

ON REFINEMENTS OF CAUCHY'S INEQUALITY AND HÖLDER'S INEQUALITY

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Abstract. The aim of this paper is to give refinements of Hölder's inequality by a sharpening of Cauchy's inequality.

1. Introduction

Let U be a real inner product space, and let u and v be arbitrary elements of U . Then Cauchy's inequality states that $|(u, v)| \leq \|u\| \|v\|$, where $\|u\| = \sqrt{(u, u)}$ and (u, v) denotes the inner product of u and v . X. Gao, M. Gao and X. Shang showed (see Lemma 2.1 in [2]) that if $w \in U$ and $\|u\| \|v\| \|w\| \neq 0$, then

$$\left| \frac{(u, v)}{\|u\| \|v\|} \right| \leq \left(1 - \left(\frac{|(u, w)|}{\|u\| \|w\|} - \frac{|(w, v)|}{\|w\| \|v\|} \right)^2 \right)^{\frac{1}{2}}, \quad (1)$$

which is clearly a sharpening of Cauchy's inequality. We will give an improvement of (1) in order to get refinements of Hölder's inequality.

Let $p > 1$, $1/p + 1/q = 1$, $f \in L^p$, where $L^p = L^p(S, \mu)$ is a Lebesgue space, and let $g \in L^q$. Then Hölder's inequality states that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \quad (2)$$

where $\|f\|_r = \left(\int_S |f|^r d\mu \right)^{\frac{1}{r}}$ for $r \geq 1$ and $f \in L^r$. Inequality (2) has many refinements and generalizations, for example, Aldaz [1] proved that if $\|f\|_p \|g\|_q \neq 0$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \left(1 - \min \left(\frac{1}{p}, \frac{1}{q} \right) \left\| \left(\frac{|f|}{\|f\|_p} \right)^{p/2} - \left(\frac{|g|}{\|g\|_q} \right)^{q/2} \right\|_2^2 \right). \quad (3)$$

Using a refinement of (1), we will prove an improvement of (3).

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2. Main results

LEMMA. *If u, v and w are elements of a real inner product space such that $\|u\|\|v\|\|w\| \neq 0$, then*

$$\begin{aligned} & \left| \frac{(u, w)}{\|u\|\|w\|} \cdot \frac{(w, v)}{\|w\|\|v\|} \right| - \sqrt{\left(1 - \left(\frac{(u, w)}{\|u\|\|w\|}\right)^2\right) \left(1 - \left(\frac{(w, v)}{\|w\|\|v\|}\right)^2\right)} \\ & \cong \left| \frac{(u, v)}{\|u\|\|v\|} \right| \\ & \cong \left| \frac{(u, w)}{\|u\|\|w\|} \cdot \frac{(w, v)}{\|w\|\|v\|} \right| + \sqrt{\left(1 - \left(\frac{(u, w)}{\|u\|\|w\|}\right)^2\right) \left(1 - \left(\frac{(w, v)}{\|w\|\|v\|}\right)^2\right)}. \end{aligned} \tag{4}$$

First we will show that the second inequality in (4) is a refinement of (1). Let $a = \frac{(u,v)}{\|u\|\|v\|}$, $b = \frac{(u,w)}{\|u\|\|w\|}$ and $c = \frac{(w,v)}{\|w\|\|v\|}$, then by Cauchy's inequality, $|a| \leq 1$, $|b| \leq 1$ and $|c| \leq 1$. Since

$$\begin{aligned} \left(|bc| + \sqrt{(1-b^2)(1-c^2)}\right)^2 &= 1 - b^2 - c^2 + 2b^2c^2 + 2|bc|\sqrt{1-b^2}\sqrt{1-c^2} \\ &= 1 - \left(|b|\sqrt{1-c^2} - |c|\sqrt{1-b^2}\right)^2 \leq 1, \end{aligned}$$

therefore

$$\begin{aligned} \left(|bc| + \sqrt{(1-b^2)(1-c^2)}\right)^2 &= 1 - b^2 - c^2 + 2|bc|\left(|bc| + \sqrt{(1-b^2)(1-c^2)}\right) \\ &\leq 1 - b^2 - c^2 + 2|bc| = 1 - (|b| - |c|)^2. \end{aligned}$$

Proof of the lemma. For any x, y, z real numbers, we have

$$\begin{aligned} 0 &\leq \left\| z \frac{u}{\|u\|} + y \frac{v}{\|v\|} + x \frac{w}{\|w\|} \right\|^2 \\ &= z^2 + y^2 + x^2 + 2yza + 2xzb + 2xyc = x^2 + 2x(yz + zb) + y^2 + z^2 + 2yza. \end{aligned} \tag{5}$$

Thus, considering the right side of (5) as a function of x , we get

$$0 \geq (2(yz + zb))^2 - 4(y^2 + z^2 + 2yza),$$

that is,

$$0 \geq y^2(c^2 - 1) + 2yz(bc - a) + z^2(b^2 - 1). \tag{6}$$

If $|c| < 1$, then the right side of (6) is a non-positive quadratic function of y , so

$$0 \geq (2z(bc - a))^2 - 4(c^2 - 1)z^2(b^2 - 1) = 4z^2((bc - a)^2 - (c^2 - 1)(b^2 - 1)).$$

Therefore $(bc - a)^2 \leq (c^2 - 1)(b^2 - 1)$, i.e.,

$$|a - bc| \leq \sqrt{(1 - b^2)(1 - c^2)}. \tag{7}$$

In case $|c| = 1$, (6) implies that $bc = a$, which means that (7) holds for $|c| = 1$. Finally, by the triangle inequality, $|a| - |bc| \leq |a - bc|$ and $|bc| - |a| \leq |a - bc|$, thus by (7),

$$|bc| - \sqrt{(1 - b^2)(1 - c^2)} \leq |a| \leq |bc| + \sqrt{(1 - b^2)(1 - c^2)}.$$

This completes the proof of the lemma. \square

THEOREM 1. *If $f, g, h \in L^2(S, \mu)$ and $\|h\|_2 \neq 0$, then*

$$\begin{aligned} & \left| \int_S |f| \frac{h}{\|h\|_2} d\mu \cdot \int_S |g| \frac{h}{\|h\|_2} d\mu \right| \tag{8} \\ & - \sqrt{\int_S |f|^2 d\mu - \left(\int_S |f| \frac{h}{\|h\|_2} d\mu \right)^2} \sqrt{\int_S |g|^2 d\mu - \left(\int_S |g| \frac{h}{\|h\|_2} d\mu \right)^2} \\ & \leq \int_S |fg| d\mu \\ & \leq \left| \int_S |f| \frac{h}{\|h\|_2} d\mu \cdot \int_S |g| \frac{h}{\|h\|_2} d\mu \right| \\ & + \sqrt{\int_S |f|^2 d\mu - \left(\int_S |f| \frac{h}{\|h\|_2} d\mu \right)^2} \sqrt{\int_S |g|^2 d\mu - \left(\int_S |g| \frac{h}{\|h\|_2} d\mu \right)^2}. \end{aligned}$$

Proof. If $\|f\|_2 \|g\|_2 \neq 0$, then applying the lemma with $u = |f|$, $v = |g|$ and $w = h$, and multiplying by $\|f\|_2 \|g\|_2$, we get

$$\begin{aligned} & \|f\|_2 \|g\|_2 \left(\left| \int_S \frac{|f|h}{\|f\|_2 \|h\|_2} d\mu \cdot \int_S \frac{|g|h}{\|g\|_2 \|h\|_2} d\mu \right| \right. \tag{9} \\ & \left. - \sqrt{\left(1 - \left(\int_S \frac{|f|h}{\|f\|_2 \|h\|_2} d\mu \right)^2 \right) \left(1 - \left(\int_S \frac{|g|h}{\|g\|_2 \|h\|_2} d\mu \right)^2 \right)} \right) \\ & \leq \int_S |fg| d\mu \end{aligned}$$

$$\begin{aligned} &\leq \|f\|_2 \|g\|_2 \left(\left| \int_S \frac{|f|h}{\|f\|_2 \|h\|_2} d\mu \cdot \int_S \frac{|g|h}{\|g\|_2 \|h\|_2} d\mu \right| \right. \\ &\quad \left. + \sqrt{\left(1 - \left(\int_S \frac{|f|h}{\|f\|_2 \|h\|_2} d\mu\right)^2\right) \left(1 - \left(\int_S \frac{|g|h}{\|g\|_2 \|h\|_2} d\mu\right)^2\right)} \right), \end{aligned}$$

which can be written as (8). If $\|f\|_2 \|g\|_2 = 0$, then $f = 0$ μ almost everywhere in S , or $g = 0$ μ almost everywhere in S , thus we obtain that the sides of the inequalities in (8) are equal to 0. This completes the proof of the theorem. \square

THEOREM 2. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$, $h \in L^2$, and $\|f\|_p \|g\|_q \|h\|_2 \neq 0$. Then

$$\begin{aligned} \int_S |fg| d\mu &\leq \|f\|_p \|g\|_q \left(\left| \int_S \frac{|f|^{p/2} h}{\|f\|_p^{p/2} \|h\|_2} d\mu \cdot \int_S \frac{|g|^{q/2} h}{\|g\|_q^{q/2} \|h\|_2} d\mu \right| \right. \\ &\quad \left. + \sqrt{\left(1 - \left(\int_S \frac{|f|^{p/2} h}{\|f\|_p^{p/2} \|h\|_2} d\mu\right)^2\right) \left(1 - \left(\int_S \frac{|g|^{q/2} h}{\|g\|_q^{q/2} \|h\|_2} d\mu\right)^2\right)} \right)^{2 \min(1/p, 1/q)}. \end{aligned} \quad (10)$$

The idea of the following argument appeared in Theorem 2.2 and Theorem 2.3 of [2].

Proof of Theorem 2. If $p > q$, then let $P = \frac{p}{2}$ and $Q = \frac{p}{p-2}$, and so $\frac{1}{P} + \frac{1}{Q} = 1$. Furthermore, $\left(\frac{|f|}{\|f\|_p}\right)^{p/2} \in L^2$, because

$$\int_S \left(\left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \right)^2 d\mu = \frac{\int_S |f|^p d\mu}{\|f\|_p^p} = 1,$$

and similarly, $\left(\frac{|g|}{\|g\|_q}\right)^{q/2} \in L^2$. Thus by Hölder's inequality, $\left(\frac{|f|}{\|f\|_p}\right)^{p/2} \left(\frac{|g|}{\|g\|_q}\right)^{q/2} \in L^1$, i.e., $\frac{|f|}{\|f\|_p} \left(\frac{|g|}{\|g\|_q}\right)^{q/p} \in L^p$. Since $1 - \frac{q}{p} = 1 - \frac{1}{p} \cdot \frac{p}{p-1} = \frac{p-2}{p-1}$, therefore $\left(\frac{|g|}{\|g\|_q}\right)^{1-q/p} = \left(\frac{|g|}{\|g\|_q}\right)^{(p-2)/(p-1)} \in L^Q$, because

$$\int_S \left(\left(\frac{|g|}{\|g\|_q} \right)^{\frac{p-2}{p-1}} \right)^Q d\mu = \int_S \left(\frac{|g|}{\|g\|_q} \right)^{\frac{p-2}{p-1} Q} d\mu = \int_S \left(\frac{|g|}{\|g\|_q} \right)^q d\mu = 1.$$

So by Hölder's inequality, we get

$$\begin{aligned} \int_S \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} d\mu &= \int_S \left(\frac{|f|}{\|f\|_p} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{p}} \right) \cdot \left(\frac{|g|}{\|g\|_q} \right)^{1-\frac{q}{p}} d\mu \\ &\leq \left(\int_S \left(\frac{|f|}{\|f\|_p} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{p}} \right)^p d\mu \right)^{\frac{1}{p}} \cdot \left(\int_S \left(\left(\frac{|g|}{\|g\|_q} \right)^{1-\frac{q}{p}} \right)^q d\mu \right)^{\frac{1}{q}} \\ &= \left(\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right)^{\frac{2}{p}} \cdot 1. \end{aligned}$$

If $q > p$, then the argument is similar, so we have

$$\begin{aligned} \int_S \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} d\mu &\leq \left(\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right)^{2\min\left(\frac{1}{p}, \frac{1}{q}\right)} \tag{11} \\ &= \left(\frac{\int_S |f|^{\frac{p}{2}} |g|^{\frac{q}{2}} d\mu}{\sqrt{\int_S |f|^p d\mu} \sqrt{\int_S |g|^q d\mu}} \right)^{2\min\left(\frac{1}{p}, \frac{1}{q}\right)}, \end{aligned}$$

which also holds for $p = q = 2$. Applying (9) for $|f|^{p/2}$, $|g|^{q/2}$ and h , Theorem 2 follows from (11). \square

We note that if $p > q$ (and $\int_S |f|^{p/2} |g|^{q/2} d\mu \neq 0$), then by (3),

$$\begin{aligned} \int_S \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} d\mu &= \int_S \left(\frac{|f|}{\|f\|_p} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{p}} \right) \cdot \left(\frac{|g|}{\|g\|_q} \right)^{1-\frac{q}{p}} d\mu \\ &\leq \left(\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right)^{\frac{2}{p}} \\ &\quad \times \left(1 - \min\left(\frac{2}{p}, 1 - \frac{2}{p}\right) \left\| \frac{\left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{4}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{4}}}{\sqrt{\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu}} - \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} \right\| \right)^2, \end{aligned}$$

which is a refinement of (11), and using (9), we can improve (10).

We will show that if $h = |g|^{q/2}$, then (10) is a refinement of (3). To this, $\|h\|_2^2 = \int_S |h|^2 d\mu = \int_S |g|^q d\mu = \|g\|_q^q$, therefore

$$\frac{\int_S |g|^{q/2} h d\mu}{\|g\|_q^{q/2} \|h\|_2} = \frac{\|g\|_q^q}{\|g\|_q^q} = 1,$$

and so (10) can be written as (11). Suppose that $p > q$ (the case $p < q$ is similar), then by Young’s inequality,

$$\begin{aligned} \left(\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right)^{\frac{2}{p}} \cdot 1 &\leq \frac{2}{p} \left(\int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right) + \left(1 - \frac{2}{p} \right) \cdot 1 \\ &= 1 - \frac{1}{p} \left(2 - 2 \int_S \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} d\mu \right) \\ &= 1 - \frac{1}{p} \int_S \left(\left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} - \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} \right)^2 d\mu \\ &= 1 - \frac{1}{p} \left\| \left(\frac{|f|}{\|f\|_p} \right)^{\frac{p}{2}} - \left(\frac{|g|}{\|g\|_q} \right)^{\frac{q}{2}} \right\|_2^2, \end{aligned}$$

and this inequality also holds for $p = q$.

The proof of the following theorem is based on Theorem 1 and the factorization of the function $|fg|$ as a product of two functions which are elements of L^2 .

THEOREM 3. *Let $p > 1$, $1/p + 1/q = 1$, $f \in L^p$, $g \in L^q$, $h \in L^2$, $\|h\|_2 \neq 0$ and $\tau \in \mathbb{R}$. If $|\tau| \leq \min(1/p, 1/q)$, and in case $|\tau| = 1/p$, $|f| > 0$ μ almost everywhere in S , and in case $|\tau| = 1/q$, $|g| > 0$ μ almost everywhere in S , then*

$$\begin{aligned} &\left| \int_S |f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \frac{h}{\|h\|_2} d\mu \cdot \int_S |f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \frac{h}{\|h\|_2} d\mu \right| \tag{12} \\ &- \sqrt{\int_S |f|^{1+p\tau} |g|^{1-q\tau} d\mu - \left(\int_S |f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\times \sqrt{\int_S |f|^{1-p\tau} |g|^{1+q\tau} d\mu - \left(\int_S |f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\leq \int_S |fg| d\mu \end{aligned}$$

$$\begin{aligned} &\leq \left| \int_S |f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \frac{h}{\|h\|_2} d\mu \cdot \int_S |f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \frac{h}{\|h\|_2} d\mu \right| \\ &\quad + \sqrt{\int_S |f|^{1+p\tau} |g|^{1-q\tau} d\mu - \left(\int_S |f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\quad \times \sqrt{\int_S |f|^{1-p\tau} |g|^{1+q\tau} d\mu - \left(\int_S |f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \frac{h}{\|h\|_2} d\mu \right)^2}. \end{aligned}$$

Proof. Clearly,

$$\int_S |fg| d\mu = \int_S \left(|f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \right) \cdot \left(|f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \right) d\mu.$$

Furthermore, in case $|\tau| < \min(1/p, 1/q)$, $p/(1+p\tau) > p/(1+p(1-1/p)) = 1$ and $(1+p\tau)/p + (1-q\tau)/q = 1$, therefore $|f|^{1+p\tau} \in L^{p/(1+p\tau)}$, $|g|^{1-q\tau} \in L^{q/(1-q\tau)}$, and by Hölder's inequality, $|f|^{1+p\tau} |g|^{1-q\tau} \in L^1$, which remains true in case $|\tau| = \min(1/p, 1/q)$ (for example, if $\tau = -1/p$, then we have $|g|^{1+q(1-1/q)} = |g|^q \in L^1$). That is, $|f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2} \in L^2$. Similarly, $|f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2} \in L^2$, thus we can apply Theorem 1 for $|f|^{(1+p\tau)/2} |g|^{(1-q\tau)/2}$, $|f|^{(1-p\tau)/2} |g|^{(1+q\tau)/2}$ and h . This completes the proof of Theorem 3. \square

We note that if $|\tau| = \min(1/p, 1/q)$, then by a similar argument we have that (12) holds without the conditions $|f| > 0$ and $|g| > 0$ if we omit the powers of zero exponent in (12). For example, if $p = q = 2$ and $\tau = 1/2$, then we get Theorem 1. If $p > q$ and $\tau = 1/p$, then the following statement follows from Theorem 3.

COROLLARY. *Let $p > q > 1$, $1/p + 1/q = 1$, $f \in L^p$, $g \in L^q$, $h \in L^2$, and $\|h\|_2 \neq 0$. Then*

$$\begin{aligned} &\left| \int_S |f| |g|^{1-q/2} \frac{h}{\|h\|_2} d\mu \cdot \int_S |g|^{q/2} \frac{h}{\|h\|_2} d\mu \right| \\ &\quad - \sqrt{\int_S |f|^2 |g|^{2-q} d\mu - \left(\int_S |f| |g|^{1-q/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\quad \times \sqrt{\int_S |g|^q d\mu - \left(\int_S |g|^{q/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\leq \int_S |fg| d\mu \end{aligned}$$

$$\begin{aligned} &\leq \left| \int_S |f||g|^{1-q/2} \frac{h}{\|h\|_2} d\mu \cdot \int_S |g|^{q/2} \frac{h}{\|h\|_2} d\mu \right| \\ &+ \sqrt{\int_S |f|^2 |g|^{2-q} d\mu - \left(\int_S |f||g|^{1-q/2} \frac{h}{\|h\|_2} d\mu \right)^2} \\ &\times \sqrt{\int_S |g|^q d\mu - \left(\int_S |g|^{q/2} \frac{h}{\|h\|_2} d\mu \right)^2}. \end{aligned}$$

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