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CALDERÓN–HARDY SPACES WITH VARIABLE EXPONENTS AND THE SOLUTION OF THE EQUATION $\Delta^m F = f$ FOR $f \in H^{p(\cdot)}(\mathbb{R}^n)$

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Abstract. In this article we define the Calderón-Hardy spaces with variable exponents on $\mathbb{R}^n$, $\mathscr{H}^{p(\cdot)}_{q,\gamma}(\mathbb{R}^n)$, and we show that for $m \in \mathbb{N}$ the operator Δ^m is a bijective mapping from $\mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$ onto $H^{p(\cdot)}(\mathbb{R}^n)$.

1. Introduction

Given a measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ such that

$$0 < \inf_{x\in\mathbb{R}^n} p(x) \leqslant \sup_{x\in\mathbb{R}^n} p(x) < \infty,$$

let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the space of all measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx < \infty.$$

We set

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

We see that $\left(L^{p(\cdot)}(\mathbb{R}^n), \|.\|_{p(\cdot)}\right)$ is a quasi normed space. These spaces are referred to as the Lebesgue spaces with variable exponents. In the last years many authors have extended the machinery of classical harmonic analysis to these spaces. See, for example [2], [3], [4], [5], [8].

In the celebrate paper [6], C. Fefferman and E. M. Stein defined the Hardy Spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$, with the norm given by

$$\|f\|_{H^p} = \left\|\sup_{t>0}\sup_{\phi\in\mathscr{F}_N} |t^{-n}\phi(t^{-1}.) * f|\right\|_p,$$

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for suitable family $\mathscr{F}_N$. In the paper [9], E. Nakai and Y. Sawano defined the Hardy spaces with variable exponents $H^{p(.)}(\mathbb{R}^n)$, replacing L^p by $L^{p(.)}$ in the above norm and they investigate their several properties.

Let $L^q_{loc}(\mathbb{R}^n)$, $1<q<\infty$, be the space of all measurable functions f on $\mathbb{R}^n$ that belong locally to L^q for compact sets of $\mathbb{R}^n$. We endowed $L^q_{loc}(\mathbb{R}^n)$ with the topology generated by the seminorms

$$|f|_{q,Q}=\left(|Q|^{-1}\int_Q |f(x)|^q\,dx\right)^{1/q},$$

where Q is a cube in $\mathbb{R}^n$ and $|Q|$ denotes its Lebesgue measure.

For $f\in L^q_{loc}(\mathbb{R}^n)$, we define a maximal function $\eta_{q,\gamma}(f;x)$ as

$$\eta_{q,\gamma}(f;x)=\sup_{r>0} r^{-\gamma}|f|_{q,Q(x,r)},$$

where γ is a positive real number and $Q(x,r)$ is the cube centered at x with side length r.

Let k a non negative integer and $\mathscr{P}_k$ the subspace of $L^q_{loc}(\mathbb{R}^n)$ formed by all the polynomials of degree at most k. We denote by E^q_k the quotient space of $L^q_{loc}(\mathbb{R}^n)$ by $\mathscr{P}_k$. If $F\in E^q_k$, we define the seminorm $\|F\|_{q,Q}=\inf\left\{|f|_{q,Q}: f\in F\right\}$. The family of all these seminorms induces on E^q_k the quotient topology.

Given a positive real number γ, we can write $\gamma=k+t$, where k is a non negative integer and $0<t\leqslant 1$. This decomposition is unique.

For $F\in E^q_k$, we define a maximal function $N_{q,\gamma}(F;x)$ as

$$N_{q,\gamma}(F;x)=\inf\left\{\eta_{q,\gamma}(f;x): f\in F\right\}.$$

This type of maximal function was introduced by A. P. Calderón in [1].

Let $p(.):\mathbb{R}^n\to(0,\infty)$ be a measurable function such that $0<p_-\leqslant p_+<\infty$, we say that an element $F\in E^q_k$ belongs to the Calderón-Hardy space with variable exponents $\mathscr{H}^{p(.)}_{q,\gamma}(\mathbb{R}^n)$ if the maximal function $N_{q,\gamma}(F;\cdot)\in L^{p(.)}(\mathbb{R}^n)$. The "norm" of F in $\mathscr{H}^{p(.)}_{q,\gamma}(\mathbb{R}^n)$ is defined as $\|F\|_{\mathscr{H}^{p(.)}_{q,\gamma}(\mathbb{R}^n)}=\|N_{q,\gamma}(F;\cdot)\|_{p(.)}$.

The Calderón-Hardy spaces were defined in the setting of the classical Lebesgue spaces by A. B. Gatto, J. G. Jiménez and C. Segovia in [7], they characterize the solutions of $\Delta^m F=f$, $m\in\mathbb{N}$, for $f\in H^p(\mathbb{R}^n)$. Moreover, they proved that the operator Δ^m is a bijective mapping from the Calderón-Hardy spaces onto $H^p(\mathbb{R}^n)$. In this work we show that this result holds in the context of the Lebesgue spaces with variable exponents. In [10], S. Ombrosi studied the weighted version of the Calderón-Hardy spaces. The Calderón-Hardy spaces were mentioned for first time with this name in [12]. A. Perini studied in [13] the boundedness of one-sided fractional integrals on these spaces. In [11], S. Ombrosi, A. Perini and R. Testoni obtain a complex interpolation theorem between weighted Calderón-Hardy spaces for weights in a Sawyer class.

Given a function $p(\cdot):\mathbb{R}^n\to(0,\infty)$ we say that $p(\cdot)$ is locally log-Hölder continuous, and denote this by $p(\cdot)\in LH_0(\mathbb{R}^n)$, if there exists a positive constant C_0 such

that

$$|p(x)-p(y)| \leqslant \frac{C_0}{-\log|x-y|}, \ |x-y| < \frac{1}{2}.$$

We say that $p(\cdot)$ is log-Hölder continuous at infinity, and denote this by $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ if there exists a positive constant C_∞ such that

$$|p(x)-p(y)| \leqslant \frac{C_\infty}{\log(e+|x|)}, \ |y| \geqslant |x|.$$

We note that the condition log-Hölder continuous at infinity is equivalent to the existence of constants C_∞ and p_∞ such that

$$|p(x)-p_\infty| \leqslant \frac{C_\infty}{\log(e+|x|)}, \ x \in \mathbb{R}^n.$$

As usual we will denote $p_+ = \sup_{x\in\mathbb{R}^n} p(x)$, $p_- = \inf_{x\in\mathbb{R}^n} p(x)$, $\underline{p} = \min\{p_-, 1\}$ and Δ stands for the Laplacian.

Let $F \in E^q_{2m-1}$ and $f \in F$. Since f belongs to $L^q_{loc}(\mathbb{R}^n)$, $\Delta^m f$ is defined in sense of distributions. On the other hand, since any two representatives of F differ in a polynomial of degree smaller than $2m$, we get that $\Delta^m f$ is independent of the representative $f \in F$ chosen. Therefore, for $F \in E^q_{2m-1}$, we shall define $\Delta^m F$ as the distribution $\Delta^m f$, where f is any representative of F.

Our main result is contained in the following

THEOREM 1. *Let $p(.)$ be a function that belongs to $LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$, $1 < q < \infty$ and $m \in \mathbb{N}$ such that $0 < p_- \leqslant p_+ < \infty$ and $n(2m+n/q)^{-1} < \underline{p}$. Then for q sufficiently large the operator Δ^m is a bijective mapping from $\mathscr{H}^{p(.)}_{q,2m}(\mathbb{R}^n)$ onto $H^{p(.)}(\mathbb{R}^n)$. Moreover, there exist two positive constant c_1 and c_2 such that*

$$c_1\|F\|_{\mathscr{H}^{p(.)}_{q,2m}} \leqslant \|\Delta^m F\|_{H^{p(.)}} \leqslant c_2\|F\|_{\mathscr{H}^{p(.)}_{q,2m}}$$

hold for all $F \in \mathscr{H}^{p(.)}_{q,2m}(\mathbb{R}^n)$.

The case $p_+ \leqslant n(2m+n/q)^{-1}$ is trivial.

THEOREM 2. *If $p(.)$ is a positive measurable function such that $p_+ \leqslant n(2m+n/q)^{-1}$, then $\mathscr{H}^{p(.)}_{q,2m}(\mathbb{R}^n) = \{0\}$.*

In Section 2 we state some auxiliary lemmas and propositions to get the main results. We also recall the definition and atomic decomposition of the Hardy spaces with variable exponents given in [9]. In Section 3 we give the proofs of Theorem 1 and Theorem 2.

NOTATION. The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ for some constant c. We denote by $Q(x_0,r)$ the cube centered at $x_0 \in \mathbb{R}^n$ with side lenght r. Given a cube $Q = Q(x_0,r)$, we set $\delta Q = Q(x_0,\delta r)$ and $l(Q) = r$. For a measurable subset $E \subseteq \mathbb{R}^n$ we denote by $|E|$ and χ_E the Lebesgue measure of E and the

characteristic function of E respectively. For a function $p(.) : \mathbb{R}^n \to (0,\infty)$ we define $p_+ = \sup_{x\in\mathbb{R}^n} p(x)$, $p_- = \inf_{x\in\mathbb{R}^n} p(x)$ and $\underline{p} = \min\{p_-, 1\}$. As usual we denote with $S(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions, with $S'(\mathbb{R}^n)$ the dual space and Δ stands for the Laplacian. If α is the multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ then $|\alpha| = \alpha_1 + ... + \alpha_n$.

Throughout this paper, c will denote a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

The function $p(.) : \mathbb{R}^n \to (0,\infty)$ is called the variable exponent. Here we adopt the standard notation in variable exponents. We write

$$p_- = \inf_{x\in\mathbb{R}^n} p(x), \quad p_+ = \sup_{x\in\mathbb{R}^n} p(x), \text{ and } \underline{p} = \min\{p_-, 1\}.$$

Recall that we assumed $0 < p_- \leqslant p_+ < \infty$.

For measurable function f, let

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

it not so hard to see the following

1. $\|f\|_{p(\cdot)} \geqslant 0$, and $\|f\|_{p(\cdot)} = 0$ if and only if $f \equiv 0$.
2. $\|cf\|_{p(\cdot)} = |c|\,\|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.
3. $\|f+g\|_{p(\cdot)}^{\underline{p}} \leqslant \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}}$.

A direct consequence of $\underline{p}$-triangle inequality is the quasi-triangle inequality

$$\|f+g\|_{p(\cdot)} \leqslant 2^{1/\underline{p}-1}\left(\|f\|_{p(\cdot)} + \|g\|_{p(\cdot)}\right),$$

for all $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$.

The following lemmas are crucial to get the principal result.

LEMMA 3. (Lemma 6 in [1]) The maximal function $N_{q;\gamma}(F;x)$ associated with a class F in E_k^q is lower semicontinuous.

LEMMA 4. Let $F \in E_k^q$ with $N_{q,\gamma}(F;x_0) < \infty$, for some $x_0 \in \mathbb{R}^n$. Then:

(i) There exists a unique $f \in F$ such that $\eta_{q,\gamma}(f;x_0) < \infty$ and, therefore, $\eta_{q,\gamma}(f;x_0) = N_{q,\gamma}(F;x_0)$.

(ii) For any cube Q, there is a constant c depending on x_0 and Q such that if f is the unique representative of F given in (i), then

$$\|F\|_{q,Q} \leqslant |f|_{q,Q} \leqslant c\,\eta_{q,\gamma}(f;x_0) = c\,N_{q,\gamma}(F;x_0).$$

The constant c can be chosen independently of x_0 provided that x_0 varies in a compact set.

Proof. The proof is similar to that of Lemma 3 in [7]. □

COROLLARY 5. *If $\{F_j\}$ is a sequence of elements of E_k^q converging to F in $\mathscr{H}_{q,\gamma}^{p(\cdot)}(\mathbb{R}^n)$, $0 < p_- \leqslant p_+ < \infty$, then $\{F_j\}$ converges to F in E_k^q.*

Proof. For any cube Q, by (ii) of Lemma 4, we have

$$\|F - F_j\|_{q,Q} \leqslant c\,\|\chi_Q\|_{p(.)}^{-1}\|\chi_Q\, N_{q,\gamma}(F - F_j;\cdot)\|_{p(\cdot)} \leqslant c\,\|F - F_j\|_{\mathscr{H}_{q,\gamma}^{p(\cdot)}},$$

which proves the corollary. □

LEMMA 6. Let $\{F_j\}$ be a sequence in E_k^q such that for a given point x_0, the series $\sum_j N_{q,\gamma}(F_j; x_0)$ is finite. Then

(i) The series $\sum_j F_j$ converges in E_k^q to an element F and

$$N_{q,\gamma}(F; x_0) \leqslant \sum_j N_{q,\gamma}(F_j; x_0).$$

(ii) If f_j is the unique representative of F_j satisfying $\eta_{q,\gamma}(f_j;x_0) = N_{q,\gamma}(F_j;x_0)$, then $\sum_j f_j$ converges in $L_{loc}^q(\mathbb{R}^n)$ to a function f that is the unique representative of F satisfying $\eta_{q,\gamma}(f;x_0) = N_{q,\gamma}(F;x_0)$

Proof. The proof is similar to that of Lemma 4 in [7]. □

On the set of the all measurable function f we define the modular function $\rho_{p(.)}$ by

$$\rho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

It is well known that if $0 < p_- \leqslant p_+ < \infty$ and $0 \neq f \in L^{p(\cdot)}(\mathbb{R}^n)$, then

$$\rho_{p(.)}\left(\|f\|_{p(.)}^{-1} f\right) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\|f\|_{p(.)}}\right)^{p(x)} dx = 1.$$

LEMMA 7. Let $p(.)$ a measurable function such that $0 < p_- \leqslant p_+ < \infty$. Then $f \in L^{p(.)}(\mathbb{R}^n)$ if and only if $\rho_{p(.)}(f) < \infty$.

Proof. Clearly, if $\rho_{p(.)}(f) < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$. Conversely, if $f \in L^{p(.)}(\mathbb{R}^n)$, then we have that $\rho_{p(.)}(f/\lambda) < \infty$ for some $\lambda > 1$. Then

$$\rho_{p(.)}(f) = \int_{\mathbb{R}^n} \left|\frac{\lambda f(x)}{\lambda}\right|^{p(x)} dx \leqslant \lambda^{p_+}\rho_{p(.)}(f/\lambda) < \infty. \quad \square$$

LEMMA 8. Let $p(.)$ a measurable function such that $0 < p_- \leqslant p_+ < \infty$. If $\{f_j\}$ is a sequence of measurable functions such that $\rho_{p(.)}(f_j) \to 0$, then $\|f_j\|_{p(.)} \to 0$.

Proof. Suppose that $\rho_{p(.)}(f_j) \to 0$. Given $0 < \varepsilon < 1$ for sufficiently large j we have $\rho_{p(.)}(f_j) \leqslant \varepsilon$ and so

$$\rho_{p(.)}\left(f_j \rho_{p(.)}(f_j)^{-1/p_+}\right) \leqslant \rho_{p(.)}(f_j)^{-1}\rho_{p(.)}(f_j) = 1,$$

from this it follows that $\|f_j\|_{p(.)} \leqslant \rho_{p(.)}(f_j)^{1/p_+} \leqslant \varepsilon^{1/p_+}$. Thus, $\|f_j\|_{p(.)} \to 0$. $\square$

PROPOSITION 9. *The space $\mathscr{H}^{p(\cdot)}_{q,\gamma}(\mathbb{R}^n)$, $0 < p_- \leqslant p_+ < \infty$, is complete.*

Proof. It is enough to show that $\mathscr{H}^{p(\cdot)}_{q,\gamma}$ has the Riesz-Fisher property: given any sequence $\{F_j\}$ in $\mathscr{H}^{p(\cdot)}_{q,\gamma}$ such that

$$\sum_j \|F_j\|^{\underline{p}}_{\mathscr{H}^{p(\cdot)}_{q,\gamma}} < \infty,$$

the series $\sum_j F_j$ converges in $\mathscr{H}^{p(\cdot)}_{q,\gamma}$.

Let $1 \leqslant l$ be fixed, then

$$\left\|\sum_{j=l}^{k} N_{q,\gamma}(F_j;.)\right\|^{\underline{p}}_{p(.)} \leqslant \sum_{j=l}^{k} \left\|N_{q,\gamma}(F_j;.)\right\|^{\underline{p}}_{p(.)} \leqslant \sum_{j=l}^{\infty} \|F_j\|^{\underline{p}}_{\mathscr{H}^{p(\cdot)}_{q,\gamma}} =: \alpha_l < \infty,$$

for all $k \geqslant l$, thus

$$\int_{\mathbb{R}^n} \left(\alpha_l^{-1/\underline{p}} \sum_{j=l}^{k} N_{q,\gamma}(F_j;x)\right)^{p(x)} dx$$

$$\leqslant \int_{\mathbb{R}^n} \left(\left\|\sum_{j=l}^{k} N_{q,\gamma}(F_j;.)\right\|^{-1}_{p(.)} \sum_{j=l}^{k} N_{q,\gamma}(F_j;x)\right)^{p(x)} dx = 1, \ \forall k \geqslant l$$

it follows from Fatou's lemma as $k \to \infty$ that

$$\int_{\mathbb{R}^n} \left(\alpha_l^{-1/\underline{p}} \sum_{j=l}^{\infty} N_{q,\gamma}(F_j;x)\right)^{p(x)} dx \leqslant 1$$

thus

$$\left\|\sum_{j=l}^{\infty} N_{q,\gamma}(F_j;\cdot)\right\|^{\underline{p}}_{p(\cdot)} \leqslant \alpha_l = \sum_{j=l}^{\infty} \|F_j\|^{\underline{p}}_{\mathscr{H}^{p(\cdot)}_{q,\gamma}} < \infty, \ \forall l \geqslant 1. \tag{1}$$

Taking $l = 1$ in (1), from Lemma 7 it follows that $\sum_j N_{q,\gamma}(F_j;x)$ is finite a.e. $x \in \mathbb{R}^n$. Then, by (i) of Lemma 6, the series $\sum_j F_j$ converges in E^q_k to an element F. Now

$$N_{q,\gamma}\left(F - \sum_{j=1}^{k} F_j; x\right) \leqslant \sum_{j=k+1}^{\infty} N_{q,\gamma}(F_j;x),$$

from this and (1) we get

$$\left\| F - \sum_{j=1}^{k} F_j \right\|_{\mathscr{H}_{q,\gamma}^{p(\cdot)}}^{\underline{p}} \leqslant \sum_{j=k+1}^{\infty} \|F_j\|_{\mathscr{H}_{q,\gamma}^{p(\cdot)}}^{\underline{p}},$$

and since the right-hand side tends to 0 as $k \to \infty$, the series $\sum_j F_j$ converges to F in $\mathscr{H}_{q,\gamma}^{p(\cdot)}(\mathbb{R}^n)$. $\square$

In the paper [9], E. Nakai and Y. Sawano give a variety of distinct approaches, based on differing definitions, all lead to the same notion of the variable Hardy space $H^{p(\cdot)}$.

We recall the definition and the atomic decomposition of the Hardy spaces with variable exponents.

Topologize $\mathscr{S}(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) = \sum_{|\beta|\leqslant N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\partial^\beta \varphi(x)\right|,$$

for each $N \in \mathbb{N}$. We set $\mathscr{F}_N = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : p_N(\varphi) \leqslant 1\}$. Let $f \in \mathscr{S}'(\mathbb{R}^n)$, we denote by $\mathscr{M}_{\mathscr{F}_N}$ the grand maximal operator given by

$$\mathscr{M}_{\mathscr{F}_N} f(x) = \sup_{t>0} \sup_{\varphi\in\mathscr{F}_N} \left|\left(t^{-n}\varphi(t^{-1}\cdot) * f\right)(x)\right|,$$

where N is a large and fix integer.

The Hardy space with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ is the set of all $f \in S'(\mathbb{R}^n)$ for which $\mathscr{M}_{\mathscr{F}_N} f \in L^{p(\cdot)}(\mathbb{R}^n)$. In this case we define $\|f\|_{H^{p(\cdot)}} = \left\|\mathscr{M}_{\mathscr{F}_N} f\right\|_{p(\cdot)}$.

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a function such that $\int \phi(x)dx \neq 0$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, we define the maximal function $M_\phi f$ by

$$M_\phi f(x) = \sup_{t>0} \left|\left(t^{-n}\phi(t^{-1}\cdot) * f\right)(x)\right|.$$

Theorem 1.2 in [9] asserts that the quantities $\|M_\phi f\|_{p(\cdot)}$ and $\left\|\mathscr{M}_{\mathscr{F}_N} f\right\|_{p(\cdot)}$ are comparable, with bounds independent of f if N is sufficiently large.

DEFINITION 10. $((p(\cdot), p_0, d) - atom)$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \leqslant p_+ < p_0 \leqslant \infty$ and $p_0 \geqslant 1$. Fix an integer $d \geqslant d_{p(\cdot)} = \min\{l \in \mathbb{N}\cup\{0\} : p_-(n+l+1) > n\}$. A function a on $\mathbb{R}^n$ is called an $(p(\cdot), p_0, d)$-atom if there exists a cube Q such that

$a_1)$ $supp\,(a) \subset Q$,

$a_2)$ $\|a\|_{p_0} \leqslant \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{p(\cdot)}}$,

$a_3)$ $\int a(x)x^\alpha dx = 0$ for all $|\alpha| \leqslant d$.

REMARK 11. Let a be an $(p(\cdot), p_0, d) - atom$ and $1 < s < p_0$, then Hölder's inequality implies $\|a\|_s \leqslant \frac{|Q|^{1/s}}{\|\chi_Q\|_{p(\cdot)}}$.

DEFINITION 12. For sequences of nonnegative numbers $\{k_j\}_{j=1}^{\infty}$ and cubes $\{Q_j\}_{j=1}^{\infty}$ and for a function $p(\cdot):\mathbb{R}^n\to(0,\infty)$, we define

$$\mathscr{A}\left(\{k_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty},p(\cdot)\right)=\left\|\left\{\sum_{j=1}^{\infty}\left(\frac{k_j\chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}}\right)^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{p(\cdot)}.$$

The space $H_{atom}^{p(\cdot),p_0,d}(\mathbb{R}^n)$ is the set of all distributions $f\in S'(\mathbb{R}^n)$ such that it can be written as

$$f=\sum_{j=1}^{\infty}k_ja_j \tag{2}$$

in $S'(\mathbb{R}^n)$, where $\{k_j\}_{j=1}^{\infty}$ is a sequence of non negative numbers, the a_j‘s are $(p(\cdot),p_0,d)$-atoms and $\mathscr{A}\left(\{k_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty},p(\cdot)\right)<\infty$. One defines

$$\|f\|_{H_{atom}^{p(\cdot),p_0,d}}=\inf\mathscr{A}\left(\{k_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty},p(\cdot)\right)$$

where the infimun is taken over all admissible expressions as in (2).

Theorem 4.6 in [9] asserts that the quantities $\|f\|_{H_{atom}^{p(\cdot),p_0,d}}$ and $\|f\|_{H^{p(\cdot)}}$ are comparable. Moreover, f admits an atomic decomposition $f=\sum\limits_{j=1}^{\infty}k_ja_j$ such that

$$\mathscr{A}\left(\{k_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty},p(\cdot)\right)\leqslant c\,\|f\|_{H^{p(\cdot)}}.$$

Let h be the function defined by

$$h(x)=\begin{cases}|x|^{2m-n}\ln|x|, & \text{if } n \text{ is even and } 2m-n\geqslant 0\\ |x|^{2m-n}, & \text{otherwise}\end{cases}. \tag{3}$$

It is well known that if a is a bounded function with compact support, its potential b, defined as

$$b(x)=\int_{\mathbb{R}^n}h(x-y)a(y)dy,$$

is a locally bounded function and $\Delta^m b=a$ in the sense of distributions. For these potentials, we have the following

LEMMA 13. Let $a(\cdot)$ be an $(p(\cdot),p_0,d)$-atom with $d=\max\{d_{p(\cdot)},2m-1\}$ and assume that $Q(x_0,r)$ is the cube containing the support of $a(.)$ in the definition of $(p(\cdot),p_0,d)$-atom. If

$$b(x)=\int_{\mathbb{R}^n}h(x-y)a(y)dy,$$

then for $|x-x_0|\geqslant\sqrt{n}r$ and every multiindex α, there exists c_α such that

$$|(\partial^\alpha b)(x)|\leqslant c_\alpha\, r^{2m+n}\|\chi_Q\|_{p(\cdot)}^{-1}|x-x_0|^{-n-\alpha}$$

holds.

Proof. Since $a(\cdot)$ has vanishing moments up to the order $d \geqslant 2m-1$, we have

$$\begin{aligned}(\partial^\alpha b)(x) &= \int_{Q(x_0,r)} (\partial^\alpha h)(x-y)a(y)\,dy \\ &= \int_{Q(x_0,r)} \left[(\partial^\alpha h)(x-y) - \sum_{|\beta|\leqslant 2m-1} (\partial^{\alpha+\beta}h)(x-x_0)\frac{(x_0-y)^\beta}{\beta!}\right] a(y)\,dy \\ &= \int_{Q(x_0,r)} \left[\sum_{|\beta|=2m} (\partial^{\alpha+\beta}h)(x-\xi)\frac{(x_0-y)^\beta}{\beta!}\right] a(y)\,dy\end{aligned}$$

where ξ is a point between y and x_0. If $|x-x_0| \geqslant \sqrt{n}r$ it follows that $|x-\xi| \geqslant \frac{|x-x_0|}{2}$ since $|x_0-\xi| \leqslant \frac{\sqrt{n}}{2}r$. Taking into account that for $|\beta| = 2m$, $\partial^{\alpha+\beta}h$ is a homogeneous function of degree $-n-\alpha$, we obtain

$$\begin{aligned}|(\partial^\alpha b)(x)| &\leqslant c\,r^{|\beta|}\int_{Q(x_0,r)} |x-\xi|^{-n-\alpha}|a(y)|dy \\ &\leqslant c\,r^{|\beta|}\|a\|_{p_0}|Q|^{1-1/p_0}|x-x_0|^{-n-\alpha} \leqslant c\,r^{2m+n}\|\chi_Q\|^{-1}_{p(\cdot)}|x-x_0|^{-n-\alpha}. \quad \square\end{aligned}$$

LEMMA 14. (Lemma 8 in [7]) If h is the kernel defined in (3) and $|\alpha| = 2m$, then $(\partial^\alpha h)(x)$ is a C^∞ homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$, and $\int_{|x|=1} (\partial^\alpha h)(x)\,dx = 0$.

We conclude this preliminaries with the following

PROPOSITION 15. *Let $a(\cdot)$ be an $(p(\cdot),p_0,d)$-atom with $d = \max\{d_{p(\cdot)}, 2m-1\}$, $p_0 > 1$, and assume that $Q = Q(x_0,r)$ is the cube containing the support of $a(\cdot)$ in the definition of $(p(\cdot),p_0,d)$-atom. If $b(x) = \int_{\mathbb{R}^n} h(x-y)a(y)dy$, then for all $x \in \mathbb{R}^n$ and all $0 < \mu < 2m$*

$$\begin{aligned}N_{q,2m}(B;x) \lesssim \|\chi_Q\|^{-1}_{p(\cdot)}[M(\chi_Q)(x)]^{\frac{2m+n/q-\mu}{n}} + \chi_{4\sqrt{n}Q}(x)M(a)(x) \\ + \chi_{4\sqrt{n}Q}(x)[M(M^q(a))(x)]^{1/q} + \chi_{4\sqrt{n}Q}(x)\sum_{|\alpha|=2m} T^*_\alpha(a)(x),\end{aligned} \tag{4}$$

where B is the class of b in E^q_{2m-1}, $T^*_\alpha(a)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}(\partial^\alpha h)(x-y)a(y)\,dy\right|$ and M is the Hardy-Littlewood maximal operator.

Proof. For an $(p(\cdot),p_0,d)$ - atom a satisfying the hypothesis of Proposition, we set

$$\begin{aligned}R(x,z) &= b(x+z) - \sum_{|\alpha|\leqslant 2m-1}(\partial^\alpha b)(x)z^\alpha/\alpha! \\ &= b(x+z) - \sum_{|\alpha|\leqslant 2m-1}\left[\int_{Q(x_0,r)}(\partial^\alpha h)(x-y)a(y)\,dy\right]\frac{z^\alpha}{\alpha!}.\end{aligned}$$

We shall estimate $|R(x,z)|$ considering the cases $|x-x_0| \geqslant 2\sqrt{n}r$ and $|x-x_0| < 2\sqrt{n}r$, and then we will obtain the estimate (4).

Case: $|x-x_0| \geqslant 2\sqrt{n}r$.

For $|x-x_0| \geqslant 2\sqrt{n}r$, $|z| < \frac{1}{2}|x-x_0|$ and $0<\theta<1$, we have $|x+\theta z-x_0| \geqslant |x-x_0|-|z| \geqslant \frac{1}{2}|x-x_0| \geqslant \sqrt{n}r$. Then, by the mean value theorem and Lemma 13, we get

$$|R(x,z)| \leqslant \sum_{|\alpha|=2m} |(\partial^\alpha b)(x+\theta z)| \frac{|z|^{|\alpha|}}{\alpha!} \leqslant c\,\|\chi_Q\|_{p(\cdot)}^{-1} \left(\frac{r}{|x-x_0|}\right)^{2m+n} |z|^{2m}, \tag{5}$$

for $|x-x_0| \geqslant 2\sqrt{n}r$ and $|z| < \frac{1}{2}|x-x_0|$.

Now, let $|z| \geqslant \frac{1}{2}|x-x_0|$. We have

$$|R(x,z)| \leqslant |b(x+z)| + \sum_{|\alpha|\leqslant 2m-1} |(\partial^\alpha b)(x)||z|^\alpha/\alpha!.$$

Since $|x-x_0| > 2\sqrt{n}r$, by Lemma 13, and observing that $|z|/|x-x_0| > 1/2$, we have

$$|(\partial^\alpha b)(x)| \frac{|z|^{|\alpha|}}{\alpha!} \leqslant c\,\|\chi_Q\|_{p(\cdot)}^{-1} \left(\frac{r}{|x-x_0|}\right)^{2m+n} |z|^{2m}.$$

As for the other term, $|b(x+z)|$, we consider the cases $|x+z-x_0| > \sqrt{n}r$ and $|x+z-x_0| \leqslant \sqrt{n}r$. In the case $|x+z-x_0| > \sqrt{n}r$, we apply Lemma 13 for $\alpha=0$, obtaining

$$|b(x+z)| \leqslant c\,\|\chi_Q\|_{p(\cdot)}^{-1} r^{2m+n} |x+z-x_0|^{-n}.$$

Then

$$|R(x,z)| \lesssim \|\chi_Q\|_{p(\cdot)}^{-1} r^{2m+n} |x+z-x_0|^{-n} + \|\chi_Q\|_{p(\cdot)}^{-1} \left(\frac{r}{|x-x_0|}\right)^{2m+n} |z|^{2m}, \tag{6}$$

holds if $|x-x_0| \geqslant 2\sqrt{n}r$, $|z| \geqslant |x-x_0|/2$ and $|x+z-x_0| > \sqrt{n}r$.

For $|x+z-x_0| \leqslant \sqrt{n}r$, we consider the cases n even and $2m-n \geqslant 0$, and n odd or $2m-n<0$. In the first case, we have that $|x|^{2m-n}$ is a polynomial of degree smaller than $2m$, and since $a(\cdot)$ is an $(p(\cdot),p_0,d)$-atom, we have

$$\begin{aligned} |b(x+z)| &= \left| \int_{Q(x_0,r)} |x+z-y|^{2m-n} [\ln(|x+z-y|) - \ln(|x-x_0|)]\, a(y)dy \right| \\ &\leqslant \|a\|_{p_0} \left(\int_{Q(x_0,r)} |x+z-y|^{(2m-n)p_0'} (\ln(|x-x_0|/|x+z-y|))^{p_0'} dy \right)^{1/p_0'}. \end{aligned}$$

Since $|x+z-y|/|x-x_0| < 1$, $\ln(t) \leqslant \mu^{-1}t^\mu$ for any $0<\mu$ and $t \geqslant 1$, and observing that $Q(x_0,r) \subset \{y : |x+z-y| < \frac{3}{2}\sqrt{n}r\}$, we take $0<\mu<2m-n$ if $2m-n>0$ or

$0<\mu<n/p_0'$ if $2m-n=0$, to get

$$|b(x+z)| \lesssim \|\chi_Q\|_{p(\cdot)}^{-1} r^{n/p_0}|x-x_0|^{\mu}\left(\int_{|x+z-y|<\frac{3}{2}\sqrt{n}r}|x+z-y|^{(2m-n-\mu)p_0'}dy\right)^{1/p_0'}$$
$$\lesssim r^{2m}\|\chi_Q\|_{p(\cdot)}^{-1}\left(\frac{r}{|x-x_0|}\right)^{-\mu}.$$

For the case n odd and $2m-n\geqslant 0$, the proof is simpler since no logarithm appears, and the estimates obtained hold with $\mu=0$. For the case $2m-n<0$, we take $n/2m<s<p_0$ thus $0<(2m-n)s'+n$, from Hölder's inequality and Remark 11 it follows the estimate with $\mu=0$.

Since $|x-x_0|\geqslant 2\sqrt{n}r$ we can conclude that

$$|R(x,z)| \lesssim r^{2m}\|\chi_Q\|_{p(\cdot)}^{-1}\left(\frac{r}{|x-x_0|}\right)^{-\mu}+\|\chi_Q\|_{p(\cdot)}^{-1}\left(\frac{r}{|x-x_0|}\right)^{2m+n}|z|^{2m}, \tag{7}$$

for all $\mu>0$, $|z|\geqslant|x-x_0|/2$ and $|x+z-x_0|\leqslant\sqrt{n}r$.

Let us the estimate

$$\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|R(x,z)|^q dz\right)^{1/q},\ \ \delta>0.$$

For them, we split the domain of integration into three subsets:

$$D_1=\{z\in Q(0,\delta):|z|<|x-x_0|/2\},$$
$$D_2=\{z\in Q(0,\delta):|z|\geqslant|x-x_0|/2,|x+z-x_0|>\sqrt{n}r\},$$

and

$$D_3=\{z\in Q(0,\delta):|z|\geqslant|x-x_0|/2,|x+z-x_0|\leqslant\sqrt{n}r\}.$$

According to the estimates obtained for $|R(x,z)|$ above, we use on D_1 the estimate (5), on D_2 the estimate (6) and on D_3 the estimate (7) to get

$$\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|R(x,z)|^q dz\right)^{1/q}\lesssim\|\chi_Q\|_{p(\cdot)}^{-1}\left(\frac{r}{|x-x_0|}\right)^{2m+n/q-\mu}.$$

Thus, for $0<\mu<2m$ we have

$$N_{q,2m}(B,x)\lesssim\|\chi_Q\|_{p(\cdot)}^{-1}\left(\frac{r}{|x-x_0|}\right)^{2m+n/q-\mu}\lesssim\|\chi_Q\|_{p(\cdot)}^{-1}[M(\chi_Q)(x)]^{\frac{2m+n/q-\mu}{n}}, \tag{8}$$

if $|x-x_0|\geqslant 2\sqrt{n}r$.

Case: $|x-x_0|<2\sqrt{n}r$.

We have

$$R(x,z)=\int\left[h(x+z-y)-\sum_{|\alpha|\leqslant 2m-1}(\partial^\alpha h)(x-y)z^\alpha/\alpha!\right]a(y)dy$$
$$=\int_{|x-y|<2|z|}+\int_{|x-y|\geqslant 2|z|}=I_1(x,z)+I_2(x,z).$$

Without loss of generality we can assume that y does not belong to the segment $[x,x+z]$, so we can write

$$\begin{aligned} U &= h(x+z-y) - \sum_{|\alpha|\leqslant 2m-1} (\partial^\alpha h)(x-y)z^\alpha/\alpha! \\ &= (2m-1)\sum_{|\alpha|=2m-1} (z^\alpha/\alpha!)\int_0^1 (\partial^\alpha h)(x+tz-y)(1-t)^{2m-2}dt \\ &\quad - \sum_{|\alpha|=2m-1} (\partial^\alpha h)(x-y)z^\alpha/\alpha! \end{aligned}$$

Since for $|\alpha| = 2m-1$ the derivatives $\partial^\alpha h$ are homogeneous functions of degree $-n+1$, we get

$$|U| \leqslant c\left(\int_0^1 |x+tz-y|^{-n+1}(1-t)^{2m-2}dt + |x-y|^{-n+1}\right)|z|^{2m-1}.$$

Observing that $|x-y| < 2|z|$ implies $|x+tz-y| < 3|z|$, we obtain

$$\begin{aligned} |I_1(x,z)| &\leqslant \int_{|x-y|<2|z|} |U||a(y)|dy \\ &\lesssim |z|^{2m-1}\int_0^1 (1-t)^{2m-2}\left(\int_{|x+tz-y|<3|z|} |x+tz-y|^{-n+1}|a(y)|dy\right)dt \\ &\quad + |z|^{2m-1}\int_{|x-y|<2|z|} |x-y|^{-n+1}|a(y)|dy \\ &= |z|^{2m-1}\int_0^1 (1-t)^{2m-2}\left(\sum_{k=0}^{\infty}\int_{3^{-k}|z|\leqslant|x+tz-y|<3^{-(k-1)}|z|} |x+tz-y|^{-n+1}|a(y)|dy\right)dt \\ &\quad + |z|^{2m-1}\sum_{k=0}^{\infty}\int_{2^{-k}|z|\leqslant|x-y|<2^{-(k-1)}|z|} |x-y|^{-n+1}|a(y)|dy \\ &\lesssim |z|^{2m}\left(\int_0^1 (1-t)^{2m-2}M(a)(x+tz)\,dt + M(a)(x)\right). \end{aligned}$$

To estimate $I_2(x,z)$, we write

$$\begin{aligned} U &= \left[h(x+z-y) - \sum_{|\alpha|\leqslant 2m} (\partial^\alpha h)(x-y)z^\alpha/\alpha!\right] + \sum_{|\alpha|=2m} (\partial^\alpha h)(x-y)z^\alpha/\alpha! \\ &= U_1 + U_2. \end{aligned}$$

We have for some $0 < s < 1$

$$|U_1| \leqslant \sum_{|\alpha|=2m+1} |(\partial^\alpha h)(x+sz-y)|\,|z|^{2m+1}/\alpha!.$$

Since $|x-y| \geqslant 2|z|$ implies $|x+sz-y| \geqslant |x-y|/2$, and recalling that for $|\alpha| = 2m+1$ the derivatives $\partial^\alpha h$ are homogeneous functions of degree $-n-1$, we get $|U_1| \leqslant c|x-$

$y|^{-(n+1)}|z|^{2m+1}$. Therefore,

$$\begin{aligned}|I_2(x,z)| &\lesssim |z|^{2m+1}\int_{|x-y|\geqslant 2|z|}|x-y|^{-(n+1)}|a(y)|dy+\left|\int_{|x-y|\geqslant 2|z|}U_2\,a(y)dy\right|\\ &\lesssim |z|^{2m}\left(M(a)(x)+\sum_{|\alpha|=2m}T^*_\alpha(a)(x)\right),\end{aligned}$$

where $T^*_\alpha(a)(x)=\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}(\partial^\alpha h)(x-y)a(y)\,dy\right|$.

Now, let us estimate $\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|I_1(x,z)|^q dz\right)^{1/q}$. We apply Minkowski's inequality for integrals to get

$$\begin{aligned}&\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|I_1(x,z)|^q dz\right)^{1/q}\\ &\lesssim \delta^{-2m}\int_0^1(1-t)^{2m-2}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|z|^{2mq}M^q(a)(x+tz)\,dz\right)^{1/q}dt\\ &\quad+M(a)(x)\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|z|^{2mq}dz\right)^{1/q}\\ &\lesssim [M(M^q(a))(x)]^{1/q}+M(a)(x).\end{aligned}$$

It is easy to check that

$$\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|I_2(x,z)|^q dz\right)^{1/q}\lesssim M(a)(x)+\sum_{|\alpha|=2m}T^*_\alpha(a)(x).$$

So

$$\begin{aligned}&\delta^{-2m}\left(|Q(0,\delta)|^{-1}\int_{Q(0,\delta)}|R(x,z)|^q dz\right)^{1/q}\\ &\lesssim [M(M^q(a))(x)]^{1/q}+M(a)(x)+\sum_{|\alpha|=2m}T^*_\alpha(a)(x).\end{aligned}$$

This estimate is global, in particular we have

$$N_{q,2m}(B;x)\lesssim [M(M^q(a))(x)]^{1/q}+M(a)(x)+\sum_{|\alpha|=2m}T^*_\alpha(a)(x), \tag{9}$$

for all $x\in 4\sqrt{n}Q$. Finally, the estimates (8) and (9) for $N_{q,2m}(B;x)$ allow us to obtain (4). $\square$

3. Proofs of the results

Proof of Theorem 1. Let $F \in \mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$. Since $N_{q,2m}(F;x)$ is finite a.e. $x \in \mathbb{R}^n$, by corollary 3 in [7], the unique representative f of F satisfying $\eta_{q,2m}(f;x) = N_{q,2m}(F;x)$ is a function in $L^q_{loc}(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n)$. Thus, if $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $\int \phi(x)\,dx \neq 0$, from Lemma 6 in [7] we get

$$M_\phi(\Delta^m F)(x) \leqslant c\, p_{2m+n}(\phi) N_{q,2m}(F;x).$$

Thus $\Delta^m F \in H^{p(\cdot)}(\mathbb{R}^n)$ and $\|\Delta^m F\|_{H^{p(\cdot)}} \leqslant c\,\|F\|_{\mathscr{H}^{p(\cdot)}_{q,2m}}$.

Now we shall see that the operator Δ^m is onto. By Theorem 4.6 in [9], given $f \in H^{p(\cdot)}(\mathbb{R}^n)$ there exists a sequence of nonnegative numbers $\{k_j\}_{j=1}^\infty$ and a sequence of cubes $\{Q_j\}_{j=1}^\infty$ and $(p(\cdot), p_0, d)$ - atoms a_j supported on Q_j, such that $f = \sum_{j=1}^\infty k_j a_j$ and

$$\mathscr{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right) \lesssim \|f\|_{H^{p(\cdot)}}.$$

For each $j \in \mathbb{N}$ we put $b_j(x) = \int_{\mathbb{R}^n} h(x-y)a_j(y)dy$, from Proposition 15 we have

$$\begin{aligned} N_{q,2m}(B_j;x) \lesssim \|\chi_{Q_j}\|^{-1}_{p(\cdot)} \left[M(\chi_{Q_j})(x)\right]^{\frac{2m+n/q-\mu}{n}} &+ \chi_{4\sqrt{n}Q_j}(x) M(a_j)(x) \\ &+ \chi_{4\sqrt{n}Q_j}(x)[M(M^q(a_j))(x)]^{1/q} + \chi_{4\sqrt{n}Q_j}(x) \sum_{|\alpha|=2m} T^*_\alpha(a_j)(x). \end{aligned}$$

So

$$\begin{aligned} \sum_{j=1}^\infty k_j N_{q,2m}(B_j;x) &\lesssim \sum_{j=1}^\infty k_j \frac{\left[M(\chi_{Q_j})(x)\right]^{\frac{2m+n/q-\mu}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}} + \sum_{j=1}^\infty k_j \chi_{4\sqrt{n}Q_j}(x) M(a_j)(x) \\ &\quad + \sum_{j=1}^\infty k_j \chi_{4\sqrt{n}Q_j}(x)[M(M^q(a_j))(x)]^{1/q} \\ &\quad + \sum_{j=1}^\infty k_j \chi_{4\sqrt{n}Q_j}(x) \sum_{|\alpha|=2m} T^*_\alpha(a_j)(x) \\ &= I + II + III + IV. \end{aligned}$$

To study I, by hypothesis, we have that $(2m+n/q)\underline{p} > n$, thus we can choose $0 < \mu < 2m$ such that $(2m+n/q-\mu)\underline{p} > n$. Then

$$\begin{aligned} \|I\|_{p(\cdot)} &= \left\| \sum_{j=1}^\infty k_j \|\chi_{Q_j}\|^{-1}_{p(\cdot)} \left[M(\chi_{Q_j})(\cdot)\right]^{\frac{2m+n/q-\mu}{n}} \right\|_{p(\cdot)} \\ &= \left\| \left\{ \sum_{j=1}^\infty k_j \|\chi_{Q_j}\|^{-1}_{p(\cdot)} \left[M(\chi_{Q_j})(\cdot)\right]^{\frac{2m+n/q-\mu}{n}} \right\}^{\frac{n}{2m+n/q-\mu}} \right\|^{\frac{2m+n/q-\mu}{n}}_{\frac{2m+n/q-\mu}{n} p(\cdot)} \end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} k_j \|\chi_{Q_j}\|_{p(\cdot)}^{-1} \chi_{Q_j}(\cdot) \right\}^{\frac{n}{2m+n/q-\mu}} \right\|_{p(\cdot)}^{\frac{2m+n/q-\mu}{n}} = \left\| \sum_{j=1}^{\infty} k_j \|\chi_{Q_j}\|_{p(\cdot)}^{-1} \chi_{Q_j}(\cdot) \right\|_{p(\cdot)}$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(k_j \|\chi_{Q_j}\|_{p(\cdot)}^{-1} \chi_{Q_j}(\cdot) \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{p(\cdot)} = \mathscr{A}\left(\{k_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, p(\cdot) \right)$$

$$\lesssim \|f\|_{H^{p(\cdot)}},$$

where the first inequality follows from Lemma 2.4 in [9], since $(2m+n/q-\mu)\underline{p} > n$. The embedding $l^{\underline{p}} = l^{\min\{p_-,1\}} \hookrightarrow l^1$ gives the second inequality.

To study II, we have that the maximal operator M is bounded on L^{p_0} for each $1 < p_0 < \infty$, thus

$$\|M(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \lesssim \|a_j\|_{p_0} \lesssim \frac{|Q_j|^{1/p_0}}{\|\chi_{Q_j}\|_{p(\cdot)}} \lesssim \frac{|4\sqrt{n}Q_j|^{1/p_0}}{\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}},$$

the third inequality holds since the quantities $\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}$ and $\|\chi_{Q_j}\|_{p(\cdot)}$ are comparable (see Lemma 2.2 in [9]). Now if $\|M(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \neq 0$ we obtain

$$\begin{aligned}
\|II\|_{p(\cdot)} &= \left\| \sum_{j=1}^{\infty} k_j \chi_{4\sqrt{n}Q_j}(\cdot) M(a_j)(\cdot) \right\|_{p(\cdot)} \\
&\lesssim \left\| \sum_{j=1}^{\infty} k_j \frac{\chi_{4\sqrt{n}Q_j}(\cdot) M(a_j)(\cdot) |4\sqrt{n}Q_j|^{1/p_0}}{\|M(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} \\
&\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(k_j \frac{\chi_{4\sqrt{n}Q_j}(\cdot) M(a_j)(\cdot) |4\sqrt{n}Q_j|^{1/p_0}}{\|M(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}} \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{p(\cdot)}
\end{aligned}$$

now take $p_0 > 1$ sufficiently large such that $\delta = \frac{1}{p_0}$ satisfies the hypothesis of Lemma 4.11 in [9] to get

$$\lesssim \mathscr{A}\left(\{k_j\}_{j=1}^{\infty}, \{4\sqrt{n}Q_j\}_{j=1}^{\infty}, p(\cdot) \right) \lesssim \mathscr{A}\left(\{k_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, p(\cdot) \right) \lesssim \|f\|_{H^{p(\cdot)}},$$

where the fourth inequality follows from Lemma 4.8 in [9].

To study III, we consider $2q < p_0$, then by Hölder's inequality and Remark 11 we obtain

$$\begin{aligned}
\|[M(M^q(a_j))]^{1/q}\|_{L^q(4\sqrt{n}Q_j)} &\lesssim \|M(M^q(a_j))\|_2^{1/q} |4\sqrt{n}Q_j|^{1/2q} \\
&\lesssim \|M^q(a_j)\|_2^{1/q} |4\sqrt{n}Q_j|^{1/2q} = \|M(a_j)\|_{2q} |4\sqrt{n}Q_j|^{1/2q} \\
&\lesssim \|a_j\|_{2q} |4\sqrt{n}Q_j|^{1/2q} \lesssim \frac{|4\sqrt{n}Q_j|^{1/q}}{\|\chi_{Q_j}\|_{p(\cdot)}} \lesssim \frac{|4\sqrt{n}Q_j|^{1/q}}{\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}}.
\end{aligned}$$

Now, if $\|[M(M^q(a_j))]^{1/q}\|_{L^q(4\sqrt{n}Q_j)} \neq 0$ and taking $q > 1$ sufficiently large such that $\delta = \frac{1}{q}$ satisfies the hypothesis of Lemma 4.11 in [9], a similar computation to done in the estimate of II allows us to get

$$\begin{aligned}\|III\|_{p(\cdot)} &= \left\| \sum_{j=1}^{\infty} k_j \chi_{4\sqrt{n}Q_j}(\cdot)[M(M^q(a_j))(\cdot)]^{1/q} \right\|_{p(\cdot)} \\ &\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(k_j \frac{\chi_{4\sqrt{n}Q_j}(\cdot)[M(M^q(a_j))(\cdot)]^{1/q}|4\sqrt{n}Q_j|^{1/q}}{\|[M(M^q(a_j))(\cdot)]^{1/q}\|_{L^{p_0}(4\sqrt{n}Q_j)}\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}} \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{p(\cdot)} \\ &\lesssim \|f\|_{H^{p(\cdot)}}.\end{aligned}$$

To study IV, by Lemma 14 and Theorem 4 in [14] p. 42, we have that the operator T_α^* is bounded on L^{p_0} for each $1 < p_0 < \infty$, so

$$\|T_\alpha^*(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \lesssim \|a_j\|_{p_0} \lesssim \frac{|Q_j|^{1/p_0}}{\|\chi_{Q_j}\|_{p(\cdot)}} \lesssim \frac{|4\sqrt{n}Q_j|^{1/p_0}}{\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}}.$$

Now, if $\|T_\alpha^*(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)} \neq 0$ we take $p_0 > 1$ sufficiently large such that $\delta = \frac{1}{p_0}$ satisfies the hypothesis of Lemma 4.11 in [9], once again a similar computation to done in the estimate of II, we get

$$\begin{aligned}\|IV\|_{p(\cdot)} &= \left\| \sum_{j=1}^{\infty} k_j \sum_{|\alpha|=2m} \chi_{4\sqrt{n}Q_j}(\cdot) T_\alpha^*(a_j)(\cdot) \right\|_{p(\cdot)} \\ &\lesssim \sum_{|\alpha|=2m} \left\| \left\{ \sum_{j=1}^{\infty} \left(k_j \frac{\chi_{4\sqrt{n}Q_j}(\cdot) T_\alpha^*(a_j)(\cdot)|4\sqrt{n}Q_j|^{1/p_0}}{\|T_\alpha^*(a_j)\|_{L^{p_0}(4\sqrt{n}Q_j)}\|\chi_{4\sqrt{n}Q_j}\|_{p(\cdot)}} \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{p(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}}.\end{aligned}$$

Thus we have

$$\left\| \sum_{j=1}^{\infty} k_j N_{q,2m}(B_j;\cdot) \right\|_{p(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}}.$$

By Lemma 7, we obtain $\rho_{p(\cdot)}\left(\sum_{j=1}^{\infty} k_j N_{q,2m}(B_j;\cdot)\right) < \infty$. Hence

$$\sum_{j=1}^{\infty} k_j N_{q,2m}(B_j;x) < \infty \quad \text{a.e.} x \in \mathbb{R}^n \tag{10}$$

and

$$\rho_{p(\cdot)}\left(\sum_{j=M+1}^{\infty} k_j N_{q,2m}(B_j;\cdot) \right) \to 0, \text{ as } M \to \infty. \tag{11}$$

From (10) and Lemma 6, there exists a function F such that $\sum_{j=1}^{\infty} k_j B_j = F$ in E_{2m-1}^q and

$$N_{q,2m}\left(\left(F - \sum_{j=1}^{M} k_j B_j \right); x \right) \leqslant c \sum_{j=M+1}^{\infty} k_j N_{q,2m}(B_j;x).$$

This estimate together with (11) and Lemma 8 implies

$$\left\| F - \sum_{j=1}^{M} k_j B_j \right\|_{\mathscr{H}^{p(\cdot)}_{q,2m}} \to 0, \text{ as } M \to \infty.$$

By proposition 9, we have that $F \in \mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$ and $F = \sum_{j=1}^{\infty} k_j B_j$ in $\mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$. Since Δ^m is a continuous operator from $\mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$ into $H^{p(\cdot)}(\mathbb{R}^n)$, we get

$$\Delta^m F = \sum_j k_j \Delta^m B_j = \sum_j k_j a_j = f,$$

in $H^{p(\cdot)}(\mathbb{R}^n)$. This shows that Δ^m is onto $H^{p(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|F\|_{\mathscr{H}^{p(\cdot)}_{q,2m}} = \left\| \sum_{j=1}^{\infty} k_j B_j \right\|_{\mathscr{H}^{p(\cdot)}_{q,2m}} \lesssim \left\| \sum_{j=1}^{\infty} k_j N_{q,2m}(B_j;\cdot) \right\|_{p(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}} = \|\Delta^m F\|_{H^{p(\cdot)}}.$$

For to conclude the proof, we will show that the operator Δ^m is injective on $\mathscr{H}^{p(\cdot)}_{q,2m}$. Let $\mathscr{O} = \{x : N_{q,2m}(F;x) > 1\}$. The set $\mathscr{O}$ is open because of that $N_{q,2m}(F;\cdot)$ is lower semicontinuous. We take a constant $r > 0$ such that $r \geqslant \{q, p_+\}$. Since $N_{q,2m}(F;\cdot) \in L^{p(\cdot)}(\mathbb{R}^n)$ it follows that $|\mathscr{O}|$ is finite and $N_{q,2m}(F;\cdot) \in L^r(\mathbb{R}^n \setminus \mathscr{O})$ thus $F \in \mathscr{N}^{r,q}_{2m}$, (for the definition of the space $\mathscr{N}^{r,q}_{2m}$ see p. 564 in [1]). Finally, the injectivity of the operator Δ^m it follows from Lemma 9 in [1]. $\square$

Proof of Theorem 2. Let $F \in \mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n)$ and assume $F \neq 0$. Then there exists $g \in F$ that is not a polynomial of degree less or equal to $2m-1$. It is easy to check that there exist a positive constant c and a cube $Q = Q(0,r)$ with $r > 1$ such that

$$\int_Q |g(y) - P(y)| dy \geqslant c > 0,$$

for every $P \in \mathscr{P}_{2m-1}$.

Let x be a point such that $|x| > \sqrt{n}r$ and let $\delta = 4|x|$. Then $Q(0,r) \subset Q(x,\delta)$. If $f \in F$, then $f = g - P$ for some $P \in \mathscr{P}_{2m-1}$ and

$$\delta^{-2m} |f|_{q,Q(x,\delta)} \geqslant c|x|^{-2m-n/q}.$$

So $N_{q,2m}(F;x) \geqslant c\,|x|^{-2m-n/q}$, for $|x| > \sqrt{n}r$. Since $p_+ \leqslant n(2m+n/q)^{-1}$, we have

$$\rho_{p(\cdot)}(N_{q,2m}(F;\cdot)) \geqslant c \int_{|x|>\sqrt{n}r} |x|^{-(2m+n/q)p_+} dx = \infty.$$

In view of Lemma 7, we get a contradiction. Thus $\mathscr{H}^{p(\cdot)}_{q,2m}(\mathbb{R}^n) = \{0\}$, if $p_+ \leqslant n(2m+n/q)^{-1}$. $\square$

REMARK 16. Theorem 1 does not hold in general for $\mathscr{H}^{p(\cdot)}_{q,\gamma}(\mathbb{R})$ when γ is not a natural number. In [10], S. Ombrosi gives an example which Theorem 1 is not true for $\mathscr{H}^{p}_{q,\gamma}(\mathbb{R})$ and the operator $\left(\frac{d}{dx}\right)^{\gamma}$, when $0 < \gamma < 1$ and $(\gamma + 1/q)p > 1$.

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