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A PROOF OF THE THREE GEOMETRIC INEQUALITIES CONJECTURED BY YU–DONG WU AND H. M. SRIVASTAVA

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(*Communicated by Hari M. Srivastava*)

Abstract. In this short note the authors give answers to the three open problems formulated by Wu and Srivastava [*Appl. Math. Lett. 25 (2012), 1347–1353*]. We disprove the Problem 1, by showing that there exists a triangle which does not satisfies the proposed inequality. We prove the inequalities conjectured in Problems 2 and 3. Furthermore, we introduce an optimal refinement of the inequality conjectured on Problem 3.

1. Introduction and conjectures of Yu-Dong Wu and H. M. Srivastava

The geometric inequalities are relevant in several areas of the science and engineering [1, 2, 3, 4, 6, 9, 13]. The methodologies to prove the geometric inequalities is disperse see for instance [5, 7, 8, 10]. In a broad sense, there exist some of methodologies which are based on analytical methods, other in integral and differential calculus, and other on geometric methods. The methodology of this paper, in spite of the basic arguments, can be considered belongs to the analytical methods since the results are strongly dependent on the analytical methodology introduced in [6, 14].

The focus of this short note is the open problems given in [6, 14]. Indeed, we introduce some notation and then we recall the conjectures. Let us consider a triangle $\triangle ABC$ with angles A, B and C, we denote by a, b, c, s and r, the lengths of the corresponding opposite sides, the semiperimeter and the inradius, respectively. Then, using the symbol $\sum$ to denote a cyclic sum, i. e.

$$\sum f(b,c) = f(a,b) + f(b,c) + f(c,a),$$

we have that, the following geometric inequality

$$2\sqrt{2}\, s \leqslant \sum \sqrt{a^2+b^2} < (2+\sqrt{2})\, s - 3\sqrt{3}(2-\sqrt{2})r, \tag{1}$$

holds. In (1), the left inequality can be proved by the Power-Mean Inequality and the right inequality was recently proved by Wu and Srivastava in [14]. It was originally proposed by Wu [12], inspired in the following inequality [12, 15]:

$$\sum \sqrt{a^2+b^2} < (2+\sqrt{2})s. \tag{2}$$

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Moreover, it is known the following generalized version of (2):

$$\sum \sqrt[n]{a^n+b^n} < (2+\sqrt[n]{2})s, \qquad n \in \mathbb{N}\setminus\{1\}, \tag{3}$$

holds true for $\mathbb{N}$ the set of positive integers, see [11]. Then, by analogy to the extension of Ye [11], Wu and Srivastava proposed a generalization of the right inequality in [14]. More precisely, they defined the following conjecture:

CONJECTURE 1. *For a given triangle $\triangle ABC$, if $n \in \mathbb{N}\setminus\{1,2\}$, then prove or disprove the following inequality:*

$$\sum \sqrt[n]{a^n+b^n} < (2+\sqrt[n]{2})s - 3\sqrt{3}(2-\sqrt[n]{2})r. \tag{4}$$

Additionally, in [14] Wu and Srivastava conjecture the following two inequalities:

CONJECTURE 2. *Let a_i $(i=1,\dots,6)$ denote the lengths of the edges of a given tetrahedron $ABCD$. Also let ρ be the inradius of the tetrahedron. Then, determine the best constant k such the following inequality holds true:*

$$\sum_{1\leqslant i,j\leqslant 6} \sqrt{a_i^2+b_j^2} \leqslant k\sum_{i=1}^{6} a_i. \tag{5}$$

CONJECTURE 3. *Let us denote by k_0 denotes the best constant k for the inequality (5) for a given tetrahedron $ABCD$. Then, prove or disprove the following inequality*

$$\sum_{1\leqslant i,j\leqslant 6} \sqrt{a_i^2+b_j^2} \leqslant k_0\sum_{i=1}^{6} a_i - 6\sqrt{6}(2k_0-5\sqrt{2})\rho. \tag{6}$$

We note that, the inequality (4) looks as a nice generalization of (1). However, unfortunately, we show that the inequality (4) is not always true, see subsection 2.1. In subsection 2.2, we determine that (5) holds true with $k=2+\sqrt{2}$. Furthermore, denoting by $k_0=2+\sqrt{2}$, in subsection 2.3, we prove that the following inequality

$$\sum_{1\leqslant i,j\leqslant 6} \sqrt{a_i^2+b_j^2} \leqslant k_0\sum_{i=1}^{6} a_i - 12\sqrt{3}(2-\sqrt{2})\rho, \tag{7}$$

which is a refinement of (6). Then, the Conjecture 3 holds true but the inequality is not optimal. Hence, summarizing the contribution of this sort note we have the following theorem:

THEOREM 1. *Let a_i $(i=1,\dots,6)$ denote the lengths of the edges of a given tetrahedron $ABCD$. Also let ρ be the inradius of the tetrahedron. Then, the inequalities (5) and (7) holds true with $k=k_0=2+\sqrt{2}$.*

2. Proofs of Conjectures

2.1. Counterexample for Conjecture 1

We given a counterexample which proves that (4) does not holds true. Indeed, let us consider $n=3$ and the right triangle $\triangle ABC$ with $a=3$, $b=1$ and $c=\sqrt{10}$. Then, $s=(4+\sqrt{10})/2$, $r=3/(4+\sqrt{10})$ and clearly the inequality (4) is reversed, since

$$\sum \sqrt[3]{a^3+b^3} \approx 10.116536541585731 \quad \text{and}$$
$$(2+\sqrt[3]{2})s-3\sqrt{3}(2-\sqrt[3]{2})r \approx 10.063472825231253.$$

Furthermore, if for the given triangle we define the function $g:\mathbb{N}\setminus\{1\}\to\mathbb{R}$ as follows

$$g(n)=(2+\sqrt[n]{2})s-3\sqrt{3}(2-\sqrt[n]{2})r-\sum \sqrt[n]{a^n+b^n},$$

we note that g is strictly decreasing with $g(2)>0>g(3)$. Thus, for the given right triangle the inequality (4) is false for all $n\in\mathbb{N}\setminus\{1,2\}$.

2.2. Proof of Conjecture 2

We apply the inequality (2) over each face of the tetrahedron $ABCD$ and, naturally, we get the following optimal estimates

$$\begin{aligned}
\sum_{1\leqslant i,j\leqslant 6}\sqrt{a_i^2+b_j^2} &= \left[\sqrt{a_1^2+a_3^2}+\sqrt{a_3^2+a_2^2}+\sqrt{a_2^2+a_1^2}\right]+\left[\sqrt{a_3^2+a_4^2}+\sqrt{a_4^2+a_5^2}+\sqrt{a_5^2+a_3^2}\right]\\
&\quad+\left[\sqrt{a_1^2+a_4^2}+\sqrt{a_4^2+a_6^2}+\sqrt{a_6^2+a_1^2}\right]+\left[\sqrt{a_6^2+a_5^2}+\sqrt{a_5^2+a_2^2}+\sqrt{a_2^2+a_4^2}\right]\\
&\leqslant \left(1+\frac{\sqrt{2}}{2}\right)(a_1+a_2+a_3)+\left(1+\frac{\sqrt{2}}{2}\right)(a_3+a_4+a_5)\\
&\quad+\left(1+\frac{\sqrt{2}}{2}\right)(a_1+a_4+a_6)+\left(1+\frac{\sqrt{2}}{2}\right)(a_2+a_5+a_6)\\
&\leqslant (2+\sqrt{2})\sum_{i=1}^{6}a_i.
\end{aligned}$$

Then, we have that the inequality (5) holds with optimal constant $k=2+\sqrt{2}$.

2.3. Proof of Conjecture 3

Let us denote by $k_0=2+\sqrt{2}$ and by r_i for $i=1,\dots,4$ the inradius of the faces of the tetrahedron $ABCD$. Then, applying the inequality (4) over each face of the tetrahedron $ABCD$ we find that (7) holds true, since

$$\sum_{1\leqslant i,j\leqslant 6}\sqrt{a_i^2+b_j^2}\leqslant k_0\sum_{i=1}^{6}a_i-3\sqrt{3}(2-\sqrt{2})\sum_{i=1}^{4}r_i\leqslant k_0\sum_{i=1}^{6}a_i-12\sqrt{3}(2-\sqrt{2})\rho.$$

Moreover, we note that

$$k_0 \sum_{i=1}^{6} a_i - 12\sqrt{3}(2-\sqrt{2})\rho \leqslant k_0 \sum_{i=1}^{6} a_i - 6\sqrt{6}(2k_0 - 5\sqrt{2})\rho.$$

Then, the inequality (6) is true, but it is not optimal.

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