x)\right) \end{aligned} \tag{15}$$

for $x > 0$ and $m \in \mathbb{N}_0$,

3. Asymptotic expansions

In this section, we provide recurrence relations to calculate the coefficients of asymptotic expansions (2) and (3). By using the results obtained, we derive recursive relations of the Bernoulli numbers.

THEOREM 3. *The following asymptotic expansion hold true:*

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty} a_j \psi^{(2j-1)}\left(x+\frac{1}{2}\right) \quad \text{as} \quad x\to\infty, \tag{16}$$

with the coefficients a_j given by the recursive relation

$$a_1 = \frac{B_2}{2}, \quad a_j = \sum_{k=1}^{j-1} a_{j-k}\left(1-\frac{1}{2^{2k-1}}\right)\frac{B_{2k}}{(2k)!} + \frac{B_{2j}}{(2j)!}, \qquad j \geqslant 2, \tag{17}$$

where B_n are the Bernoulli numbers.

Proof. The psi function has the following asymptotic expansion (see [5, p. 33]):

$$\psi(x+t) \sim \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n(t)}{n x^n}, \qquad x\to\infty, \tag{18}$$

where $B_n(t)$ are the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}. \tag{19}$$

Note that the Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ are defined by (19) for $t=0$. Differentiating the relation (18) j times, we obtain

$$\psi^{(j)}(x+t) \sim \sum_{k=0}^{\infty} \frac{(-1)^{k+j-1}(k+j-1)!B_k(t)}{k!x^{k+j}}. \tag{20}$$

It follows from Stirling's series for the gamma function (see [1, p. 257, Equation (6.1.40)]) that

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}, \qquad x\to\infty. \tag{21}$$

In view of (20) and (21), we can let

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty} a_j \psi^{(2j-1)}\left(x+\frac{1}{2}\right) \quad \text{as} \quad x\to\infty, \tag{22}$$

where a_j are real numbers to be determined.

Setting $t=\frac{1}{2}$ and noting that

$$B_k(\tfrac{1}{2}) = -\left(1-\frac{1}{2^{k-1}}\right)B_k, \quad k=0,1,2,\dots,$$

we obtain from (20) that

$$\psi^{(2j-1)}\left(x+\frac{1}{2}\right) \sim -\sum_{k=0}^{\infty}\frac{(2k+2j-2)!}{(2k)!x^{2k+2j-1}}\left(1-\frac{1}{2^{2k-1}}\right)B_{2k}. \tag{23}$$

Substituting (23) into (22), we obtain that, as $x\to\infty$,

$$\begin{aligned}\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) &\sim -\sum_{j=1}^{\infty}\frac{a_j}{x^{2j-1}}\sum_{k=0}^{\infty}\frac{(2k+2j-2)!}{(2k)!x^{2k}}\left(1-\frac{1}{2^{2k-1}}\right)B_{2k}\\ &\sim \sum_{j=1}^{\infty}\left(-\sum_{k=0}^{j-1}a_{j-k}\frac{(2j-2)!}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)B_{2k}\right)\frac{1}{x^{2j-1}}.\end{aligned} \tag{24}$$

Equating the coefficients of $1/x^{2j-1}$ in (21) and (24) yields

$$-\sum_{k=0}^{j-1}a_{j-k}\frac{(2j-2)!}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)B_{2k} = \frac{B_{2j}}{2j(2j-1)}, \quad j\geqslant 1.$$

We then obtain the recursive formula

$$a_1=\frac{B_2}{2}, \quad a_j=\sum_{k=1}^{j-1}a_{j-k}\left(1-\frac{1}{2^{2k-1}}\right)\frac{B_{2k}}{(2k)!}+\frac{B_{2j}}{(2j)!}, \qquad j\geqslant 2.$$

The proof of Theorem 3 is complete. $\square$

REMARK 2. It is know that

$$B_0=1, \quad B_1=-\frac{1}{2}, \quad B_2=\frac{1}{6}, \quad B_4=-\frac{1}{30}, \quad B_6=\frac{1}{42}, \quad B_8=-\frac{1}{30},$$
$$B_{10}=\frac{5}{66},\dots, \quad \text{and} \quad B_{2n+1}=0, \quad n\in\mathbb{N}.$$

Substituting (4) into (17), we find the following recursive relation of the Bernoulli numbers:

$$B_{2n} = -\sum_{k=0}^{n-1}\binom{2n}{2k}\frac{(2^{2k+1}-1)(n-k)}{2^{2n-2}(2n-2k+1)}B_{2k}, \qquad n\in\mathbb{N}. \tag{25}$$

REMARK 3. From (21) and (16), we obtain the following approximation formulas:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\exp\left(\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}\right) = u_n \tag{26}$$

and

$$n! \sim \sqrt{2\pi n}\left(\frac{x}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)+\frac{1}{480}\psi'''\left(n+\frac{1}{2}\right)+\frac{1}{53760}\psi^{(5)}\left(n+\frac{1}{2}\right)\right)$$
$$= v_n, \tag{27}$$

respectively. The following numerical computations (see Table 1) would show that, for $n \in \mathbb{N}$, the formula (27) is sharper than the formula (26).

Table 1. *Comparison between approximation formulas* (26) *and* (27).

n	$\frac{u_n-n!}{n!}$	$\frac{v_n-n!}{n!}$
1	2.87×10^{-4}	1.89×10^{-5}
10	5.87×10^{-11}	6.06×10^{-12}
100	5.95×10^{-18}	6.19×10^{-19}
1000	5.95×10^{-25}	6.2×10^{-26}
10000	5.95×10^{-32}	6.2×10^{-33}

THEOREM 4. *The following asymptotic expansion hold true:*

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty} b_j\psi^{(j)}(x) \quad \textit{as} \quad x\to\infty, \tag{28}$$

with the coefficients b_j given by the recursive relation

$$b_1=\frac{B_2}{2},\quad b_j=\frac{b_{j-1}}{2}-\sum_{k=0}^{j-3}\frac{(-1)^k b_{j-k-2}B_{k+2}}{(k+2)!}+\frac{(-1)^{j-1}B_{j+1}}{(j+1)!},\qquad j\geqslant 2. \tag{29}$$

where B_n are the Bernoulli numbers.

Proof. Write (21) as

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty}\frac{B_{j+1}}{j(j+1)x^j},\qquad x\to\infty. \tag{30}$$

It is well-known (see [1, p. 260, Equation (6.4.11)]) that

$$\psi^{(j)}(x) \sim (-1)^{j-1}\left[\frac{(j-1)!}{x^j}+\frac{j!}{2x^{j+1}}+\sum_{k=2}^{\infty}B_k\frac{(k+j-1)!}{k!x^{k+j}}\right] \tag{31}$$

for $x\to\infty$ and $j\in\mathbb{N}$. In view of (30) and (31), we can let

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) \sim \sum_{j=1}^{\infty} b_j\psi^{(j)}(x) \quad \text{as} \quad x\to\infty, \tag{32}$$

where b_j are real numbers to be determined. Substituting (31) into (32), we obtain that, as $x\to\infty$,

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)\sim\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(j-1)!b_j}{x^j}+\sum_{j=1}^{\infty}\frac{(-1)^{j-1}j!b_j}{2x^{j+1}}+\sum_{j=1}^{\infty}\frac{(-1)^{j-1}b_j}{x^j}\sum_{k=2}^{\infty}\frac{(k+j-1)!B_k}{k!x^k}.$$

It is not difficult to see that

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}b_j}{x^j}\sum_{k=2}^{\infty}\frac{(k+j-1)!B_k}{k!x^k}=\sum_{j=3}^{\infty}\left(\sum_{k=0}^{j-3}b_{j-k-2}(-1)^{j-k-1}B_{k+2}\frac{(j-1)!}{(k+2)!}\right)\frac{1}{x^j}.$$

Hence, we have

$$\begin{aligned}\ln&\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)\\&\sim\frac{b_1}{x}+\left(\frac{b_1}{2}-b_2\right)\frac{1}{x^2}\\&\quad+\sum_{j=3}^{\infty}(-1)^{j-1}(j-1)!\left(b_j-\frac{b_{j-1}}{2}+\sum_{k=0}^{j-3}(-1)^kb_{j-k-2}B_{k+2}\frac{1}{(k+2)!}\right)\frac{1}{x^j}.\end{aligned}\tag{33}$$

Equating coefficients of the term $1/x^j$ on the right sides of (30) and (33) yields

$$\begin{aligned}&b_1=\frac{B_2}{2},\\&\frac{b_1}{2}-b_2=0,\\&(-1)^{j-1}(j-1)!\left(b_j-\frac{b_{j-1}}{2}+\sum_{k=0}^{j-3}(-1)^kb_{j-k-2}B_{k+2}\frac{1}{(k+2)!}\right)=\frac{B_{j+1}}{j(j+1)},\quad j\geqslant 3.\end{aligned}$$

We then obtain the recursive formula

$$b_1=\frac{B_2}{2},\quad b_j=\frac{b_{j-1}}{2}-\sum_{k=0}^{j-3}\frac{(-1)^kb_{j-k-2}B_{k+2}}{(k+2)!}+\frac{(-1)^{j-1}B_{j+1}}{(j+1)!},\qquad j\geqslant 2.$$

The proof of Theorem 3 is complete. □

REMARK 4. Substituting (5) into (29), we find the following recursive relation of the Bernoulli numbers:

$$B_n=(-1)^n\sum_{k=0}^{n-2}(-1)^k\binom{n}{k}\frac{n-k-1}{n-k+1}B_k,\qquad n\in\mathbb{N}.\tag{34}$$

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