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MULTIPLE SINGULAR INTEGRALS AND MAXIMAL OPERATORS WITH MIXED HOMOGENEITY ALONG COMPOUND SURFACES

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Abstract. In this paper we present the L^p mapping properties for a class of multiple singular integral operators along polynomial compound surfaces provided that the integral kernels are given by the radial function $h \in \Delta_\gamma$(or $h \in U_\gamma$) for some $\gamma > 1$ and the sphere function $\Omega \in \tilde{\mathscr{F}}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 0$, which is distinct from $L(\log^+ L)^2(S^{m-1} \times S^{n-1})$. In addition, the L^p bounds for the related maximal operators are also established. Some previous results are greatly extended and improved.

1. Introduction

Let $\mathbb{R}^d$ $(d = m$ or$n)$, $d \geqslant 2$, be the d-dimensional Euclidean space and S^{d-1} be the unit sphere in $\mathbb{R}^d$ equipped with the induced Lebesgue measure $d\sigma_d$. Let $\alpha_{d,1}, \alpha_{d,2}, \dots, \alpha_{d,d}$ be fixed real numbers, $\alpha_{d,j} \geqslant 1$ $(j = 1, \dots, d)$. Define the function $F : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ by $F(x,\rho_d) = \sum_{j=1}^d x_j^2 \rho_d^{-2\alpha_{d,j}}$, $x = (x_1, x_2, \dots, x_d)$. It is clear that for each fixed $x \in \mathbb{R}^d$, the function $F(x,\rho_d)$ is a decreasing function in $\rho_d > 0$. We let $\rho_d(x)$ denote the unique solution of the equation $F(x,\rho_d) = 1$. Fabes and Riviére [12] showed that $(\mathbb{R}^d, \rho_d)$ is a metric space, which is often called the mixed homogeneity space related to $\{\alpha_{d,j}\}_{j=1}^d$. For $\lambda > 0$, we let $A_{d,\lambda}$ be the diagonal $d \times d$ matrix $A_{d,\lambda} = \mathrm{diag}\{\lambda^{\alpha_{d,1}}, \dots, \lambda^{\alpha_{d,d}}\}$. Let $\mathbb{R}^+ := (0,\infty)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, we denote $A_d^\phi(y)$ by $A_d^\phi(y) = A_{d,\phi(\rho_d(y))} y'$ for $y \in \mathbb{R}^d$, where $y' = A_{d,\rho_d(y)^{-1}} y \in S^{d-1}$.

The change of variables related to the spaces $(\mathbb{R}^d, \rho_d)$ is given by the transformation

$$\begin{aligned}
x_1 &= \rho_d^{\alpha_{d,1}} \cos\theta_1 \cdots \cos\theta_{n-2} \cos\theta_{n-1},\\
x_2 &= \rho_d^{\alpha_{d,2}} \cos\theta_1 \cdots \cos\theta_{n-2} \sin\theta_{n-1},\\
&\cdots\\
x_{d-1} &= \rho_d^{\alpha_{d,d-1}} \cos\theta_1 \sin\theta_2,\\
x_d &= \rho_d^{\alpha_{d,d}} \sin\theta_1.
\end{aligned}$$

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Thus $dx = \rho_d^{\alpha_d - 1} J_d(x') d\rho_d d\sigma_d(x')$, where $\rho_d^{\alpha_d-1} J_d(x')$ is the Jacobian of the above transform and $\alpha_d = \sum_{j=1}^d \alpha_{d,j}, J_d(x') = \sum_{j=1}^d \alpha_{d,j}(x_j')^2$. Obviously, $J_d(x') \in C^\infty(S^{d-1})$ and there exists $M_d > 0$ such that

$$1 \leqslant J_d(x') \leqslant M_d, \qquad \forall\, x' \in S^{d-1}.$$

It is easy to check that

$$\rho_d(x) = |x|, \text{ if } \alpha_{d,1} = \alpha_{d,2} = \ldots = \alpha_{d,d}.$$

Let $\Omega \in L^1(S^{m-1} \times S^{n-1})$ satisfying the conditions

$$\Omega(A_{m,s}x, A_{n,t}y) = \Omega(x,y), \quad \forall\, s,t > 0 \text{ and } (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1.1}$$

$$\int_{S^{m-1}} \Omega(u',\cdot) J_m(u') d\sigma_m(u') = \int_{S^{n-1}} \Omega(\cdot, v') J_n(v') d\sigma_n(v') = 0. \tag{1.2}$$

The multiple singular integral operator with mixed homogeneity $T_{h,\Omega}$ is defined by

$$T_{h,\Omega}(f)(x,y) := p.v. \iint_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Omega(u,v) h(\rho_m(u), \rho_n(v))}{\rho_m(u)^{\alpha_m} \rho_n(v)^{\alpha_n}} f(x-u, y-v) du dv, \tag{1.3}$$

where $h \in \Delta_1$. Here $\Delta_\gamma (\gamma \geqslant 1)$ denote the set of all measurable functions $h(r,s)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma} = \sup_{R_1>0, R_2>0} \left(R_1^{-1} R_2^{-1} \int_0^{R_1} \int_0^{R_2} |h(r,s)|^\gamma dr ds \right)^{1/\gamma} < \infty.$$

It is obvious that $\Delta_{\gamma_1} \subsetneq \Delta_{\gamma_2}$ for $\gamma_1 > \gamma_2$, and $\Delta_\infty = L^\infty$.

For $h = 1$, we denote $T_{h,\Omega}$ by T. For $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, the operator T is the classical multiple singular integral operator denoted by $\tilde{T}$, which was first considered by Fefferman and Stein (see [14, 15]) and has been studied extensively by many authors (see [11, 25, 26] for examples). In particular, Ying [26] (also see [25] for the multiple singular integrals along polynomial curves) proved that $\tilde{T}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$ and $\beta > 1$, provided that Ω satisfies the following condition:

$$\sup_{(\xi', \eta') \in S^{m-1} \times S^{n-1}} \iint_{S^{m-1} \times S^{n-1}} |\Omega(u', v')| \{G(\xi', \eta')\}^\beta d\sigma_m(u') d\sigma_n(v') < \infty, \tag{1.4}$$

where

$$G(\xi', \eta') = \log \frac{1}{|\xi' \cdot u'|} + \log \frac{1}{|\eta' \cdot v'|} + \log \frac{1}{|\xi' \cdot u'|} \cdot \log \frac{1}{|\eta' \cdot v'|}.$$

It should be pointed out that the condition (1.4) for one parameter case was originally defined in Walsh's paper [23] and developed by Grafakos and Stefanov [16] (also see [6] for its variant). For the sake of simplicity, we denote that for $\beta > 0$,

$$\mathscr{F}_\beta(S^{m-1} \times S^{n-1}) = \{\Omega \in L^1(S^{m-1} \times S^{n-1}) : \ \Omega \text{ satisfies } (1.4)\}.$$

We remark that the variant of $\mathscr{F}_\beta(S^{m-1}\times S^{n-1})$ was introduced and studied by Al-Salman in [4]. Employing the ideas in [16], one can verify that $\mathscr{F}_{\beta_1}(S^{m-1}\times S^{n-1})\subsetneq \mathscr{F}_{\beta_2}(S^{m-1}\times S^{n-1})$ for $\beta_1>\beta_2$, and

$$\bigcap_{\beta>1}\mathscr{F}_\beta(S^{m-1}\times S^{n-1})\nsubseteq L(\log^+L)^2(S^{m-1}\times S^{n-1})\subsetneq L\log^+L(S^{m-1}\times S^{n-1})$$
$$\nsubseteq \bigcup_{\beta>1}\mathscr{F}_\beta(S^{m-1}\times S^{n-1}).$$

For $\alpha_{m,i}\geqslant 1$ and $\alpha_{n,j}\geqslant 1$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, Chen and Le [9] showed that if $\Omega\in L(\log^+L)^2(S^{m-1}\times S^{n-1})$, then T is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for $1<p<\infty$. Subsequently, Lan et al. [17] extended the result of [9] to the multiple singular integrals along certain compound curves. Recently, Liu and Wu [19] proved that T (in more general form) is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for $2\beta/(2\beta-1)<p<2\beta$, if $\Omega\in\mathscr{F}_\beta(S^{m-1}\times S^{n-1})$ for some $\beta>1$.

For $\alpha_{m,i}=\alpha_{n,j}=1$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$, we denote $T_{h,\Omega}$ by T_h. Duoandikoetxea [11] obtained the L^p $(1<p<\infty)$ boundedness of T_h under the weaker condition $\Omega\in L^q(S^{m-1}\times S^{n-1})$ with $q>1$ and $h\in L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$. Subsequently, the result of [11] was improved by many authors (see [5, 8, 21, 24] et al.). In particular, Al-Salman et al. [5] showed that if $\Omega\in L(\log^+L)^2(S^{m-1}\times S^{n-1})$ and $h\in\Delta_\gamma$ for some $\gamma>1$, then T_h is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for p with satisfying $|1/p-1/2|<\min\{1/2,1/\gamma'\}$. Recently, Ma et al. [21] introduced the following size condition:

$$\sup_{(\xi',\eta')\in S^{m-1}\times S^{n-1}}\iint_{(S^{m-1}\times S^{n-1})^2}|\Omega(u',v')\Omega(\theta,w)|\times\{G_{\xi',\eta'}(\theta,w)\}^\beta d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(w)<\infty, \tag{1.5}$$

where

$$G_{\xi',\eta'}(\theta,w)=\log\frac{1}{|\langle u'-\theta,\xi'\rangle|}\log\frac{1}{|\langle v'-w,\eta'\rangle|}+\log\frac{1}{|\langle u'-\theta,\xi'\rangle|}+\log\frac{1}{|\langle v'-w,\eta'\rangle|}.$$

For simplicity, we set

$$\tilde{\mathscr{F}}_\beta(S^{m-1}\times S^{n-1}):=\{\Omega\in L^1(S^{m-1}\times S^{n-1}):\Omega\text{ satisfies }(1.5)\},\ \ \forall\,\beta>0.$$

Ma et al. [21] showed that the following result.

THEOREM A. *Let Ω be homogeneous of degree zero, integrable on $S^{m-1}\times S^{n-1}$ and satisfy*

$$\int_{S^{m-1}}\Omega(x',y')d\sigma_m(x')=\int_{S^{n-1}}\Omega(x',y')d\sigma_n(y')=0.$$

Let P_{N_1},P_{N_2} be two real polynomials on $\mathbb{R}$ with $P_{N_1}(0)=P_{N_2}(0)=0$ and $\deg(P_{N_i})=N_i$ $(i=1,2)$. Suppose that $h\in\Delta_\gamma$ for some $\gamma>1$ and $\Omega\in\tilde{\mathscr{F}}_\beta(S^{m-1}\times S^{n-1})$ with

$\beta > \max\{2, \gamma'\}$. *Then the multiple singular integral operators along polynomial curves* $\tilde{T}_{h,P}$ *defined by*

$$\tilde{T}_{h,P}(f)(x,y) = p.v. \iint_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Omega(u,v)h(|u|,|v|)}{|u|^m|v|^n} f(x - P_{N_1}(|u|)u', y - P_{N_2}(|v|)v')dudv$$

is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ *for* p *with satisfying* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$. *The bound is independent of the coefficients of* P_{N_1} *and* P_{N_2}, *but depends on* N_1 *and* N_2.

REMARK 1. We remark that the condition (1.5) for one parameter was originally introduced by Fan and Sato in more general form in [15], and it was introduced by Ma et al. in more general form in [21]. It follows from [21, Proposition 2.1] that $\mathscr{F}_\beta(S^1 \times S^1) \subset \tilde{\mathscr{F}}_\beta(S^1 \times S^1)$. When $m > 2$ or $n > 2$, the relation between $\mathscr{F}_\beta(S^{m-1} \times S^{n-1})$ and $\tilde{\mathscr{F}}_\beta(S^{m-1} \times S^{n-1})$ remains to be open.

A natural question, which arises from the above results, is the following:

QUESTION. *For the general case* $\alpha_{m,i} \geqslant 1\,(i = 1, \dots, m)$ *and* $\alpha_{n,j} \geqslant 1\,(j = 1, \dots, n)$, *is* $T_{h,\Omega}$ *bounded on* $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ *under the same assumptions on* Ω *and* h *as in Theorem A?*

In this paper, we will give a affirmative answer to this question by treating a family of operators, which is broader than $T_{h,\Omega}$. Precise, let P_{N_1} and P_{N_2} be two non-negative polynomials on $\mathbb{R}$ with $P_{N_i}(0) = 0$ and $\deg(P_{N_i}) = N_i$ $(i = 1, 2)$. For suitable functions $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and h defined on $\mathbb{R}^+ \times \mathbb{R}^+$, we define the multiple singular integral operators $T_{h,\Omega}^{P,\varphi,\psi}$ along surfaces $S(P_{N_1}(\varphi), P_{N_2}(\psi))$ by

$$T_{h,\Omega}^{P,\varphi,\psi}(f)(x,y) = p.v. \iint_{\mathbb{R}^m \times \mathbb{R}^n} \frac{\Omega(u,v)h(\rho_m(u),\rho_n(v))}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}} f\big(x - A_m^{P_{N_1}(\varphi)}(u), y - A_n^{P_{N_2}(\psi)}(v)\big)dudv,$$

where

$$S(P_{N_1}(\varphi), P_{N_2}(\psi)) := \big\{\big(A_m^{P_{N_1}(\varphi)}(u), A_n^{P_{N_2}(\psi)}(v)\big) : \ (u,v) \in \mathbb{R}^m \times \mathbb{R}^n\big\}.$$

Obviously, $T_{h,\Omega}$ is the special case of $T_{h,\Omega}^{P,\varphi,\psi}$ for $P_{N_i}(s) = \varphi(s) = \psi(s) = s$ $(i = 1, 2)$. Also, in the special case $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, \dots, m;\ j = 1, \dots, n)$,

$$S(P_{N_1}(\varphi), P_{N_2}(\psi)) = \big\{(P_{N_1}(\varphi(|u|))u', P_{N_2}(\psi(|v|))v') : \ (u,v) \in \mathbb{R}^m \times \mathbb{R}^n\big\}.$$

Moreover, for $\varphi(s) = \psi(s) = s$ and $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, \dots, m;\ j = 1, \dots, n)$, that is,

$$S(P_{N_1}(\varphi), P_{N_2}(\psi)) = \big\{(P_{N_1}(|u|)u', P_{N_2}(|v|)v') : \ (u,v) \in \mathbb{R}^m \times \mathbb{R}^n\big\}.$$

In this paper we shall establish the following results.

THEOREM 1. *Let P_{N_1} and P_{N_2} be two real polynomials on $\mathbb{R}$ with $P_{N_i}(0)=0$ and $P_{N_i}(t)>0$ for $t\neq 0$, where N_i is the degree of P_{N_i} $(i=1,2)$, and let $\varphi,\psi\in\mathfrak{F}$, where $\mathfrak{F}$ is the set of functions ϕ satisfying the following properties:*

(i) *$\phi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous strictly increasing and $\phi\in\mathscr{C}^1(\mathbb{R}^+)$ satisfying that ϕ' is monotonous;*

(ii) *there exist constants $C_\phi, c_\phi>0$ such that $t\phi'(t)\geqslant C_\phi\phi(t)$ and $\phi(2t)\leqslant c_\phi\phi(t)$ for all $t>0$.*

Suppose that $h\in\Delta_\gamma$ for some $\gamma>1$ and $\Omega\in\tilde{\mathscr{F}}_\beta(S^{m-1}\times S^{n-1})$ for some $\beta>\max\{\gamma',2\}$ with satisfying (1.1)–(1.2). Then $T_{h,\Omega}^{P,\varphi,\psi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ for p with satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$. The bounds are independent of the coefficients of P_{N_1} and P_{N_2}, but depend on φ,ψ,N_1,N_2,m,n and β.

THEOREM 2. *Let P_{N_1},P_{N_2},φ and ψ be as in Theorem 1. Suppose that $h\in\Delta_\gamma$ for some $\gamma>1$, Ω satisfies (1.1)–(1.2) and $\Omega\in\mathscr{F}_\beta(S^1\times S^1)$ for some $\beta>\max\{\gamma',2\}$. Then $T_{h,\Omega}^{P,\varphi,\psi}$ is bounded on $L^p(\mathbb{R}^2\times\mathbb{R}^2)$ for p with satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$. The bounds are independent of the coefficients of P_{N_1} and P_{N_2}, but depend on φ,ψ,N_1,N_2 and β.*

REMARK 2. Clearly, Theorem 2 directly follows from Theorem 1 and Remark 1. We remark that Theorem 1.1 extends Theorem A to the mixed homogeneity setting, even for the special case $\varphi(s)=\psi(s)=s$. It should be pointed out that our results are new even in the case $\alpha_{m,i}=\alpha_{n,j}=1$ $(i=1,\dots,m;\ j=1,\dots,n)$, i.e., the Euclidean setting.

REMARK 3. For any $\phi\in\mathfrak{F}$, there exists a constant $B_\phi>1$ such that $\phi(2r)\geqslant B_\phi\phi(r)$ for all $r>0$ (see [1, 3, 19]). We remark that there are some model examples in the class $\mathfrak{F}$, such as t^α $(\alpha>0)$, $t^\alpha(\ln(1+t))^\beta$ $(\alpha,\beta>0)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0)=0$ (see [1]).

In Theorem 1 or 2, for $\gamma\geqslant 2$, the range of β is $[2,\infty)$ and the range of p is (β',β), but for $1<\gamma<2$, the range of β is $[\gamma',\infty)$ and the range of p is shrunk to $|1/p-1/2|<1/\gamma'-1/\beta$. It is natural to ask the following question.

QUESTION. *Can the range of β and p in Theorems 1-2 be enlarged for the case $1<\gamma<2$?*

The next aim of this paper is to address the above question by imposing some more restrictive conditions on h. Precisely, for $1\leqslant\gamma\leqslant\infty$, let U_γ be the set of all measurable functions h on $\mathbb{R}^+\times\mathbb{R}^+$ satisfying

$$\|h\|_{U_\gamma}=\left(\int_0^\infty\int_0^\infty|h(r,s)|^\gamma\frac{drds}{rs}\right)^{1/\gamma}<\infty. \tag{1.6}$$

Obviously, $U_\gamma\subsetneq\Delta_\gamma$ for $0<\gamma<\infty$ and $U_\infty=\Delta_\infty=L^\infty$. The second one of our main results can be formulated as follows.

THEOREM 3. *Let P_{N_1}, P_{N_2}, φ and ψ be as in Theorem 1. Suppose that $h \in U_\gamma$ for some $\gamma \geqslant 1$, Ω satisfies (1.1)–(1.2) and $\Omega \in \tilde{\mathscr{F}}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 2$. Then $T_{h,\Omega}^{P,\varphi,\psi}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ if one of the following conditions holds:*

(i) $\gamma = 1, p = \infty$;

(ii) $\gamma > 1$, *p satisfies* $|1/p - 1/2| < 1/2 - \min\{2/\gamma', 1\}/\beta$.

The bounds are independent of the coefficients of P_{N_1} and P_{N_2}, but depend on $\varphi, \psi, N_1, N_2, m, n$ and β.

REMARK 4. Obviously, for $\gamma > 2$, the ranges of β and p in Theorem 3 are coincidence with ones of Theorem 1. However, for $1 < \gamma < 2$, the range of p in Theorem 3 is larger than one in Theorem 1 since $1/\gamma' - 1/\beta < 1/2 - 2/(\gamma'\beta)$, and the range of β is extended to $(2, \infty]$. Meanwhile, we also obtain the result at the endpoint case $\gamma = 1$. Therefore, it is worth to impose the above restriction on h. It is not clear whether the restriction on h can be removed, which is interesting.

To prove Theorem 3, we need to establish the following L^p-boundedness for the related maximal operator $M_{P,\varphi,\psi}^{(\gamma)}$ by

$$M_{P,\varphi,\psi}^{(\gamma)}(f)(x,y) = \sup_{\|h\|_{U_\gamma} \leqslant 1} |T_{h,\Omega}^{P,\varphi,\psi}(f)(x,y)|,$$

which is interesting itself. When $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, $P_{N_1}(t) = P_{N_2}(t) = \varphi(t) = \psi(t) = t$, we shall denote $M_{P,\varphi,\psi}^{(\gamma)}$ by $M^{(\gamma)}$, Historically, in 1999, Ding [10] proved that the operator $M^{(2)}$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ provided that $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$. This result was greatly improved by Al-Salman in [2]. In [2], Al-Salman obtained the L^p boundedness of $M^{(2)}$ for $2 \leqslant p < \infty$ under the weaker condition that $\Omega \in L(\log^+ L)(S^{m-1} \times S^{n-1})$. Moreover, Al-Salman showed that the condition $\Omega \in L(\log^+ L)(S^{m-1} \times S^{n-1})$ cannot be replaced by any condition of the form $\Omega \in L(\log^+ L)^{1-\varepsilon}(S^{m-1} \times S^{n-1}), \varepsilon > 0$. Especially, Al-Qassem and Pan [7] proved that the operator $M^{(\gamma)}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $\gamma' \leqslant p < \infty$ (for $\gamma = 1$, $p = \infty$) provided that $\Omega \in L(\log^+ L)^{2/\gamma'}(S^{m-1} \times S^{n-1})$ and $1 \leqslant \gamma \leqslant 2$ (also see [18] for the non-isotropic case).

For the operator $M_{P,\varphi,\psi}^{(\gamma)}$, we will prove the following result.

THEOREM 4. *Let P_{N_1}, P_{N_2}, φ and ψ be as in Theorem 1. Suppose that Ω satisfies (1.1)–(1.2) and $\Omega \in \tilde{\mathscr{F}}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 2$. Then $M_{P,\varphi,\psi}^{(\gamma)}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < \gamma \leqslant 2$ with $\gamma' \leqslant p < \gamma'\beta/2$, and it is bounded on $L^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ for $\gamma = 1$. The bounds are independent of the coefficients of P_{N_1} and P_{N_2}, but depend on $\varphi, \psi, N_1, N_2, m, n$ and β.*

By Theorems 3–4 and Remark 1, we have the following results.

THEOREM 5. *Let P_{N_1}, P_{N_2}, φ and ψ be as in Theorem 1 and $h \in U_\gamma$ for some $\gamma \geqslant 1$. Suppose that Ω satisfies (1.1)–(1.2) and $\Omega \in \mathscr{F}_\beta(S^1 \times S^1)$ for $\beta > 2$. Then $T_{h,\Omega}^P$ is bounded on $L^p(\mathbb{R}^2 \times \mathbb{R}^2)$ if one of the following conditions holds:*

(i) $\gamma = 1$, $p = \infty$;

(ii) $\gamma > 1$, p *satisfies* $|1/p - 1/2| < 1/2 - \min\{2/\gamma', 1\}/\beta$.

The bounds are independent of the coefficients of P_{N_1} *and* P_{N_2}, *but depend on* φ, ψ, N_1, N_2 *and* β.

THEOREM 6. *Let* P_{N_1}, P_{N_2}, φ *and* ψ *be as in Theorem* 1. *Suppose that* Ω *satisfies* (1.1)–(1.2) *and* $\Omega \in \mathscr{F}_\beta(S^1 \times S^1)$ *for some* $\beta > 2$. *Then* $M^{(\gamma)}_{P,\varphi,\psi}$ *is bounded on* $L^p(\mathbb{R}^2 \times \mathbb{R}^2)$ *for* $1 < \gamma \leqslant 2$ *with* $\gamma' \leqslant p < \gamma'\beta/2$, *and it is bounded on* $L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ *for* $\gamma = 1$. *The bound is independent of the coefficients of* P_{N_1} *and* P_{N_2}, *but depends on* φ, ψ, N_1, N_2 *and* β.

REMARK 5. We remark that it is still an open problem whether the L^p-boundedness of $M^{(\gamma)}_{P,\varphi,\psi}$ holds for $2 < \gamma < \infty$, even for the case $P_{N_1}(t) = P_{N_2}(t) = \varphi(t) = \psi(t) = t$. Also, by Remark 1, all of our results are new, even in the special case: $P_{N_1}(t) = P_{N_2}(t) = \varphi(t) = \psi(t) = t$, moreover, even in the Euclidean setting.

REMARK 6. It should be pointed out that all of main results are the multiple-parameter case of the results in [20].

The rest of this paper is organized as follows. In Section 2 we will recall some notation and establishing some preliminary lemmas. The proofs of main results will be proved in Section 3. We remark that some ideas of our methods are taken from [7, 11, 20], but our methods and technique are more delicate and complex than those used in [7, 11, 20]. Throughout this paper, let p' denote the conjugate index of p; that is, $1/p + 1/p' = 1$ the letter C or c, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but independent of the essential variables. We also set $\sum_{j\in\emptyset} a_j = 0$ and $\prod_{j\in\emptyset} a_j = 1$.

2. Some notations and auxiliary lemmas

For given positive polynomials $P_{N_1}(t) = \sum_{i=1}^{N_1} \beta_i t^i$, $P_{N_2}(t) = \sum_{i=1}^{N_2} \gamma_i t^i$ and $l \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, n\}$, we set $(P_{N_1}(t))^{\alpha_{m,l}} := \sum_{i=1}^{N_1\alpha_{m,l}} a_{i,l} t^i$ and $(P_{N_2}(t))^{\alpha_{n,k}} := \sum_{j=1}^{N_2\alpha_{n,k}} b_{j,k} t^j$. Then for $x, \xi \in \mathbb{R}^m$; $y, \eta \in \mathbb{R}^n$ and $\varphi, \psi \in \mathfrak{F}$, we can write

$$A_m^{P_{N_1}(\varphi)}(x) \cdot \xi = \sum_{l=1}^{m} P_{N_1}(\varphi(\rho_m(x)))^{\alpha_{m,l}} x_l{}' \cdot \xi_l = \sum_{l=1}^{m} \sum_{i=1}^{N_1\alpha_{m,l}} a_{i,l} \varphi(\rho_m(x))^i x_l{}' \cdot \xi_l,$$

$$A_n^{P_{N_2}(\psi)}(y) \cdot \eta = \sum_{k=1}^{n} P_{N_2}(\psi(\rho_n(y)))^{\alpha_{n,k}} y_k' \cdot \eta_k = \sum_{k=1}^{n} \sum_{j=1}^{N_2\alpha_{n,k}} b_{j,k} \psi(\rho_n(y))^j y_k' \cdot \eta_k.$$

We denote $\mathscr{N}_1 := \max\{N_1\alpha_{m,l} : 1 \leqslant l \leqslant m\}$, $\mathscr{N}_2 := \max\{N_2\alpha_{n,k} : 1 \leqslant k \leqslant n\}$ and set $a_{i,l} = 0$ whenever $i > N_1\alpha_{m,l}$; $b_{j,k} = 0$ whenever $j > N_2\alpha_{n,k}$. Thus

$$A_m^{P_{N_1}(\varphi)}(x) \cdot \xi = \sum_{l=1}^{m} \sum_{i=1}^{N_1\alpha_{m,l}} a_{i,l} \varphi(\rho_m(x))^i x_l{}' \cdot \xi_l = \sum_{i=1}^{\mathscr{N}_1} (L_i(\xi) \cdot x') \varphi(\rho_m(x))^i,$$

where $L_i(\xi) = (a_{i,1}\xi_1, a_{i,2}\xi_2, \ldots, a_{i,m}\xi_m)$. Similarly,

$$A_n^{P_{N_2}(\psi)}(y)\cdot\eta = \sum_{j=1}^{\mathcal{N}_2}(I_j(\eta)\cdot y')\psi(\rho_n(y))^j,$$

where $I_j(\eta) = (b_{j,1}\eta_1, b_{j,2}\eta_2, \ldots, b_{j,n}\eta_n)$. For any $\mu \in \{0,1,\ldots,\mathcal{N}_1\}$ and $\nu \in \{0,1,\ldots,\mathcal{N}_2\}$, we set

$$Q_\mu(x) = \Big(\sum_{i=1}^{\mu} a_{i,1}x_1'\varphi(\rho_m(x))^i, \ldots, \sum_{i=1}^{\mu} a_{i,m}x_m'\varphi(\rho_m(x))^i\Big),$$

$$R_\nu(y) = \Big(\sum_{j=1}^{\nu} b_{j,1}y_1'\psi(\rho_n(y))^j, \ldots, \sum_{j=1}^{\nu} b_{j,n}y_n'\psi(\rho_n(y))^j\Big).$$

Hence,

$$Q_\mu(x)\cdot\xi = \sum_{i=1}^{\mu}(L_i(\xi)\cdot x')\varphi(\rho_m(x))^i,\ 0 \leqslant \mu \leqslant \mathcal{N}_1;$$

$$R_\nu(y)\cdot\eta = \sum_{j=1}^{\nu}(I_j(\eta)\cdot y')\psi(\rho_n(y))^j,\ 0 \leqslant \nu \leqslant \mathcal{N}_2.$$

For any $r, s > 0$, we define the measures $\{\sigma_{r,s}^{\mu,\nu}\}$ as follows.

$$\begin{aligned}\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta) = \iint_{S^{m-1}\times S^{n-1}} &\exp\big(-2\pi i(\xi\cdot Q_\mu(A_{m,r}u') + \eta\cdot R_\nu(A_{n,s}v'))\big)\\ &\times \Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v');\end{aligned}$$

Now we introduce the following result, which will paly key roles in the proofs of our main results.

LEMMA 1. ([19, Lemma 2.5]) *Suppose $\varphi, \psi \in \mathfrak{F}$. Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$, $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$ and $r > 0$,*

$$\int_{r/2}^{r} \exp(-i\xi\cdot Q_\mu(A_{m,\rho_m}x'))\frac{d\rho_m}{\rho_m} \leqslant C(\varphi)|\varphi(r)^\mu L_\mu(\xi)\cdot x'|^{-1/\mu};$$

$$\int_{r/2}^{r} \exp(-i\eta\cdot R_\nu(A_{n,\rho_n}y'))\frac{d\rho_n}{\rho_n} \leqslant C(\psi)|\psi(r)^\nu I_\nu(\eta)\cdot y'|^{-1/\nu}.$$

The constant $C(\varphi)$ is independent of the coefficients of P_{N_1}, but depends on φ; The constant $C(\psi)$ is independent of the coefficients of P_{N_2}, but depends on ψ.

Applying Lemma 1 and combining with the similar arguments as in getting estimates of measures in [25, 26] we have

LEMMA 2. *Suppose* $\Omega \in \tilde{\mathscr{F}}_\beta(S^{m-1}\times S^{n-1})$ *for some* $\beta > 0$ *and satisfies* (1.1)–(1.2). *Then for* $\mu \in \{1,2,\dots,\mathscr{N}_1\}$, $\nu \in \{1,2,\dots,\mathscr{N}_2\}$, $k,l \in \mathbb{Z}$ *and* $(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n$, *there exists* $C > 0$ *such that*

(i)

$$\sup_{r,s>0} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta)| \leqslant C; \tag{2.1}$$

(ii)

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu-1,\nu}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs} \leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|^2; \tag{2.2}$$

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu,\nu-1}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs} \leqslant C|\psi(2^{l+1})^\nu I_\nu(\eta)|^2; \tag{2.3}$$

(iii) *for* $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, *then*

$$\begin{aligned}&\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu-1,\nu}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs}\\ &\leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|^2 (\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta};\end{aligned} \tag{2.4}$$

(iv) *for* $|\varphi(2^{k+1})^\nu L_\mu(\xi)| > 1$, *then*

$$\begin{aligned}&\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu,\nu-1}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs}\\ &\leqslant C|\psi(2^{l+1})^\nu I_\nu(\eta)|^2 (\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta};\end{aligned} \tag{2.5}$$

(v) *for* $|\varphi(2^{k+1})^\mu L_\mu(\xi)| > 1$, *then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs} \leqslant C(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta}; \tag{2.6}$$

for $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, *then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs} \leqslant C(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta}; \tag{2.7}$$

for $|\varphi(2^{k+1})^\mu L_\mu(\xi)| > 1$ *and* $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, *then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs} \leqslant C(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta}(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta}; \tag{2.8}$$

(vi)

$$\begin{aligned}&\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}} |\widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu,\nu-1}_{r,s}}(\xi,\eta) - \widehat{\sigma^{\mu-1,\nu}_{r,s}}(\xi,\eta) + \widehat{\sigma^{\mu-1,\nu-1}_{r,s}}(\xi,\eta)|^2 \frac{drds}{rs}\\ &\leqslant C\min\{|\varphi(2^{k+1})^\mu L_\mu(\xi)|^2, |\psi(2^{l+1})^\nu I_\nu(\eta)|^2, |\varphi(2^{k+1})^\mu L_\mu(\xi)|^2|\psi(2^{l+1})^\nu I_\nu(\eta)|^2\}.\end{aligned} \tag{2.9}$$

The constant C *is independent of the coefficients of* P_{N_1} *and* P_{N_2}.

Proof. (2.1) is obvious. Let

$$H_{k,\mu}(u',\theta,\xi)=\int_{2^k}^{2^{k+1}}\exp\Big(-2\pi i\sum_{j=1}^{\mu}L_j(\xi)\cdot(u'-\theta)\varphi(r)^j\Big)\frac{dr}{r};$$

$$J_{l,\nu}(v',\omega,\eta)=\int_{2^l}^{2^{l+1}}\exp\Big(-2\pi i\sum_{k=1}^{\nu}I_k(\eta)\cdot(v'-\omega)\psi(s)^k\Big)\frac{ds}{s}.$$

By Lemma 1, we have

$$|H_{k,\mu}(u',\theta,\xi)|\leqslant C\min\{1,|\varphi(2^{k+1})^{\mu}L_\mu(\xi)\cdot(u'-\theta)|^{-1/\mu}\}. \tag{2.10}$$

When $|\varphi(2^{k+1})^{\mu}L_\mu(\xi)|>1$, since $\frac{t}{(\log t)^{\beta}}$ is increasing in $(\mathbf{e}^{\beta},\infty)$, we have

$$|H_{k,\mu}(u',\theta,\xi)|\leqslant C\frac{(\log \mathbf{e}^{\beta}2^{1/\mu}|(L_\mu(\xi))'\cdot(u'-\theta)|^{-1/\mu})^{\beta}}{(\log|\varphi(2^{k+1})^{\mu}L_\mu(\xi)|)^{\beta}}. \tag{2.11}$$

Similarly, when $|\psi(2^{l+1})^{\nu}L_\nu(\eta)|>1$, we have

$$|J_{l,\nu}(v',\omega,\eta)|\leqslant C\frac{(\log \mathbf{e}^{\beta}2^{1/\nu}|(I_\nu(\eta))'\cdot(v'-\omega)|^{-1/\nu})^{\beta}}{(\log|\psi(2^{\ell+1})^{\nu}I_\nu(\eta)|)^{\beta}}. \tag{2.12}$$

By the definition of $\sigma_{r,s}^{\mu,\nu}$, we have

$$\begin{aligned}
&|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)|\\
&=\Big|\iint_{S^{m-1}\times S^{n-1}}(\exp(-2\pi i\xi\cdot Q_\mu(A_{m,r}u'))-\exp(-2\pi i\xi\cdot Q_{\mu-1}(A_{m,r}u')))\\
&\quad\times\exp(-2\pi i\eta\cdot R_\nu(A_{n,s}v'))\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\Big|\\
&\leqslant C|\varphi(r)^{\mu}L_\mu(\xi)|.
\end{aligned} \tag{2.13}$$

This together with the fact that φ is increasing in $(0,\infty)$ implies (2.2). Similarly, (2.3) holds. On the other hand,

$$\begin{aligned}
&|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)|^2\\
&=\Big|\iint_{S^{m-1}\times S^{n-1}}(\exp(-2\pi iL_\mu(\xi)\cdot u'\varphi(r)^{\mu})-1)\exp(-2\pi i\xi\cdot Q_{\mu-1}(A_{m,r}u'))\\
&\quad\times\exp(-2\pi i\eta\cdot R_\nu(A_{n,s}v'))\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\Big|^2\\
&=\Big|\iint_{(S^{m-1}\times S^{n-1})^2}(\exp(-2\pi iL_\mu(\xi)\cdot u'\varphi(r)^{\mu})-1)(\exp(2\pi iL_\mu(\xi)\cdot\theta\varphi(r)^{\mu})-1)\\
&\quad\times\exp\Big(-2\pi i\sum_{j=1}^{\mu-1}L_j(\xi)\cdot(u'-\theta)\varphi(r)^j\Big)\exp\Big(-2\pi i\sum_{k=1}^{\nu}I_k(\eta)\cdot(v'-\omega)\psi(s)^k\Big)\\
&\quad\times\Omega(u',v')\overline{\Omega(\theta,\omega)}J_m(u')J_n(v')\overline{J_m(\theta)J_n(\omega)}d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(\omega).
\end{aligned}$$

Then

$$\begin{aligned}&\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\\&\leqslant C|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)|^2\iint_{(S^{m-1}\times S^{n-1})^2}|J_{l,\nu}(v',w,\eta)|\\&\quad\times|\Omega(u',v')\overline{\Omega(\theta,\omega)}|d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(\omega).\end{aligned}$$

Combining (2.12) with the fact that $\Omega\in\tilde{\mathscr{F}}_{\beta}(S^{m-1}\times S^{n-1})$ implies (2.4). Similarly, (2.5) holds. On the other hand,

$$\begin{aligned}&|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\\&=\Big|\iint_{S^{m-1}\times S^{n-1}}\exp(-2\pi i(\xi\cdot Q_{\mu}(A_{m,r}u')+\eta\cdot R_{\nu}(A_{n,s}v')))\\&\quad\times\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\Big|^2\\&=\iint_{(S^{m-1}\times S^{n-1})^2}\exp\Big(-2\pi i\sum_{j=1}^{\mu}L_j(\xi)\cdot(u'-\theta)\varphi(r)^j\Big)\Omega(u',v')\overline{\Omega(\theta,\omega)J_m(\theta)J_n(\omega)}\\&\quad\times\exp\Big(-2\pi i\sum_{k=1}^{\nu}I_k(\eta)\cdot(v'-\omega)\psi(s)^k\Big)J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(\omega).\end{aligned}$$

Thus we have

$$\begin{aligned}&\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\\&\leqslant\iint_{(S^{m-1}\times S^{n-1})^2}|H_{k,\mu}(u',\theta,\xi)||J_{l,\nu}(v',\omega,\eta)||\Omega(u',v')\overline{\Omega(\theta,\omega)}|\\&\quad\times J_m(u')J_n(v')\overline{J_m(\theta)J_n(\omega)}d\sigma_m(u')d\sigma_n(v')d\sigma_m(\theta)d\sigma_n(\omega).\end{aligned}$$

Combining (2.11)–(2.12) with the fact that $\Omega\in\tilde{\mathscr{F}}_{\beta}(S^{m-1}\times S^{n-1})$ yields (v). (vi) follows from

$$\begin{aligned}&|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu,\nu-1}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)+\widehat{\sigma_{r,s}^{\mu-1,\nu-1}}(\xi,\eta)|\\&=\Big|\iint_{S^{m-1}\times S^{n-1}}\exp(-2\pi i(\xi\cdot Q_{\mu-1}(A_{m,r}u')+\eta\cdot R_{\nu-1}(A_{n,s}v')))\Omega(u',v')J_m(u')J_n(v')\\&\quad\times(\exp(-2\pi iL_{\mu}(\xi)\cdot u'\varphi(r)^{\mu})-1)(\exp(-2\pi iL_{\mu}(\xi)\cdot\theta\varphi(r)^{\mu})-1)d\sigma_m(u')d\sigma_n(v')\Big|\\&\leqslant C|\varphi(r)^{\mu}L_{\mu}(\xi)||\psi(s)^{\nu}I_{\nu}(\eta)|.\end{aligned}$$

This proves Lemma 2. $\square$

For any $k,l\in\mathbb{Z}$ and $\mu\in\{0,1,\dots,\mathscr{N}_1\}$, $\nu\in\{0,1,\dots,\mathscr{N}_2\}$, we define the measures $\{\tau_{k,l}^{\mu,\nu}\}$ and the related maximal operators $\tau_{\mu,\nu}^{*}$ as follows:

$$\widehat{\tau_{k,l}^{\mu,\nu}}(\xi,\eta)=\iint_{\triangle_{k,l}}\frac{\Omega(u,v)h(\rho_m(u),\rho_n(v))}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}}\exp(-2\pi i(\xi\cdot Q_{\mu}(x)+\eta\cdot R_{\nu}(y)))dudv,$$

$$\tau^*_{\mu,\nu}(f)(x,y) = \sup_{\kappa,\ell\in\mathbb{Z}} \big||\tau^{\mu,\nu}_{\kappa,\ell}| * f(x,y)\big|,$$

where $\triangle_{k,l} = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^n : 2^k \leqslant \rho_m(u) < 2^{k+1}, 2^l \leqslant \rho_n(v) < 2^{l+1}\}$ and $|\tau^{\mu,\nu}_{\kappa,\ell}|$ is defined in the same way as $\tau^{\mu,\nu}_{\kappa,\ell}$, but with h and Ω replaced by $|h|$ and $|\Omega|$, respectively. One can easily check that for any $\mu \in \{0,1,\dots,\mathscr{N}_1\}$ and $\nu \in \{0,1,\dots,\mathscr{N}_2\}$,

$$\widehat{\tau^{0,\nu}_{\kappa,\ell}}(\xi,\eta) = \widehat{\tau^{\mu,0}_{\kappa,\ell}}(\xi,\eta) = 0 \tag{2.14}$$

and

$$T^{P,\varphi,\psi}_{h,\Omega}(f)(x,y) = \sum_{k,l\in\mathbb{Z}} \tau^{\mathscr{N}_1,\mathscr{N}_2}_{k,l} * f(x,y). \tag{2.15}$$

LEMMA 3. *Let $h \in \Delta_\gamma$ for some $\gamma > 1$ and $\tilde{\gamma} = \max\{2,\gamma'\}$. Suppose that Ω satisfies (1.1)–(1.2) and $\Omega \in \tilde{\mathscr{F}}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 0$. Then for $\mu \in \{1,2,\dots,\mathscr{N}_1\}$, $\nu \in \{1,2,\dots,\mathscr{N}_2\}$ and $k,l \in \mathbb{Z}$, there exists a constant $C > 0$ such that*

(i)

$$\sup_{k,l\in\mathbb{Z}} |\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta)| \leqslant C; \tag{2.16}$$

(ii)

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta) - \widehat{\tau^{\mu-1,\nu}_{k,l}}(\xi,\eta)| \leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|; \tag{2.17}$$

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta) - \widehat{\tau^{\mu,\nu-1}_{k,l}}(\xi,\eta)| \leqslant C|\psi(2^{\ell+1})^\nu I_\nu(\eta)|; \tag{2.18}$$

(iii) *for $|\varphi(2^{k+1})^\mu L_\mu(\xi)| > 1$, then*

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta) - \widehat{\tau^{\mu,\nu-1}_{k,l}}(\xi,\eta)| \leqslant C|\psi(2^{l+1})^\nu I_\nu(\eta)|(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta/\tilde{\gamma}}; \tag{2.19}$$

(iv) *for $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, then*

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta) - \widehat{\tau^{\mu-1,\nu}_{k,l}}(\xi,\eta)| \leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta/\tilde{\gamma}}; \tag{2.20}$$

(v) *for $|\varphi(2^{k+1})^\mu L_\mu(\xi)| > 1$, then*

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta)| \leqslant C(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta/\tilde{\gamma}}; \tag{2.21}$$

for $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, then

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta)| \leqslant C(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta/\tilde{\gamma}}; \tag{2.22}$$

for $|\varphi(2^{k+1})^\mu L_\mu(\xi)| > 1$ and $|\psi(2^{l+1})^\nu I_\nu(\eta)| > 1$, then

$$|\widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta)| \leqslant C(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta/\tilde{\gamma}}(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta/\tilde{\gamma}}; \tag{2.23}$$

(vi)

$$\begin{aligned}&|\widehat{\tau_{k,l}^{\mu,\nu}}(\xi,\eta)-\widehat{\tau_{k,l}^{\mu,\nu-1}}(\xi,\eta)-\widehat{\tau_{k,l}^{\mu-1,\nu}}(\xi,\eta)+\widehat{\tau_{k,l}^{\mu-1,\nu-1}}(\xi,\eta)|\\ \leqslant\ & C\min\{|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)|,|\psi(2^{l+1})^{\nu}I_{\nu}(\eta)|,|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)||\psi(2^{l+1})^{\nu}I_{\nu}(\eta)|\}.\end{aligned} \tag{2.24}$$

The constant C is independent of the coefficients of P_{N_1} and P_{N_2}.

Proof. By a change of variable and (2.1) we have

$$\begin{aligned}|\widehat{\tau_{k,l}^{\mu,\nu}}(\xi,\eta)| =& \Big|\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\iint_{S^{m-1}\times S^{n-1}}\exp(-2\pi i(\xi\cdot Q_{\mu}(A_{m,r}u')+\eta\cdot R_{\nu}(A_{n,s}v')))\\ &\times\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')h(r,s)\frac{drds}{rs}\Big|\\ &\leqslant C\|h\|_{\Delta_\gamma}\Big(\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)|^{\gamma'}\frac{drds}{rs}\Big)^{1/\gamma'}\\ &\leqslant C\Big(\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)|^{2}\frac{drds}{rs}\Big)^{1/\tilde\gamma},\end{aligned}$$

which combining (2.1) with (2.6)-(2.8) implies (i) and (v). On the other hand, by a change of variable and Hölder's inequality, we get form (2.13) that

$$\begin{aligned}&|\widehat{\tau_{k,l}^{\mu,\nu}}(\xi,\eta)-\widehat{\tau_{k,l}^{\mu-1,\nu}}(\xi,\eta)|\\ =&\Big|\iint_{\Delta_{k,l}}(\exp(-2\pi i\xi\cdot Q_{\mu}(u))-\exp(-2\pi i\xi\cdot Q_{\mu-1}(u)))\exp(-2\pi i\eta\cdot R_{\nu}(v))\\ &\times\frac{h(\rho_m(u),\rho_n(v))\Omega(u',v')}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}}dudv\Big|\\ =&\Big|\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\iint_{S^{m-1}\times S^{n-1}}(\exp(-2\pi i\xi\cdot Q_{\mu}(A_{m,r}u'))-\exp(-2\pi i\xi\cdot Q_{\mu-1}(A_{m,r}u')))\\ &\times\exp(-2\pi i\eta\cdot R_{\nu}(A_{n,s}v'))\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')h(r,s)\frac{drds}{rs}\Big|\\ =&\Big|\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}(\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta))h(r,s)\frac{drds}{rs}\Big|\\ \leqslant&\ C\|h\|_{\Delta_\gamma}\Big(\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)|^{\gamma'}\frac{drds}{rs}\Big)^{1/\gamma'}\\ \leqslant&\ C|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)|^{\max\{1-2/\gamma',0\}}\Big(\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\sigma_{r,s}^{\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{r,s}^{\mu-1,\nu}}(\xi,\eta)|^{2}\frac{drds}{rs}\Big)^{1/\tilde\gamma}.\end{aligned}$$

This together with (2.2) and (2.4) implies (2.17) and (2.20). Similarly, (2.18) and (2.19) hold. (2.24) follows from the inequality

$$\begin{aligned}
&|\widehat{\tau_{k,l}^{\mu,\nu}}(\xi,\eta)-\widehat{\tau_{k,l}^{\mu,\nu-1}}(\xi,\eta)-\widehat{\tau_{k,l}^{\mu-1,\nu}}(\xi,\eta)+\widehat{\tau_{k,l}^{\mu-1,\nu-1}}(\xi,\eta)|\\
&=\Big|\iint_{S^{m-1}\times S^{n-1}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\exp(-2\pi i(\xi\cdot Q_{\mu-1}(A_{m,r}u')+\eta\cdot R_{\nu-1}(A_{n,s}v')))\\
&\quad\times(\exp(-2\pi iL_\mu(\xi)\cdot u'\varphi(r)^\mu)-1)(\exp(-2\pi iI_\nu(\eta)\cdot v'\psi(s)^\nu)-1)h(r,s)\frac{drds}{rs}\\
&\quad\times\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\Big|\\
&\leqslant C\iint_{S^{m-1}\times S^{n-1}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\exp(-2\pi iL_\mu(\xi)\cdot u'\varphi(r)^\mu)-1|\\
&\quad\times|\exp(-2\pi iI_\nu(\eta)\cdot v'\psi(s)^\nu)-1||h(r,s)|\frac{drds}{rs}|\Omega(u',v')|d\sigma_m(u')d\sigma_n(v').
\end{aligned}$$

This completes the proof of Lemma 3. $\square$

LEMMA 4. *([19, Lemma 2.2]) Let* $P(t)=(P_1(t),P_2(t),\dots,P_d(t))$ *with* P_i $(i=1,\dots,d)$ *being real polynomials defined on* $\mathbb{R}^+$ *and* $\phi\in\mathfrak{F}$*. Then the maximal function* $M_{P,\phi}(f)(x)$ *defined by*

$$M_{P,\phi}(f)(x)=\sup_{r>0}\int_r^{2r}f(x-P(\phi(t)))\frac{dt}{t}$$

is bounded on $L^p(\mathbb{R}^d)$ *for* $1<p<\infty$*. The bound is independent of the coefficients of* P_i $(i=1,2,\dots,d)$ *and* f*, but depends on* ϕ*.*

Applying lemma 4, we have

LEMMA 5. *Let* $\Omega\in L^1(S^{m-1}\times S^{n-1})$ *with satisfying* (1.1)–(1.2) *and* $h\in\Delta_\gamma$ *for some* $\gamma>1$*. Then for* $\mu\in\{1,2,\dots,\mathcal{N}_1\}$ *and* $\nu\in\{1,2,\dots,\mathcal{N}_2\}$*, there exists* $C_p>0$ *such that*

$$\|\tau^*_{\mu,\nu}(f)\|_p\leqslant C_p\|f\|_p,\ \text{ for }\ \gamma'<p\leqslant\infty.$$

The constant C_p *is independent of the coefficients of* P_{N_1} *and* P_{N_2}*, but depends on* φ*,* ψ*.*

Proof. We define the measures $\{|\Lambda_{k,l}^{\mu,\nu}|\}$ and maximal operator $\Lambda^*_{\mu,\nu}$ by

$$\widehat{|\Lambda_{k,l}^{\mu,\nu}|}(\xi,\eta)=\iint_{\Delta_{k,l}}\exp(-2\pi i(\xi\cdot Q_\mu(u)+\eta\cdot R_\nu(v)))\frac{|\Omega(u',v')|}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}}dudv$$

and

$$\Lambda^*_{\mu,\nu}(f)(x,y)=\sup_{k,l\in\mathbb{Z}}\big||\Lambda_{k,l}^{\mu,\nu}|*f(x,y)\big|.$$

By a change of variable we have

$$\begin{aligned}\Lambda^*_{\mu,\nu}(f)(x,y) &= \sup_{k,l\in\mathbb{Z}}\Big|\iint_{\Delta_{k,l}} f(x-Q_\mu(u),y-R_\nu(v))\frac{|\Omega(u,v)|}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}}dudv\Big| \\ &\leqslant \sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\iint_{S^{m-1}\times S^{n-1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))| \\ &\quad\times|\Omega(u',v')|J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\frac{drds}{rs} \\ &\leqslant C\iint_{S^{m-1}\times S^{n-1}}\sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))|\frac{drds}{rs} \\ &\quad\times|\Omega(u',v')|d\sigma_m(u')d\sigma_n(v').\end{aligned}$$

By Lemma 4, using iterated integration and Minkowski's inequality, we have

$$\|\Lambda^*_{\mu,\nu}(f)\|_p \leqslant C_p\|f\|_p, \quad 1<p<\infty. \tag{2.25}$$

In addition, by Hölder's inequality

$$\begin{aligned}||\tau^{\mu,\nu}_{k,l}|*f(x,y)| &= \Big|\iint_{\Delta_{k,l}} f(x-Q_\mu(u),y-R_\nu(v))\frac{|\Omega(u',v')h(\rho_m(u),\rho_n(v))|}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}}dudv\Big| \\ &\leqslant \int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\iint_{S^{m-1}\times S^{n-1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))||\Omega(u',v')| \\ &\quad\times J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')|h(r,s)|\frac{drds}{rs} \\ &\leqslant C\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\Big(\iint_{S^{m-1}\times S^{n-1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))|^{\gamma'} \\ &\quad\times|\Omega(u',v')|J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\Big)^{1/\gamma'}|h(r,s)|\frac{drds}{rs} \\ &\leqslant C\|h\|_{\Delta_\gamma}\Big(\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\iint_{S^{m-1}\times S^{n-1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))|^{\gamma'} \\ &\quad\times|\Omega(u',v')|J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\frac{drds}{rs}\Big)^{1/\gamma'} \\ &\leqslant C(\Lambda^*_{\mu,\nu}(|f|^{\gamma'})(x,y))^{1/\gamma'},\end{aligned}$$

which together with (2.25) completes the proof of Lemma 5. □

Now we take two radial functions $\phi_1\in C_0^\infty(\mathbb{R}^m)$ and $\phi_2\in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1(t)=\phi_2(s)\equiv 1$ for $\max\{|t|,|s|\}\leqslant 1$ and $\phi_1(t)=\phi_2(s)\equiv 0$ for $\min\{|t|,|s|\}>\min\{B_\varphi,B_\psi\}$, where B_φ,B_ψ are as in Remark 1.3. For $\mu\in\{1,2,\ldots,\mathcal{N}_1\}$ and $\nu\in\{1,2,\ldots,\mathcal{N}_2\}$, we define the measures $\{\omega^{\mu,\nu}_{r,s}\}$ and $\{\lambda^{\mu,\nu}_{k,l}\}$ by

$$\begin{aligned}\widehat{\omega^{\mu,\nu}_{r,s}}(\xi,\eta) &= \widehat{\sigma^{\mu,\nu}_{r,s}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu)-\widehat{\sigma^{\mu-1,\nu}_{r,s}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu) \\ &\quad-\widehat{\sigma^{\mu,\nu-1}_{r,s}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu-1)+\widehat{\sigma^{\mu-1,\nu-1}_{r,s}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu-1),\end{aligned}$$

and

$$\begin{aligned}\widehat{\lambda^{\mu,\nu}_{k,l}}(\xi,\eta) &= \widehat{\tau^{\mu,\nu}_{k,l}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu)-\widehat{\tau^{\mu-1,\nu}_{k,l}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu) \\ &\quad-\widehat{\tau^{\mu,\nu-1}_{k,l}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu-1)+\widehat{\tau^{\mu-1,\nu-1}_{k,l}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu-1),\end{aligned}$$

where $\Pi_1(\mu)=\prod_{i=\mu+1}^{\mathscr{N}_1}\phi_1(\varphi(2^{k+1})^i L_i(\xi))$, $\Pi_2(\nu)=\prod_{j=\nu+1}^{\mathscr{N}_2}\phi_2(\psi(2^{l+1})^j I_j(\eta))$. It is easy to see that

$$\sigma_{r,s}^{\mathscr{N}_1,\mathscr{N}_2}=\sum_{\mu=1}^{\mathscr{N}_1}\sum_{\nu=1}^{\mathscr{N}_2}\omega_{r,s}^{\mu,\nu} \tag{2.26}$$

and

$$\tau_{k,l}^{\mathscr{N}_1,\mathscr{N}_2}=\sum_{\mu=1}^{\mathscr{N}_1}\sum_{\nu=1}^{\mathscr{N}_2}\lambda_{k,l}^{\mu,\nu}. \tag{2.27}$$

Applying Lemmas 2 and 3, combining with the arguments which are similar to those in the proof of [19, Lemma 2.7], we can obtain

LEMMA 6. *Let Ω be as in Lemma 2. Then for $\mu\in\{1,2,\dots,\mathscr{N}_1\}$, $\nu\in\{1,2,\dots,\mathscr{N}_2\}$ and any $k,l\in\mathbb{Z}$, there exists a constant $C>0$ such that*

(i)

$$\sup_{r,s>0}\|\omega_{r,s}^{\mu,\nu}\|\leqslant C; \tag{2.28}$$

(ii) *for $|\varphi(2^{k+1})^\nu L_\mu(\xi)|>B_\varphi$, then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\leqslant C|\psi(2^{l+1})^\nu I_\nu(\eta)|^2\Big(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|\Big)^{-\beta}; \tag{2.29}$$

(iii) *for $|\psi(2^{l+1})^\nu I_\nu(\eta)|>B_\psi$, then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|^2\Big(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|\Big)^{-\beta}; \tag{2.30}$$

(iv) *for $|\varphi(2^{k+1})^\mu L_\mu(\xi)|>B_\varphi$ and $|\psi(2^{l+1})^\nu I_\nu(\eta)|>B_\psi$, then*

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\leqslant C(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta}(\log|\psi(2^{l+1})^\nu I_\nu(\eta)|)^{-\beta}; \tag{2.31}$$

(v)

$$\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}\leqslant C|\varphi(2^{k+1})^\mu L_\mu(\xi)|^2|\psi(2^{l+1})^\nu I_\nu(\eta)|^2. \tag{2.32}$$

The constant C is independent of the coefficients of P_{N_1} and P_{N_2}.

LEMMA 7. *Let h and Ω be as in Lemma 3. Then for $\mu\in\{1,2,\dots,\mathscr{N}_1\}$, $\nu\in\{1,2,\dots,\mathscr{N}_2\}$ and $k,l\in\mathbb{Z}$, there exists a constant $C>0$ such that*

(i)

$$\sup_{k,l\in\mathbb{Z}}\|\lambda_{k,l}^{\mu,\nu}\|\leqslant C; \tag{2.33}$$

(ii) *for $|\varphi(2^{k+1})^\mu L_\mu(\xi)|>B_\varphi$, then*

$$|\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)|\leqslant C|\psi(2^{l+1})^\nu I_\nu(\eta)|(\log|\varphi(2^{k+1})^\mu L_\mu(\xi)|)^{-\beta/\tilde{\gamma}}; \tag{2.34}$$

(iii) *for* $|\psi(2^{l+1})^{\nu}I_{\nu}(\eta)| > B_{\psi}$, *then*

$$|\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)| \leqslant C|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)|(\log|\psi(2^{l+1})^{\nu}I_{\nu}(\eta)|)^{-\beta/\tilde{\gamma}}; \tag{2.35}$$

(iv) *for* $|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)| > B_{\varphi}$ *and* $|\psi(2^{l+1})I_{\nu}(\eta)| > B_{\psi}$, *then*

$$|\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)| \leqslant C(\log|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)|)^{-\beta/\tilde{\gamma}}(\log|\psi(2^{l+1})^{\nu}I_{\nu}(\eta)|)^{-\beta/\tilde{\gamma}}; \tag{2.36}$$

(v)

$$|\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)| \leqslant C|\varphi(2^{k+1})^{\mu}L_{\mu}(\xi)||\psi(2^{l+1})^{\nu}I_{\nu}(\eta)|. \tag{2.37}$$

The constant C *is independent of the coefficients of* P_{N_1} *and* P_{N_2}.

Applying Lemma 5 and the definition of $\lambda_{k,l}^{\mu,\nu}$, we can establish the following lemma.

LEMMA 8. *Let* Ω, h *be as in Lemma* 5. *Then for* $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ *and* $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, *there exists a constant* $C > 0$ *such that*

$$\Big\|\sup_{k,l\in\mathbb{Z}}||\lambda_{k,l}^{\mu,\nu}|*f|\Big\|_p \leqslant C\|f\|_p, \quad \gamma' < p \leqslant \infty.$$

The constant C *is independent of the coefficients of* P_{N_1} *and* P_{N_2}, *but depends on* φ, ψ.

Applying Lemma 8, by similar arguments to those used in the proof of [13, Theorem 7.5], we have

LEMMA 9. *Let* Ω, h *be as in Lemma 5. Then for* $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$, $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$ *and any suitable functions* $\{g_{k,l}\}$, *there exists a constant* $C > 0$ *such that*

$$\Big\|\Big(\sum_{k,l\in\mathbb{Z}}|\lambda_{k,l}^{\mu,\nu}*g_{k,l}|^2\Big)^{1/2}\Big\|_p \leqslant C\Big\|\Big(\sum_{k,l\in\mathbb{Z}}|g_{k,l}|^2\Big)^{1/2}\Big\|_p$$

for p *with satisfying* $|1/p-1/2| < \min\{1/\gamma', 1/2\}$. *The constant* C *is independent of the coefficients of* P_{N_1} *and* P_{N_2}, *but depends on* φ, ψ.

LEMMA 10. *Let* $\Omega \in L^1(S^{m-1}\times S^{n-1})$. *Then for* $\mu \in \{1,\ldots,\mathcal{N}_1\}$ *and* $\nu \in \{1,\ldots,\mathcal{N}_2\}$, *the operator* $\mathcal{U}$ *defined by*

$$\mathcal{U}(f)(x,y) = \sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}||\omega_{r,s}^{\mu,\nu}|*f(x,y)|\frac{drds}{rs}$$

is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ *for* $1 < p < \infty$. *The bound is independent of the coefficients of* P_{N_1} *and* P_{N_2}, *but depends on* φ, ψ.

Proof. We define the operator $\mathscr{H}$ by

$$\mathscr{H}(f)(x,y)=\sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}||\sigma_{r,s}^{\mu,\nu}|*f(x,y)|\frac{drds}{rs},$$

where $|\sigma_{r,s}^{\mu.\nu}|$ is defined in the same way as $\sigma_{r,s}^{\mu,\nu}$, but with Ω replaced by $|\Omega|$. Then we have

$$\begin{aligned}\mathscr{H}(f)(x,y)&=\sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\Big|\iint_{S^{m-1}\times S^{n-1}}f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))\\&\quad\times|\Omega(u',v')|J_m(u')J_n(v')d\sigma(u')d\sigma(v')\Big|\frac{drds}{rs}\\&\leqslant C\int_{S^{m-1}\times S^{n-1}}\sup_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))|\frac{drds}{rs}\\&\quad\times|\Omega(u',v')|d\sigma(u')d\sigma(v').\end{aligned}$$

Invoking Lemma 4, using iterated integration and Minkowski inequality, one can obtain that

$$\|\mathscr{H}(f)\|_p\leqslant C\|f\|_p,\quad 1<p<\infty.$$

This together with the definition of $\omega_{r,s}^{\mu,\nu}$ implies Lemma 10. $\square$

3. Proofs of main results

We will first prove Theorem 1.

Proof. It follows from (2.15) and (2.27) that

$$T_{h,\Omega}^{P,\varphi,\psi}(f)=\sum_{\mu=1}^{\mathscr{N}_1}\sum_{\nu=1}^{\mathscr{N}_2}\sum_{k,l\in\mathbb{Z}}\lambda_{k,l}^{\mu,\nu}*f:=\sum_{\mu=1}^{\mathscr{N}_1}\sum_{\nu=1}^{\mathscr{N}_2}T_{\mu,\nu}^{P,\varphi,\psi}(f).\tag{3.1}$$

It suffices to show that for any $\mu\in\{1,2,\dots,\mathscr{N}_1\}$ and $\nu\in\{1,2,\dots,\mathscr{N}_2\}$,

$$\|T_{\mu,\nu}^{P,\varphi,\psi}(f)\|_p\leqslant C\|f\|_p\qquad\text{for } |1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta.\tag{3.2}$$

For fixed $\mu\in\{1,2,\dots,\mathscr{N}_1\}$ and $\nu\in\{1,2,\dots,\mathscr{N}_2\}$, we can choose two collections of C^∞ functions $\{\lambda_i\}_{i\in\mathbb{Z}}$ and $\{\eta_j\}_{j\in\mathbb{Z}}$ on $(0,\infty)$ with the following properties:

(i) $\operatorname{supp}(\lambda_i)\subset[\varphi(2^{i+1})^{-\mu},\varphi(2^{i-1})^{-\mu}]$, $\operatorname{supp}(\eta_j)\subset[\psi(2^{j+1})^{-\nu},\psi(2^{j-1})^{-\nu}]$;

(ii) $0\leqslant\lambda_i,\,\eta_j\leqslant1$, $\sum_{i\in\mathbb{Z}}\lambda_i(t)^2=\sum_{j\in\mathbb{Z}}\eta_j(t)^2=1$;

(iii) $|(d/dt)^\iota\lambda_i(t)|\leqslant C_1/t$, $|(d/dt)^\iota\eta_j(t)|\leqslant C_2/t$, where C_1,C_2 are independent of i,j,ι.

Define the multiplier operator $S_{i,j}$ on $\mathbb{R}^m\times\mathbb{R}^n$ by

$$\widehat{S_{i,j}f}(x,y)=\lambda_i(|L_\mu(x)|)\eta_j(|I_\nu(y)|)\hat{f}(x,y).\tag{3.3}$$

Then

$$
\begin{aligned}
T_{\mu,\nu}^{P,\varphi,\psi}(f) &= \sum_{k,l\in\mathbb{Z}} \lambda_{k,l}^{\mu,\nu} * \Big(\textstyle\sum_{i,j\in\mathbb{Z}} S_{i+k,j+l}S_{i+k,j+l}f\Big) \\
&= \sum_{i,j\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}} S_{i+k,j+l}(\lambda_{k,l}^{\mu,\nu} * S_{i+k,j+l}f) \\
&:= \sum_{i,j\in\mathbb{Z}} T_{i,j}f.
\end{aligned}
\tag{3.4}
$$

Now we consider the L^p-boundedness of $T_{i,j}$. By the Littlewood-Paley theory and Lemma 9, we have for any p with satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}$,

$$
\begin{aligned}
\|T_{i,j}f\|_p &\leqslant C\Big\|\Big(\sum_{k,l\in\mathbb{Z}} |\lambda_{k,l}^{\mu,\nu} * S_{i+k,j+l}f|^2\Big)^{1/2}\Big\|_p \\
&\leqslant C\Big\|\Big(\sum_{k,l\in\mathbb{Z}} |S_{i+k,j+l}f|^2\Big)^{1/2}\Big\|_p \\
&\leqslant C\|f\|_p
\end{aligned}
\tag{3.5}
$$

On the other hand, by the Littlewood-Paley theory and Plancherel's theorem, we have

$$
\begin{aligned}
\|T_{i,j}f\|_2^2 &\leqslant C\Big\|\Big(\sum_{k,l\in\mathbb{Z}} |\lambda_{k,l}^{\mu,\nu} * S_{i+k,j+l}f|^2\Big)^{1/2}\Big\|_2^2 \\
&= C\sum_{k,l}\iint_{\mathbb{R}^m\times\mathbb{R}^n} |\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)|^2|\lambda_{i+k}(|L_\mu(\xi)|)\eta_{j+l}(|I_\nu(\eta)|)|^2|\hat{f}(\xi,\eta)|^2 d\xi d\eta \\
&\leqslant C\sum_{k,l}\iint_{E_{i+k,j+l}} |\widehat{\lambda_{k,l}^{\mu,\nu}}(\xi,\eta)|^2|\hat{f}(\xi,\eta)|^2 d\xi d\eta,
\end{aligned}
$$

where

$$
\begin{aligned}
E_{i+k,j+l} = \{(\xi,\eta)\in\mathbb{R}^m\times\mathbb{R}^n : \varphi(2^{i+k+1})^{-\mu} \leqslant |L_\mu(\xi)| \leqslant \varphi(2^{i+k-1})^{-\mu}, \\
\psi(2^{j+l+1})^{-\nu} \leqslant |I_\nu(\eta)| \leqslant \psi(2^{j+l-1})^{-\nu}\}.
\end{aligned}
\tag{3.6}
$$

Using Lemma 7 and Remark 3, we have

$$
\|T_{i,j}f\|_2 \leqslant C(\varphi,\psi,\mu,\nu)B_{i,j}\|f\|_2, \tag{3.7}
$$

where

$$
B_{i,j} = \begin{cases}
B_\varphi^{-i\mu}B_\psi^{-j\nu}, & i,j>-2; \\
B_\varphi^{-i\mu}|j|^{-\beta/\tilde{\gamma}}, & i>-2,\ j\leqslant -2; \\
|i|^{-\beta/\tilde{\gamma}}B_\psi^{-j\nu}, & i\leqslant -2,\ j>-2; \\
|ij|^{-\beta/\tilde{\gamma}}, & i,j\leqslant -2.
\end{cases}
\tag{3.8}
$$

where $\tilde{\gamma}=\max\{2,\gamma'\}$. Interpolation between (3.5) and (3.7) yields that for any p satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$, there exists $\delta\in(0,1]$ such that $\delta\beta/\tilde{\gamma}>1$ and

$$
\|T_{i,j}f\|_p \leqslant C(\varphi,\psi,\mu,\nu)^{1-\delta}B_{i,j}^{\delta}\|f\|_p, \qquad |1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta.
$$

Then we have for any p with satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$,

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}}\|T_{i,j}f\|_p &\leqslant C(\varphi,\psi,\mu,\nu)\Big(\sum_{i,j>-2}B_\varphi^{-i\mu\delta}B_\psi^{-j\nu\delta}+\sum_{i>-2,j\leqslant-2}B_\varphi^{-i\mu\delta}|j|^{-\delta\beta/\tilde{\gamma}}\\ &\quad+\sum_{i\leqslant-2,j>-2}|i|^{-\delta\beta/\tilde{\gamma}}B_\psi^{-j\nu\delta}+\sum_{i,j\leqslant-2}|ij|^{-\delta\beta/\tilde{\gamma}}\Big)\|f\|_p\\ &\leqslant C(\varphi,\,\psi,\,\mu,\,\nu)\|f\|_p\end{aligned}$$

This combining (3.1) with (3.4) completes the proof of Theorem 1. $\square$

Next, we will prove Theorem 4.

Proof. By duality, Hölder's inequality, Minkowski's inequality and (2.26) we have

$$\begin{aligned}M_{P,\varphi,\psi}^{(\gamma)}(f)(x,y) &= \sup_{\|h\|_{U^\gamma}\leqslant1}\Big|\int_0^\infty\int_0^\infty f*\sigma_{r,s}^{\mathcal{N}_1,\mathcal{N}_2}(x,y)h(r,s)\frac{drds}{rs}\Big|\\ &\leqslant\Big(\int_0^\infty\int_0^\infty|f*\sigma_{r,s}^{\mathcal{N}_1,\mathcal{N}_2}(x,y)|^{\gamma'}\frac{drds}{rs}\Big)^{1/\gamma'}\\ &\leqslant\sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2}\Big(\int_0^\infty\int_0^\infty|f*\omega_{r,s}^{\mu,\nu}(x,y)|^{\gamma'}\frac{drds}{rs}\Big)^{1/\gamma'}\\ &:=\sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2}M_{\mu,\nu}^{(\gamma')}(f)(x,y).\end{aligned}\tag{3.9}$$

So it suffices to obtain the $L^p(\mathbb{R}^m\times\mathbb{R}^n)$-bounds of $M_{\mu,\nu}^{(\gamma')}$ for $1\leqslant\gamma\leqslant2$.

Case 1 $(\gamma=2)$. Let $S_{i,j}$ be as in (3.3). Then by Minkowski's inequality we have

$$\begin{aligned}M_{\mu,\nu}^{(2)}(f)(x,y) &= \Big(\int_0^\infty\int_0^\infty|f*\omega_{r,s}^{\mu,\nu}(x,y)|^2\frac{drds}{rs}\Big)^{1/2}\\ &=\Big(\sum_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|f*\omega_{r,s}^{\mu,\nu}(x,y)|^2\frac{drds}{rs}\Big)^{1/2}\\ &=\Big(\sum_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\Big|\omega_{r,s}^{\mu,\nu}*\Big(\sum_{i,j\in\mathbb{Z}}S_{i+k,j+l}S_{i+k,j+l}f\Big)(x,y)\Big|^2\frac{drds}{rs}\Big)^{1/2}\\ &\leqslant\sum_{i,j\in\mathbb{Z}}\Big(\sum_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\Big|\omega_{r,s}^{\mu,\nu}*S_{i+k,j+l}S_{i+k,j+l}f(x,y)\Big|^2\frac{drds}{rs}\Big)^{1/2}\\ &:=\sum_{i,j\in\mathbb{Z}}G_{i,j}(f)(x,y).\end{aligned}\tag{3.10}$$

By Plancherel's theorem we have

$$\begin{aligned}\|G_{i,j}(f)\|_2^2 &= \int_{\mathbb{R}^m\times\mathbb{R}^n}\sum_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\Big|\omega_{r,s}^{\mu,\nu}*S_{i+k,j+l}S_{i+k,j+l}f(x,y)\Big|^2\frac{drds}{rs}dxdy\\ &=\sum_{k,l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}\int_{E_{i+k,j+l}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2|\hat{f}(\xi,\eta)|^2d\xi d\eta\frac{drds}{rs}\\ &=\sum_{k,l\in\mathbb{Z}}\int_{E_{i+k,j+l}}\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}|\widehat{\omega_{r,s}^{\mu,\nu}}(\xi,\eta)|^2\frac{drds}{rs}|\hat{f}(\xi,\eta)|^2d\xi d\eta.\end{aligned}$$

where $E_{i+\kappa,j+\ell}$ is as in (3.6). Then by Lemma 6 we get

$$\|G_{i,j}f\|_2 \leqslant C(\varphi,\psi,\mu,\nu)\tilde{B}_{i,j}\|f\|_2, \tag{3.11}$$

where

$$\tilde{B}_{i,j} = \begin{cases} B_\varphi^{-i\mu}B_\psi^{-j\nu}, & i,j > -2; \\ B_\varphi^{-i\mu}|j|^{-\beta/2}, & i > -2,\ j \leqslant -2; \\ |i|^{-\beta/2}B_\psi^{-j\nu}, & i \leqslant -2,\ j > -2; \\ |ij|^{-\beta/2}, & i,j \leqslant -2. \end{cases} \tag{3.12}$$

Next, for $p > 2$, let $q = (p/2)'$, there exists a function $g \in L^q(\mathbb{R}^m \times \mathbb{R}^n)$ with $\|g\|_q \leqslant 1$ such that

$$\begin{aligned}
&\|G_{i,j}(f)\|_p^2 \\
&= \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} \sum_{k,l\in\mathbb{Z}} \int\limits_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} \left|\omega_{r,s}^{\mu,\nu} * (S_{i+k,j+l}S_{i+k,j+l}f)(x,y)\right|^2 \frac{drds}{rs}|g(x,y)|dxdy \\
&\leqslant \sup_{r,s>0}\|\omega_{r,s}^{\mu,\nu}\| \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} \sum_{k,l\in\mathbb{Z}} \int\limits_{2^l}^{2^{l+1}} \int\limits_{2^k}^{2^{k+1}} \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} |(S_{i+k,j+l}S_{i+k,j+l}f)(x-u,y-v)|^2 d\omega_{r,s}^{\mu,\nu}(u,v)\frac{drds}{rs} \\
&\qquad \times |g(x,y)|dxdy \\
&\leqslant C \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} \sum_{k,l\in\mathbb{Z}} |S_{i+k,j+l}S_{i+k,j+l}f(x,y)|^2 \mathscr{U}\left(|g|\right)(-x,-y)dxdy \\
&\leqslant C\left\|\Big(\sum_{k,l\in\mathbb{Z}} |S_{i+k,j+l}S_{i+k,j+l}f|^2\Big)^{1/2}\right\|_p^2 \|\mathscr{U}(|g|)\|_q,
\end{aligned}$$

where the operator $\mathscr{U}$ is as in Lemma 10. Using (2.28), Littlewood-Paley theorem and Lemma 10, we have

$$\|G_{i,j}(f)\|_p \leqslant C\|f\|_p, \qquad p > 2. \tag{3.13}$$

Interpolating between (3.11) and (3.13), for some $\beta > 2$ and any fixed $p \in [2,\beta)$, we can choose $\delta_p \in (0,1]$ such that $\delta_p\beta/2 > 1$ and

$$\|G_{i,j}(f)\|_p \leqslant C\tilde{B}_{i,j}^{\delta_p}\|f\|_p.$$

This combing (3.10) with Minkowski's inequality yields

$$\begin{aligned}
\|M_{\mu,\nu}^{(2)}(f)\|_p &\leqslant C\Big(\sum_{i,j>-2} B_\varphi^{-i\mu\delta_p}B_\psi^{-j\nu\delta_p} + \sum_{i>-2,j\leqslant-2} B_\varphi^{-i\mu\delta_p}|j|^{-\delta_p\beta/2} \\
&\qquad + \sum_{i\leqslant-2,j>-2} |i|^{-\delta_p\beta/2}B_\psi^{-j\nu\delta_p} + \sum_{i,j\leqslant-2} |ij|^{-\delta_p\beta/2}\Big)\|f\|_p \\
&\leqslant C\|f\|_p.
\end{aligned}$$

This together with (3.9) implies

$$\|\mathscr{M}^{(2)}_{P,\varphi,\psi}(f)\|_p \leqslant C\|f\|_p, \qquad p\in[2,\beta). \tag{3.14}$$

Case 2 $(\gamma=1)$. For $f\in L^\infty(\mathbb{R}^m\times\mathbb{R}^n)$ and $h\in U_1$, we have

$$\begin{aligned}|T^{P,\varphi,\psi}_{h,\Omega}(f)(x,y)| &= \Big|\int_0^\infty\int_0^\infty h(r,s)\int_{S^{m-1}\times S^{n-1}} f(x-Q_\mu(A_{m,r}u'),y-R_\nu(A_{n,s}v'))\\ &\qquad\times\Omega(u',v')J_m(u')J_n(v')d\sigma_m(u')d\sigma_n(v')\frac{drds}{rs}\Big|\\ &\leqslant C\|\Omega\|_{L^1(S^{m-1}\times S^{n-1})}\|h\|_{U_1}\|f\|_\infty\end{aligned}$$

holds for every $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$. Thus for every $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$,

$$|M^{(1)}_{P,\varphi,\psi}(f)(x,y)| = \sup_{h\in U_1}|T^P_{\Omega,h}(f)(x,y)| \leqslant C\|f\|_\infty,$$

which implies

$$\|M^{(1)}_{P,\varphi,\psi}(f)\|_\infty \leqslant C\|f\|_\infty.$$

Case 3 $(1<\gamma<2)$. For convenience, let $\digamma(f)=f*\sigma^{\mathscr{N}_1,\mathscr{N}_2}_{r,s}$. By Cases 1 and 2, we have

$$\|\digamma(f)\|_{L^{p_0}(\mathbb{R}^m\times\mathbb{R}^n,L^2(\mathbb{R}^+\times\mathbb{R}^+,r^{-1}s^{-1}drds))} \leqslant C_p\|f\|_{p_0}, \quad p_0\in[2,\beta);$$

$$\|\digamma(f)\|_{L^\infty(\mathbb{R}^m\times\mathbb{R}^n,L^\infty(\mathbb{R}^+\times\mathbb{R}^+,r^{-1}s^{-1}drds))} \leqslant C_p\|f\|_\infty.$$

The real interpolation theorem for Lebesgue mixed norm spaces tells us that

$$\|\digamma(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n,L^{\gamma'}(\mathbb{R}^+\times\mathbb{R}^+,r^{-1}s^{-1}drds))} \leqslant C_p\|f\|_p, \quad p\in[\gamma',\beta\gamma'/2).$$

This together with (3.9) completes the proof of Theorem 4. $\square$

Finally, we will prove Theorem 3.

Proof. Case 1 $(1\leqslant\gamma\leqslant 2)$. Without loss of generality, we may assume that $\|h\|_{U_\gamma}=1$. Then

$$\|T^{P,\varphi,\psi}_{h,\Omega}(f)\|_p \leqslant \|M^{(\gamma)}_{P,\varphi,\psi}(f)\|_p \leqslant C\|f\|_p.$$

By Theorem 4, we have

$$\|T^{P,\varphi,\psi}_{h,\Omega}(f)\|_\infty \leqslant C\|f\|_\infty, \quad \text{for} \quad \gamma=1.$$

and

$$\|T^{P,\varphi,\psi}_{h,\Omega}(f)\|_p \leqslant C\|f\|_p, \quad p\in[\gamma',\gamma'\beta/2), \quad \text{for} \quad 1<\gamma\leqslant 2.$$

By a standard duality argument, we get

$$\|T^{P,\varphi,\psi}_{h,\Omega}(f)\|_p \leqslant C\|f\|_p, \quad p\in(\gamma'\beta/(\gamma'\beta-2),\gamma), \quad \text{for} \quad 1<\gamma\leqslant 2.$$

Hence, the interpolation theorem tell us that

$$\|T_{h,\Omega}^{P,\varphi,\psi}(f)\|_p \leqslant C\|f\|_p, \quad |1/p-1/2|<1/2-2/(\gamma'\beta), \quad \text{for} \quad 1<\gamma\leqslant 2.$$

Case 2 $(\gamma>2)$. Note that $U_\gamma \subsetneq \Delta_\gamma$ for $\gamma>1$. The rest result of Theorem 3 directly follows from Theorem 1. □

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