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WEIGHTED HARDY TYPE INEQUALITIES ON THE HEISENBERG GROUP $\mathbb{H}^n$

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Abstract. In the present article, we provide a sufficient condition on a pair of nonnegative weight functions V and W on the Heisenberg group $\mathbb{H}^n$, so that we establish a general L^p Hardy type inequality involving these weights with a remainder term. The method we use here is practical enough to get more weighted Hardy type inequalities. We also obtain new results on two-weight Hardy and Hardy-Poincaré type inequalities with remainder terms on $\mathbb{H}^n$. Our findings improve and include many previously known results in special cases.

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