

SOME NEW HARDY TYPE INEQUALITIES WITH GENERAL KERNELS II

KRISTINA KRULIĆ HIMMELREICH AND JOSIP PEČARIĆ

(Communicated by N. Elezović)

Abstract. In this paper we prove new Hardy type inequalities with general kernels. We give new results that involve the Hardy–Hilbert and the Pólya–Knopp inequality. We also prove new results that involve n -convex functions.

1. Introduction

The classical Hardy inequality reads:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad p > 1, \quad (1.1)$$

where f is nonnegative function such that $f \in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0, \infty)$. The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. The almost dramatic period of research in at least 10 years until G. H. Hardy [2] stated and proved (1.1) was recently described in details in [7] and [8].

By putting $f(t) = g\left(t^{\frac{p-1}{p}}\right)t^{\frac{-1}{p}}$ and making some obvious substitutions we find that (1.1) is equivalent to

$$\int_0^\infty \left(\frac{1}{x} \int_0^x g(t) dt \right)^p \frac{dx}{x} \leq \int_0^\infty g^p(x) \frac{dx}{x}. \quad (1.2)$$

Note that (1.2) holds also for $p = 1$ (with equality) while (1.1) has no meaning for $p = 1$. Of course this proof also shows that the following more general inequality

$$\int_0^\infty \Phi \left(\frac{1}{x} \int_0^x f(t) dt \right) \frac{dx}{x} \leq \int_0^\infty \Phi(f(x)) \frac{dx}{x} \quad (1.3)$$

holds for each convex function Φ on the interval I with $Imf \subseteq I$. This observation can be found in the papers [4] by Kaijser et al. but was known even before, see e.g. Godunova [1].

Mathematics subject classification (2010): 26D10, 26D15.

Keywords and phrases: The Hardy inequality, the Hardy–Hilbert inequality, inequalities, Hardy type inequalities, the Green function, the Taylor theorem, convex function, kernel.

Let H_k be defined by

$$H_k f(x) = \frac{1}{K(x)} \int_0^x f(t)k(x,t) dt, \quad (1.4)$$

where

$$K(x) := \int_0^x k(x,t) dt < \infty$$

Here $k(x,y)$ is a general measurable and nonnegative function, a so called kernel.

The following result was proved by Kaijser et al. [3]:

THEOREM 1.1. *Let u be a weight function on $(0,b)$, $0 < b \leq \infty$, and let $k(x,y) \geq 0$ on $(0,b) \times (0,b)$. Assume that $\frac{k(x,y)u(x)}{xK(x)}$ is locally integrable on $(0,b)$ for each fixed $y \in (0,b)$ and define v by*

$$v(y) = y \int_y^b \frac{k(x,y)}{K(x)} u(x) \frac{dx}{x} < \infty, \quad y \in (0,b).$$

If Φ is a positive and convex function on (a,c) , $-\infty \leq a < c \leq \infty$, then

$$\int_0^b \Phi(H_k f(x)) u(x) \frac{dx}{x} \leq \int_0^b \Phi(f(x)) v(x) \frac{dx}{x}, \quad (1.5)$$

for all f with $a < f(x) < c$, $0 \leq x \leq b$, where H_k is defined by (1.4).

In the same paper the dual operator $H_{\bar{k}}$, defined by

$$H_{\bar{k}} f(x) := \frac{1}{\bar{K}(x)} \int_x^\infty k(x,y) f(y) dy, \quad (1.6)$$

where $\bar{K}(x) = \int_x^\infty k(x,y) dy < \infty$, was studied and the following result was proved:

THEOREM 1.2. *For $0 \leq b < \infty$, let u be a weight function such that $\frac{k(x,y)u(x)}{x\bar{K}(x)}$ is locally integrable on (b,∞) for every fixed $y \in (b,\infty)$. Let the function v be defined by*

$$v(y) = y \int_b^y \frac{k(x,y)}{\bar{K}(x)} u(x) \frac{dx}{x} < \infty, \quad y \in (b,\infty).$$

If Φ is a positive and convex function on (a,c) , $-\infty \leq a < c \leq \infty$, then

$$\int_b^\infty \Phi(H_{\bar{k}} f(x)) u(x) \frac{dx}{x} \leq \int_b^\infty \Phi(f(x)) v(x) \frac{dx}{x}, \quad (1.7)$$

for all f with $a < f(x) < c$, $b \leq x < \infty$, where $H_{\bar{k}}$ is defined by (1.6).

In the sequel let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and let A_k be defined as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{1.8}$$

where $f : \Omega_2 \rightarrow \mathbb{R}$ is measurable, $k : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ is a measurable and nonnegative kernel and

$$K(x) := \int_{\Omega_2} k(x, y) d\mu_2(y) < \infty, \quad x \in \Omega_1. \tag{1.9}$$

The following result was given in [5] see also [6].

THEOREM 1.3. *Let u be a weight function, $k(x, y) \geq 0$. Assume that $\frac{k(x, y)}{K(x)}u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} \frac{k(x, y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{1.10}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \Phi(A_k f(x)) u(x) d\mu_1(x) \leq \int_{\Omega_2} \Phi(f(y)) v(y) d\mu_2(y) \tag{1.11}$$

holds for all measurable functions $f : \Omega_2 \rightarrow \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (1.8)–(1.9).

This result unifies and generalizes most results of this type.

Now consider the Green function G defined on $[\alpha, \beta] \times [\alpha, \beta]$ by

$$G(t, s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(t-\alpha)(s-\beta)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{1.12}$$

The function G is convex under s , it is symmetric nonpositive function and it is also convex under t . It is continuous under s and continuous under t .

For any function $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$, $\phi \in C^2([a, b])$, we can easily show by integrating by parts that the following is valid

$$\phi(x) = \frac{\beta-x}{\beta-\alpha} \phi(\alpha) + \frac{x-\alpha}{\beta-\alpha} \phi(\beta) + \int_{\alpha}^{\beta} G(x, s) \phi''(s) ds, \tag{1.13}$$

where the function G is defined as above in (1.12).

Next we give a well known Taylor’s theorem with the integral remainder.

THEOREM 1.4. *Let n be a positive integer and $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, then for all $x \in [\alpha, \beta]$ the Taylor formula at the point $c \in [\alpha, \beta]$ is*

$$\phi(x) = T_{n-1}(\phi; c, x) + R_{n-1}(\phi; c, x)$$

where

$$T_{n-1}(\phi; c, x) = \sum_{k=0}^{n-1} \frac{\phi^k(c)}{k!} (x-c)^k$$

and the remainder is given by

$$R_{n-1}(\phi; c, x) = \frac{1}{(n-1)!} \int_c^x \phi^{(n)}(t) (x-t)^{n-1} dt.$$

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a nonnegative measurable function on the actual interval or more general set.

2. The main results

We give the first result.

THEOREM 2.1. *Let $A_k f(x)$, $K(x)$ be defined by (1.8) and (1.9) respectively. Let u and v be weight functions and $\phi^{(n-1)}$ be absolutely continuous for some $n \geq 3$ and G is the Green function defined by (1.12). Then*

$$\begin{aligned} & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y)d\mu_2(y) \\ & \quad - \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x)d\mu_1(x) \\ & \quad + \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ & \quad \times (s - \alpha)^l ds \\ & \quad + \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ & \quad \times (s - t)^{n-3} ds \phi^{(n)}(t) dt \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y)d\mu_2(y) \\ & \quad - \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x)d\mu_1(x) \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=0}^{n-3} \frac{(-1)^k \phi^{(l+2)}(\beta)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) \\
& \times (\beta - s)^l ds \\
& - \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_{\alpha}^t \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) \\
& \times (s-t)^{n-3} ds \phi^{(n)}(t) dt \tag{2.2}
\end{aligned}$$

where $f : \Omega_2 \rightarrow \mathbb{R}$ is a measurable function.

Proof. Using (1.13) in $\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$ we have

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
& = \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y) d\mu_2(y) \\
& \quad - \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x) d\mu_1(x) \\
& \int_{\alpha}^{\beta} \phi''(s) \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) ds \tag{2.3}
\end{aligned}$$

Now applying Taylor's formula on the function ϕ'' at the point α and replacing n by $n-2$ ($n \geq 3$) we have

$$\phi''(s) = \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} (s-\alpha)^l + \frac{1}{(n-3)!} \int_{\alpha}^s \phi^{(n)}(t) (s-t)^{n-3} dt. \tag{2.4}$$

Similarly, applying Taylor's formula for ϕ'' at the point β and replacing n by $n-2$ ($n \geq 3$) we have

$$\phi''(s) = \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\beta)}{l!} (s-\beta)^l + \frac{1}{(n-3)!} \int_s^{\beta} \phi^{(n)}(t) (s-t)^{n-3} dt. \tag{2.5}$$

Using (2.4) in (2.3) we get

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
& = \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y) d\mu_2(y) \\
& \quad - \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x) d\mu_1(x) \\
& \quad + \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right)
\end{aligned}$$

$$\begin{aligned}
& \times (s - \alpha)^l ds \\
& + \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) \\
& \times \left(\int_{\alpha}^s \phi^{(n)}(t) (s-t)^{n-3} dt \right) ds \tag{2.6}
\end{aligned}$$

By applying Fubini's theorem in the last term of (2.6) we obtain (2.1). Similarly using (2.5) in (2.3) we obtain (2.2). \square

First we give result involving convex functions.

COROLLARY 2.1. *Let the assumptions of Theorem 2.1 be satisfied. If ϕ is a convex function and*

$$\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \geq 0,$$

then

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\
& \geq \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y) d\mu_2(y) \\
& - \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x) d\mu_1(x) \tag{2.7}
\end{aligned}$$

Proof. By assumptions ϕ'' exists almost everywhere and $\phi'' \geq 0$. Then in (2.3) the last integral

$$\int_{\alpha}^{\beta} \phi''(s) \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) ds \geq 0$$

and (2.7) follows. \square

We continue with the following result that involves n -convex functions.

COROLLARY 2.2. *Let the assumptions of Theorem 2.1 be satisfied.*

(i) *If ϕ is n -convex and*

$$\int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s) v(y) d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s) u(x) d\mu_1(x) \right) (s-t)^{n-3} ds \geq 0,$$

then

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\
& - \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y) d\mu_2(y) \\
& + \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x) d\mu_1(x)
\end{aligned}$$

$$\begin{aligned} &\geq \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ &\quad \times (s - \alpha)^l ds \end{aligned} \quad (2.8)$$

(ii) If ϕ is n -convex and

$$\int_{\alpha}^t \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) (s - t)^{n-3} ds \leq 0,$$

then

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &\quad - \int_{\Omega_2} \left(\frac{\beta - f(y)}{\beta - \alpha} \phi(\alpha) + \frac{f(y) - \alpha}{\beta - \alpha} \phi(\beta) \right) v(y)d\mu_2(y) \\ &\quad + \int_{\Omega_1} \left(\frac{\beta - A_k f(x)}{\beta - \alpha} \phi(\alpha) + \frac{A_k f(x) - \alpha}{\beta - \alpha} \phi(\beta) \right) u(x)d\mu_1(x) \\ &\geq \sum_{l=0}^{n-3} \frac{(-1)^l \phi^{(l+2)}(\beta)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ &\quad \times (\beta - s)^l ds \end{aligned} \quad (2.9)$$

Proof. By assumptions $\phi^{(n)}$ exists almost everywhere and $\phi^{(n)} \geq 0$. We can apply Theorem 2.1 to obtain (2.8) and (2.9) respectively. \square

COROLLARY 2.3. *Let the assumptions of Theorem 2.1 be satisfied. If v is defined by (1.10), then*

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ &\quad \times (s - \alpha)^l ds \\ &\quad + \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ &\quad \times (s - t)^{n-3} ds \phi^{(n)}(t) dt \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \sum_{l=0}^{n-3} \frac{(-1)^k \phi^{(l+2)}(\beta)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ &\quad \times (\beta - s)^l ds \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_{\alpha}^t \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\
& \times (s-t)^{n-3} ds \phi^{(n)}(t) dt \tag{2.11}
\end{aligned}$$

where $f : \Omega_2 \rightarrow \mathbb{R}$ is a measurable function.

Proof. The first and second integrals on the right-hand sides of (2.1) and (2.2) are equal to 0 when we apply the definition of the function v . \square

Next result involving n -convex functions is given in the following Corollary.

COROLLARY 2.4. *Let the assumptions of Theorem 2.1 be satisfied. Let v be defined by (1.10). If ϕ is n -convex and*

$$\int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) (s-t)^{n-3} ds \geq 0, \tag{2.12}$$

then

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
& \geq \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\
& \times (s-\alpha)^l ds. \tag{2.13}
\end{aligned}$$

If ϕ is n -convex and

$$\int_{\alpha}^t \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) (s-t)^{n-3} ds \leq 0,$$

then

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
& \geq \sum_{l=0}^{n-3} \frac{(-1)^k \phi^{(l+2)}(\beta)}{l!} \int_{\alpha}^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\
& \times (\beta-s)^l ds \tag{2.14}
\end{aligned}$$

where $f : \Omega_2 \rightarrow \mathbb{R}$ is a measurable function.

Proof. Since (2.12) holds and the function ϕ is n -convex so by assumptions $\phi^{(n)}$ exists almost everywhere and $\phi^{(n)} \geq 0$, so

$$\begin{aligned}
& \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\
& \times (s-t)^{n-3} ds \phi^{(n)}(t) dt \geq 0
\end{aligned}$$

and (2.13) follows. Similarly we obtain (2.14). \square

COROLLARY 2.5. *Let the assumptions of Theorem 2.1 be satisfied. Let $k(x, y) \geq 0$ and v be defined by (1.10). If ϕ is n -convex, then (2.13) follows. If ϕ is n -concave, then (2.13) holds with the reversed sign of inequality.*

Proof. Since G is a convex function, $k(x, y) \geq 0$ and v is defined by (1.10) then we can apply Theorem 1.3 and obtain that

$$\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \geq 0.$$

Since the function ϕ is n -convex function, so by assumptions $\phi^{(n)}$ exists almost everywhere and $\phi^{(n)} \geq 0$, so

$$\begin{aligned} & \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_t^{\beta} \left(\int_{\Omega_2} G(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x), s)u(x)d\mu_1(x) \right) \\ & \times (s-t)^{n-3} ds \phi^{(n)}(t) dt \geq 0 \end{aligned}$$

and (2.13) follows. Similarly we obtain (2.14). \square

EXAMPLE 2.1. By applying Corollary 2.5 with $\Omega_1 = \Omega_2 = (0, \infty)$ and $k(x, y) = 1, 0 \leq y \leq x, k(x, y) = 0, y > x, d\mu_1(x) = dx, d\mu_2(y) = dy$ and $u(x) = \frac{1}{x}$ (so that $v(y) = \frac{1}{y}$) we obtain the following result

$$\begin{aligned} & \int_0^{\infty} \phi(f(x)) \frac{dx}{x} - \int_0^{\infty} \phi \left(\frac{1}{x} \int_0^x f(t) dt \right) \frac{dx}{x} \\ & \geq \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_{\alpha}^{\beta} \left(\int_0^{\infty} G(f(y), s) \frac{dy}{y} - \int_0^{\infty} G(A_k f(x), s) \frac{dx}{x} \right) (s-\alpha)^l ds \end{aligned}$$

where A_k is defined by

$$A_k f(x) = \frac{1}{x} \int_0^x f(t) dt.$$

EXAMPLE 2.2. By arguing as in Example 2.1 but only with $\phi(x) = x^p, \prod_{i=1}^n (p-i+1) \geq 0$ we obtain the following result

$$\begin{aligned} & \int_0^{\infty} f^p(x) \frac{dx}{x} - \int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p \frac{dx}{x} \\ & \geq \sum_{l=0}^{n-3} \frac{(\prod_{i=0}^{l+1} (p-i)) \alpha^{p-l-2}}{l!} \int_{\alpha}^{\beta} \left(\int_0^{\infty} G(f(y), s) \frac{dy}{y} - \int_0^{\infty} G(A_k f(x), s) \frac{dx}{x} \right) (s-\alpha)^l ds. \end{aligned} \tag{2.15}$$

If $\prod_{i=1}^n (p-i+1) < 0$, then (2.15) holds with the reversed sign of inequality.

EXAMPLE 2.3. Let $\Omega_1 = \Omega_2 = (0, b)$, $0 < b \leq \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy , respectively, and let $k(x, y) = 0$ for $x < y \leq b$. Then A_k coincides with the operator H_k defined by (1.4) and if also $u(x)$ is replaced by $u(x)/x$ and $v(x)$ by $v(x)/x$, then we obtain the following result

$$\int_0^b \phi(f(y)v(y)) \frac{dy}{y} - \int_0^b \phi(H_k f(x)u(x)) \frac{dx}{x} \\ \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_\alpha^\beta \left(\int_0^b G(f(y), s)v(y) \frac{dy}{y} - \int_0^b G(H_k f(x), s)u(x) \frac{dx}{x} \right) (s - \alpha)^l ds \quad \square$$

EXAMPLE 2.4. By arguing as in Example 2.3 but $\Omega_1 = \Omega_2 = (b, \infty)$, $0 \leq b < \infty$ and with kernels such that $k(x, y) = 0$ for $b \leq y < x$ we obtain the following result

$$\int_b^\infty \phi(f(y)v(y)) \frac{dy}{y} - \int_b^\infty \phi(H_{\bar{k}} f(x)u(x)) \frac{dx}{x} \\ \sum_{l=0}^{n-3} \frac{\phi^{(l+2)}(\alpha)}{l!} \int_\alpha^\beta \left(\int_b^\infty G(f(y), s)v(y) \frac{dy}{y} - \int_b^\infty G(H_{\bar{k}} f(x), s)u(x) \frac{dx}{x} \right) (s - \alpha)^l ds$$

where $H_{\bar{k}} f$ is defined by (1.6). \square

We continue with the result that involves Hardy–Hilbert’s inequality.

If $p > 1$ and f is a nonnegative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \leq \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y) dy. \tag{2.16}$$

Inequality (2.16) is sometimes called Hilbert’s inequality even if Hilbert himself only considered the case $p = 2$.

EXAMPLE 2.5. Let $\Omega_1 = \Omega_2 = (0, \infty)$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy , respectively, let $k(x, y) = \frac{(\frac{x}{y})^{-1/p}}{x+y}$, $p > 1$ and $u(x) = \frac{1}{x}$. Then $K(x) = K = \frac{\pi}{\sin(\pi/p)}$ and $v(y) = \frac{1}{y}$. Let $\Phi(u) = u^p$, $\prod_{i=1}^n (p - i + 1) \geq 0$ then the following result follows

$$\int_0^\infty f^p(y) dy - K^{-p} \int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y} dy \right)^p dx \\ \sum_{l=0}^{n-3} \frac{(\prod_{i=0}^{l+1} (p - i)) \alpha^{p-l-2}}{l!} \int_\alpha^\beta \left(\int_b^\infty G(f(y)y^{\frac{1}{p}}, s) \frac{dy}{y} - \int_b^\infty G(A_k f(x), s) \frac{dx}{x} \right) (s - \alpha)^l ds \tag{2.17}$$

where

$$A_k f(x) = \frac{\sin(\pi/p)}{\pi} \int_0^\infty \frac{f(y)}{x+y} x^{\frac{1}{p}} dy.$$

If $\prod_{i=1}^n p - i + 1 < 0$, then (2.17) holds with the reversed sign of inequality. \square

We mention Pólya–Knopp’s inequality,

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(t) dt\right) dx < e \int_0^\infty f(x) dx, \quad (2.18)$$

for positive functions $f \in L^1(\mathbb{R}_+)$. Since (2.18) can be obtained from (1.1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p \rightarrow \infty$, Pólya–Knopp’s inequality may be considered as a limiting case of Hardy’s inequality.

EXAMPLE 2.6. Let the assumptions in Theorem 2.1 be satisfied. Then, by applying (2.13) and (2.14) with $\Phi(x) = e^x$, and f replaced by $\ln f^p$, $p > 0$ we obtain that

$$\begin{aligned} & \int_{\Omega_2} f^p(y)v(y)d\mu_2(y) - \int_{\Omega_1} \left[\exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \ln f(y)d\mu_2(y)\right) \right]^p u(x)d\mu_1(x) \\ & \geq \sum_{l=0}^{n-3} \frac{\exp \alpha}{l!} \int_\alpha^\beta \left(\int_{\Omega_2} G(\ln f^p(y),s)v(y)d\mu_2(y) - \int_{\Omega_1} G(A_k f(x),s)u(x)d\mu_1(x) \right) \\ & \quad \times (s - \alpha)^l ds \end{aligned} \quad (2.19)$$

where $k(x,y)$, $K(x)$, $u(x)$ and $v(y)$ are defined as in Theorem 1.3 and

$$A_k f(x) = \frac{p}{K(x)} \int_{\Omega_2} k(x,y) \ln f(y)d\mu_2(y).$$

In particular, if $p = 1$, $\Omega_1 = \Omega_2 = (0, \infty)$, $k(x,y) = 1$, $0 < y < x$, $k(x,y) = 0$, $y \geq x$. (so that $K(x) = x$), $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, $u(x) = 1/x$ (so that $v(x) = 1/x$) replacing $f(x)/x$ by $f(x)$ and making a simple calculation we find that (2.19) is equal to

$$\begin{aligned} & \int_0^\infty f(y) dy - e \int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(y) dy\right) dx \\ & \geq \sum_{l=0}^{n-3} \frac{\exp \alpha}{l!} \int_\alpha^\beta \left(\int_0^\infty G(\ln f(y),s) \frac{dy}{y} - \int_0^\infty G(P_k f(x),s) \frac{dx}{x} \right) (s - \alpha)^l ds, \end{aligned}$$

where

$$P_k f(x) = \frac{1}{x} \int_0^x \ln(f(y)) dy.$$

Acknowledgements. This work has been supported by Croatian Science Foundation under the project 5435.

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(Received November 11, 2014)

Kristina Krulić Himmelreich
Faculty of Textile Technology, University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
e-mail: kkrulic@ttf.hr

Josip Pečarić
Faculty of Textile Technology, University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
pecaric@element.hr