

WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED-TYPE SPACES TO ZYGMUND-TYPE SPACES

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Abstract. Some criteria for the boundedness and the compactness of weighted composition operators from weighted-type spaces into Zygmund-type spaces are given in this paper. Moreover, we give some estimates for the essential norm of these operators.

1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} and $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} . An analytic self-map φ of \mathbb{D} induces the composition operator C_φ , defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ , is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

There has been a great interest in studying the operator on various domains, see, e.g., [2, 3, 4, 6, 7, 8, 10, 12, 15, 16, 17, 18, 22, 26, 27, 28, 32] and the related references therein.

Let H^∞ be the space of bounded analytic functions. The Bloch space, denoted by \mathcal{B} , is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

See [35] for more information on the Bloch space.

Let $\alpha > 0$. The weighted-type space, denoted by H_α^∞ , is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

It is easy to check that H_α^∞ is a Banach space with the norm $\|\cdot\|_{H_\alpha^\infty}$. Composition operators, weighted composition operators and related concrete operators from or into

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weighted-type spaces and their generalizations have been studied a lot, see, for example, [3, 15, 17, 18, 19, 22, 23, 25, 29, 32, 36]. For the case of the upper half-plane, see, for example, [26, 27]. On these and related operators from or into the Bloch-type spaces, see, for example, [4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19, 20, 21, 22, 24, 25, 31, 32, 34, 36, 37] and the references therein.

We say that a function $v : \mathbb{D} \rightarrow \mathbb{R}_+$ is a weight, if v is a continuous, strictly positive and bounded function. The general weighted-type space, denoted by H_v^∞ , is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty.$$

H_v^∞ is a Banach space under the norm $\|\cdot\|_v$. The weight v is called radial if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The associated weight \tilde{v} of v is defined by

$$\tilde{v} = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\})^{-1}, z \in \mathbb{D}.$$

When $v = v_\alpha(z) = (1 - |z|^2)^\alpha$ ($0 < \alpha < \infty$), it is well-known that $\tilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$, which is in fact H_α^∞ .

For $0 < \beta < \infty$, the Zygmund-type space, denoted by \mathcal{Z}^β , is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}^\beta} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f''(z)| < \infty.$$

\mathcal{Z}^β is a Banach space with the above norm. When $\beta = 1$, $\mathcal{Z}^1 = \mathcal{Z}$ is the classical Zygmund space. See [1, 5] for more information on the Zygmund space on the unit disk. Composition operators, weighted composition operators and related operators on Zygmund-type spaces and their generalizations, including n -dimensional ones, were studied, for example, in [1, 2, 5, 9, 10, 11, 20, 25, 33].

Madigan and Matheson studied the compactness of the operator $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ in [13]. In [14], Montes-Rodriguez obtained the exact value for the essential norm of the operator $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$, i.e.,

$$\|C_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{B}} = \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)}.$$

Recall that the essential norm of a bounded linear operator $T : X \rightarrow Y$ is its distance to the set of compact operators K mapping X into Y , that is,

$$\|T\|_{e, X \rightarrow Y} = \inf \{\|T - K\|_{X \rightarrow Y} : K \text{ is compact}\},$$

where X, Y are Banach spaces and $\|\cdot\|_{X \rightarrow Y}$ is the operator norm. Tjani in [30] proved that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\lim_{|a| \rightarrow 1} \|C_\varphi(\frac{a-z}{1-\bar{a}z})\|_{\mathcal{B}} = 0$. Wulan, Zheng and Zhu in [31] showed that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\lim_{j \rightarrow \infty} \|\varphi^j\|_{\mathcal{B}} = 0$. In [34], Zhao showed that

$$\|C_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{B}} = \frac{e}{2} \limsup_{n \rightarrow \infty} \|\varphi^n\|_{\mathcal{B}}.$$

The boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ were studied in [16], while the essential norm of the operator $uC_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ was studied in [4, 12].

Motivated by the above statements, in this paper, we completely characterize the boundedness, compactness and the essential norm of the operator uC_φ from weighted-type spaces to Zygmund-type spaces. In particular, we use three families of functions and φ^j to characterize the operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need two lemmas as follows. The following lemma is well-known.

LEMMA 2.1. *Assume that $0 < \alpha < \infty$. Let n be a nonnegative integer and $f \in H_\alpha^\infty$. Then there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H_\alpha^\infty}}{(1-|z|^2)^{\alpha+n}}.$$

LEMMA 2.2. [4] *For $\alpha > 0$, we have $\lim_{k \rightarrow \infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

LEMMA 2.3. [15] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Then the weighted composition operator $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} |u(z)| < \infty.$$

LEMMA 2.4. [3] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Then the weighted composition operator $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded if and only if*

$$\sup_{k \geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty.$$

Now we are in a position to give various characterizations for the boundedness of the operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$. Some of the methods and ideas in the following result are closely related to those in [25], which was one of our motivations.

THEOREM 2.1. *Let $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} . Then the following statements are equivalent:*

- (a) *The operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded.*
 (b) $u \in \mathcal{Z}^\beta$, $Q := \sup_{z \in \mathbb{D}} (1-|z|^2)^\beta |u(z)| |\varphi'(z)|^2 < \infty$,

$$M := \sup_{z \in \mathbb{D}} (1-|z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty,$$

$$A := \max \left\{ \sup_{w \in \mathbb{D}} \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{Z}^\beta}, \sup_{w \in \mathbb{D}} \|uC_\varphi g_{\varphi(w)}\|_{\mathcal{Z}^\beta}, \sup_{w \in \mathbb{D}} \|uC_\varphi h_{\varphi(w)}\|_{\mathcal{Z}^\beta} \right\} < \infty;$$

where

$$f_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^{\alpha+1}}, \quad g_a(z) = \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{\alpha+2}}, \quad h_a(z) = \frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{\alpha+3}}, \quad a \in \mathbb{D}.$$

(c)

$$M_1 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty, \tag{1}$$

$$M_2 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty \tag{2}$$

and

$$M_3 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} < \infty. \tag{3}$$

(d)

$$\sup_{j \geq 1} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta} < \infty, \quad \sup_{j \geq 1} j^\alpha \|u''\varphi^{j-1}\|_{v_\beta} < \infty \tag{4}$$

and

$$\sup_{j \geq 1} j^{\alpha+2} \|u\varphi'^2\varphi^{j-1}\|_{v_\beta} < \infty. \tag{5}$$

Proof. (c) \Rightarrow (a). Suppose that (c) holds. For arbitrary z in \mathbb{D} and $f \in H_\alpha^\infty$, by Lemma 2.1 we have

$$\begin{aligned} & (1 - |z|^2)^\beta |(uC_\varphi f)''(z)| \\ & \leq (1 - |z|^2)^\beta |u''(z)||f(\varphi(z))| + (1 - |z|^2)^\beta |f''(\varphi(z))||u(z)(\varphi'(z))^2| \\ & \quad + (1 - |z|^2)^\beta |f'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ & \leq C \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} \|f\|_{H_\alpha^\infty} + C \frac{(1 - |z|^2)^\beta |u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} \|f\|_{H_\alpha^\infty} \\ & \quad + C \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}} \|f\|_{H_\alpha^\infty} \\ & \lesssim (M_1 + M_2 + M_3) \|f\|_{H_\alpha^\infty}. \end{aligned} \tag{6}$$

In addition, by Lemma 1.1,

$$|(uC_\varphi f)(0)| = |u(0)||f(\varphi(0))| \leq \frac{C|u(0)|}{(1 - |\varphi(0)|^2)^\alpha} \|f\|_{H_\alpha^\infty} \tag{7}$$

and

$$|(uC_\varphi f)'(0)| \leq \frac{C|u'(0)|}{(1-|\varphi(0)|^2)^\alpha} \|f\|_{H_\alpha^\infty} + \frac{C|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\alpha+1}} \|f\|_{H_\alpha^\infty}. \quad (8)$$

Taking the supremum in (6) over \mathbb{D} and then using the condition in (c) we see that $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded.

(a) \Rightarrow (b). Assume $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded. Taking the functions $1, z, z^2$ and using the boundedness of uC_φ and the fact that $|\varphi(z)| \leq 1$ we see that $u \in \mathcal{Z}^\beta$, $Q < \infty$ and $M < \infty$. For each $a \in \mathbb{D}$, it is easy to check that $f_a, g_a, h_a \in H_\alpha^\infty$. Moreover $\|f_a\|_{H_\alpha^\infty}$, $\|g_a\|_{H_\alpha^\infty}$ and $\|h_a\|_{H_\alpha^\infty}$ are bounded by $2^{\alpha+1}$, $2^{\alpha+2}$ and $2^{\alpha+3}$, respectively. By the boundedness of $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$, we get

$$\sup_{w \in \mathbb{D}} \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{Z}^\beta} \leq \|uC_\varphi\| \sup_{w \in \mathbb{D}} \|f_{\varphi(w)}\|_{H_\alpha^\infty} \leq 2^{\alpha+1} \|uC_\varphi\| < \infty,$$

$$\sup_{w \in \mathbb{D}} \|uC_\varphi g_{\varphi(w)}\|_{\mathcal{Z}^\beta} \leq \|uC_\varphi\| \sup_{w \in \mathbb{D}} \|g_{\varphi(w)}\|_{H_\alpha^\infty} \leq 2^{\alpha+2} \|uC_\varphi\| < \infty$$

and

$$\sup_{w \in \mathbb{D}} \|uC_\varphi h_{\varphi(w)}\|_{\mathcal{Z}^\beta} \leq \|uC_\varphi\| \sup_{w \in \mathbb{D}} \|h_{\varphi(w)}\|_{H_\alpha^\infty} \leq 2^{\alpha+3} \|uC_\varphi\| < \infty,$$

as desired.

(b) \Rightarrow (c). Suppose that $u \in \mathcal{Z}^\beta$, Q, M , and A are finite. A calculation shows that

$$f_a^{(n)}(a) = \prod_{j=1}^n (\alpha + j) \frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n}}, \quad g_a^{(n)}(a) = \prod_{j=2}^{n+1} (\alpha + j) \frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n}} \quad (9)$$

and

$$h_a^{(n)}(a) = \prod_{j=3}^{n+2} (\alpha + j) \frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n}}. \quad (10)$$

For the simplicity, we denoted $2u'(z)\varphi'(z) + u(z)\varphi''(z)$ by $v(z)$. From (9) and (10), for $w \in \mathbb{D}$, we have

$$\begin{aligned} (uC_\varphi f_{\varphi(w)})''(w) &= \frac{u''(w)}{(1-|\varphi(w)|^2)^\alpha} + (\alpha+1) \frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} \\ &\quad + (\alpha+1)(\alpha+2) \frac{u(w)(\varphi'(w))^2 \overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}}, \end{aligned} \quad (11)$$

$$\begin{aligned} (uC_\varphi g_{\varphi(w)})''(w) &= \frac{u''(w)}{(1-|\varphi(w)|^2)^\alpha} + (\alpha+2) \frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} \\ &\quad + (\alpha+2)(\alpha+3) \frac{u(w)(\varphi'(w))^2 \overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}} \end{aligned} \quad (12)$$

and

$$\begin{aligned}
 (uC_{\varphi}h_{\varphi(w)})''(w) &= \frac{u''(w)}{(1-|\varphi(w)|^2)^{\alpha}} + (\alpha+3)\frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} \\
 &\quad + (\alpha+3)(\alpha+4)\frac{u(w)(\varphi'(w))^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.
 \end{aligned}
 \tag{13}$$

From (11) and (12), we get

$$\begin{aligned}
 &-(uC_{\varphi}f_{\varphi(w)})''(w) + (uC_{\varphi}g_{\varphi(w)})''(w) \\
 &= \frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} + 2(\alpha+2)\frac{u(w)(\varphi'(w))^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.
 \end{aligned}
 \tag{14}$$

From (11) and (13), we obtain

$$\begin{aligned}
 &-(uC_{\varphi}f_{\varphi(w)})''(w) + (uC_{\varphi}h_{\varphi(w)})''(w) \\
 &= \frac{2v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} + \frac{(4\alpha+10)u(w)(\varphi'(w))^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.
 \end{aligned}
 \tag{15}$$

Multiplying (14) by 2, we get

$$\begin{aligned}
 &-2(uC_{\varphi}f_{\varphi(w)})''(w) + 2(uC_{\varphi}g_{\varphi(w)})''(w) \\
 &= \frac{2v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} + \frac{(4\alpha+8)u(w)(\varphi'(w))^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.
 \end{aligned}
 \tag{16}$$

Subtracting (16) from (15), we obtain

$$\begin{aligned}
 \frac{2u(w)(\varphi'(w))^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2+\alpha}} &= (uC_{\varphi}f_{\varphi(w)})''(w) - 2(uC_{\varphi}g_{\varphi(w)})''(w) \\
 &\quad + (uC_{\varphi}h_{\varphi(w)})''(w),
 \end{aligned}
 \tag{17}$$

which implies that

$$\begin{aligned}
 &\frac{(1-|w|^2)^{\beta}|u(w)(\varphi'(w))^2||\varphi(w)|^2}{(1-|\varphi(w)|^2)^{2+\alpha}} \\
 &\leq \frac{1}{2}(1-|w|^2)^{\beta}|(uC_{\varphi}f_{\varphi(w)})''(w)| + (1-|w|^2)^{\beta}|(uC_{\varphi}g_{\varphi(w)})''(w)| \\
 &\quad + \frac{1}{2}(1-|w|^2)^{\beta}|(uC_{\varphi}h_{\varphi(w)})''(w)| \\
 &\leq \frac{1}{2}\|uC_{\varphi}f_{\varphi(w)}\|_{\mathcal{X}^{\beta}} + \|uC_{\varphi}g_{\varphi(w)}\|_{\mathcal{X}^{\beta}} + \frac{1}{2}\|uC_{\varphi}h_{\varphi(w)}\|_{\mathcal{X}^{\beta}}
 \end{aligned}
 \tag{18}$$

$$\leq 2A.
 \tag{19}$$

From (15) and (17), we obtain

$$\frac{v(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1+\alpha}} = -(\alpha+3)(uC_{\varphi}f_{\varphi(w)})''(w) + (2\alpha+5)(uC_{\varphi}g_{\varphi(w)})''(w) \\ -(\alpha+2)(uC_{\varphi}h_{\varphi(w)})''(w), \quad (20)$$

which implies that

$$\frac{(1-|w|^2)^{\beta}|v(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^{1+\alpha}} \\ \leq (\alpha+3)(1-|w|^2)^{\beta}|(uC_{\varphi}f_{\varphi(w)})''(w)| + (2\alpha+5)(1-|w|^2)^{\beta}|(uC_{\varphi}g_{\varphi(w)})''(w)| \\ + (\alpha+2)(1-|w|^2)^{\beta}|(uC_{\varphi}h_{\varphi(w)})''(w)| \\ \leq (\alpha+3)\|uC_{\varphi}f_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} + (2\alpha+5)\|uC_{\varphi}g_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} + (\alpha+2)\|uC_{\varphi}h_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} \quad (21) \\ \leq (4\alpha+10)A. \quad (22)$$

By (11), (17) and (20), we have

$$\frac{u''(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{\alpha}} = \frac{2+(\alpha+1)(\alpha+4)}{2}(uC_{\varphi}f_{\varphi(w)})''(w) \\ -(\alpha+1)(\alpha+3)(uC_{\varphi}g_{\varphi(w)})''(w) \\ + \frac{(\alpha+1)(\alpha+2)}{2}(uC_{\varphi}h_{\varphi(w)})''(w), \quad (23)$$

which implies that

$$\frac{(1-|w|^2)^{\beta}|u''(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^{\alpha}} \\ \leq \frac{2+(\alpha+1)(\alpha+4)}{2}(1-|w|^2)^{\beta}|(uC_{\varphi}f_{\varphi(w)})''(w)| \\ + (\alpha+1)(\alpha+3)(1-|w|^2)^{\beta}|(uC_{\varphi}g_{\varphi(w)})''(w)| \\ + \frac{(\alpha+1)(\alpha+2)}{2}(1-|w|^2)^{\beta}|(uC_{\varphi}h_{\varphi(w)})''(w)| \\ \leq \frac{2+(\alpha+1)(\alpha+4)}{2}\|uC_{\varphi}f_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} + (\alpha+1)(\alpha+3)\|uC_{\varphi}g_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} \\ + \frac{(\alpha+1)(\alpha+2)}{2}\|uC_{\varphi}h_{\varphi(w)}\|_{\mathcal{Z}^{\beta}} \quad (24) \\ \leq [1+2(\alpha+1)(\alpha+3)]A. \quad (25)$$

Fix $r \in (0, 1)$. If $|\varphi(w)| > r$, then from (25) we obtain

$$\frac{(1-|w|^2)^{\beta}|u''(w)|}{(1-|\varphi(w)|^2)^{\alpha}} \leq \frac{1}{r}[1+2(\alpha+1)(\alpha+3)]A < \infty. \quad (26)$$

On the other hand, if $|\varphi(w)| \leq r$, we get

$$\frac{(1 - |w|^2)^\beta |u''(w)|}{(1 - |\varphi(w)|^2)^\alpha} \leq \frac{1}{(1 - r^2)^\alpha} \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u''(z)| < \infty. \tag{27}$$

From (26) and (27) we see that M_1 is finite. Using similar arguments, by (19) and (22) we can obtain that M_2 and M_3 are finite as well.

(d) \Leftrightarrow (a). We have proved that $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded if and only if (c) holds. By Lemma 2.3, $M_2 < \infty$ is equivalent to the weighted composition operator $(2u'\varphi' + u\varphi'')C_\varphi : H_{\nu_{\alpha+1}}^\infty \rightarrow H_{\nu_\beta}^\infty$ is bounded. By Lemma 2.4, this is equivalent to

$$\sup_{j \geq 1} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{\nu_\beta}}{\|z^{j-1}\|_{\nu_{\alpha+1}}} < \infty.$$

$M_1 < \infty$ is equivalent to the operator $u''C_\varphi : H_{\nu_\alpha}^\infty \rightarrow H_{\nu_\beta}^\infty$ is bounded. By Lemma 2.4, this is equivalent to

$$\sup_{j \geq 1} \frac{\|u''\varphi^{j-1}\|_{\nu_\beta}}{\|z^{j-1}\|_{\nu_\alpha}} < \infty.$$

$M_3 < \infty$ is equivalent to the operator $u\varphi'^2C_\varphi : H_{\nu_{\alpha+2}}^\infty \rightarrow H_{\nu_\beta}^\infty$ is bounded. By Lemma 2.4, this is equivalent to

$$\sup_{j \geq 1} \frac{\|u\varphi'^2\varphi^{j-1}\|_{\nu_\beta}}{\|z^{j-1}\|_{\nu_{\alpha+2}}} < \infty.$$

By Lemma 2.2, we see that $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded if and only if

$$\sup_{j \geq 1} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{\nu_\beta} \approx \sup_{j \geq 1} \frac{j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{\nu_\beta}}{j^{\alpha+1} \|z^{j-1}\|_{\nu_{\alpha+1}}} < \infty,$$

$$\sup_{j \geq 1} j^\alpha \|u''\varphi^{j-1}\|_{\nu_\beta} \approx \sup_{j \geq 1} \frac{j^\alpha \|u''\varphi^{j-1}\|_{\nu_\beta}}{j^\alpha \|z^{j-1}\|_{\nu_\alpha}} < \infty$$

and

$$\sup_{j \geq 1} j^{\alpha+2} \|u\varphi'^2\varphi^{j-1}\|_{\nu_\beta} \approx \sup_{j \geq 1} \frac{j^{\alpha+2} \|u\varphi'^2\varphi^{j-1}\|_{\nu_\beta}}{j^{\alpha+2} \|z^{j-1}\|_{\nu_{\alpha+2}}} < \infty.$$

The proof is completed. \square

3. Essential norm of $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$

In this section, we give some estimates of the essential norm for the operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$. Hence, we first state some lemmas which will be used in the proofs of the main results in this section.

LEMMA 3.1. [30] *Let X, Y be two Banach spaces of analytic functions on \mathbb{D} . Suppose that*

- (1) *The point evaluation functionals on Y are continuous.*
- (2) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
- (3) *$T : X \rightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \rightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y .

LEMMA 3.2. [15] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Suppose $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Then*

$$\|uC_\varphi\|_{e, H_v^\infty \rightarrow H_w^\infty} = \lim_{s \rightarrow 1^-} \sup_{|\varphi(z)| > s} \frac{w(z)}{\bar{v}(\varphi(z))} |u(z)|.$$

LEMMA 3.3. [3] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Suppose $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Then*

$$\|uC_\varphi\|_{e, H_v^\infty \rightarrow H_w^\infty} = \limsup_{k \rightarrow \infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

THEOREM 3.1. *Let $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} such that $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded. Then*

$$\|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \approx \max\{A, B, C\} \approx \max\{E, F, G\},$$

where

$$A := \limsup_{|a| \rightarrow 1} \left\| uC_\varphi \left(\frac{1 - |a|^2}{(1 - \bar{a}z)^{\alpha+1}} \right) \right\|_{\mathcal{Z}^\beta}, \quad B := \limsup_{|a| \rightarrow 1} \left\| uC_\varphi \left(\frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{\alpha+2}} \right) \right\|_{\mathcal{Z}^\beta},$$

$$C := \limsup_{|a| \rightarrow 1} \left\| uC_\varphi \left(\frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{\alpha+3}} \right) \right\|_{\mathcal{Z}^\beta}, \quad F := \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha},$$

$$E := \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}}$$

and

$$G := \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}}.$$

Proof. When $\|\varphi\|_\infty < 1$. It is easy to see that $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is compact by using Lemma 3.1. In this case, the asymptotic relations vacuously holds.

Now we consider the case $\|\varphi\|_\infty = 1$. First we prove that

$$\max \{A, B, C\} \leq \|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta}.$$

Let $a \in \mathbb{D}$. From the proof of Theorem 2.1, we see that $f_a, g_a, h_a \in H_\alpha^\infty$ and f_a, g_a, h_a converges to 0 uniformly on compact subsets of \mathbb{D} . Thus, for any compact operator $K : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$, by Lemma 3.1 we have

$$\lim_{|a| \rightarrow 1} \|Kf_a\|_{\mathcal{Z}^\beta} = 0, \quad \lim_{|a| \rightarrow 1} \|Kg_a\|_{\mathcal{Z}^\beta} = 0, \quad \lim_{|a| \rightarrow 1} \|Kh_a\|_{\mathcal{Z}^\beta} = 0.$$

Hence

$$\begin{aligned} \|uC_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{|a| \rightarrow 1} \|(uC_\varphi - K)f_a\|_{\mathcal{Z}^\beta} \\ &\geq \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\beta} - \limsup_{|a| \rightarrow 1} \|Kf_a\|_{\mathcal{Z}^\beta} = A, \end{aligned}$$

$$\begin{aligned} \|uC_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{|a| \rightarrow 1} \|(uC_\varphi - K)g_a\|_{\mathcal{Z}^\beta} \\ &\geq \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{Z}^\beta} - \limsup_{|a| \rightarrow 1} \|Kg_a\|_{\mathcal{Z}^\beta} = B \end{aligned}$$

and

$$\begin{aligned} \|uC_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{|a| \rightarrow 1} \|(uC_\varphi - K)h_a\|_{\mathcal{Z}^\beta} \\ &\geq \limsup_{|a| \rightarrow 1} \|uC_\varphi h_a\|_{\mathcal{Z}^\beta} - \limsup_{|a| \rightarrow 1} \|Kh_a\|_{\mathcal{Z}^\beta} = C. \end{aligned}$$

Therefore, from the definition of the essential norm, we obtain

$$\|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} = \inf_K \|uC_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \gtrsim \max \{A, B, C\}.$$

Next, let $\{z_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{D} such that $|\varphi(z_j)| \rightarrow 1$ as $j \rightarrow \infty$. Define

$$\begin{aligned} k_j(z) &= \frac{1 - |\varphi(z_j)|^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+1}} - \frac{2\alpha + 5}{\alpha + 3} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+2}} \\ &\quad + \frac{\alpha + 2}{\alpha + 3} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \overline{\varphi(z_j)}z)^{\alpha+3}}, \end{aligned}$$

$$\begin{aligned} p_j(z) &= \frac{1 - |\varphi(z_j)|^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+1}} - \frac{2(\alpha + 1)(\alpha + 3)}{2 + (\alpha + 1)(\alpha + 4)} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+2}} \\ &\quad + \frac{(\alpha + 2)(\alpha + 1)}{2 + (\alpha + 1)(\alpha + 4)} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \overline{\varphi(z_j)}z)^{\alpha+3}} \end{aligned}$$

and

$$q_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+1}} - 2 \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \overline{\varphi(z_j)}z)^{\alpha+2}} + \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \overline{\varphi(z_j)}z)^{\alpha+3}}.$$

Similarly, all k_j, p_j and q_j belong to H_α^∞ and converges to 0 uniformly on compact subsets of \mathbb{D} . Moreover,

$$k_j(\varphi(z_j)) = 0, \quad k_j''(\varphi(z_j)) = 0, \quad |k_j'(\varphi(z_j))| = \frac{1}{\alpha+3} \frac{|\varphi(z_j)|}{(1 - |\varphi(z_j)|^2)^{\alpha+1}},$$

$$p_j'(\varphi(z_j)) = 0, \quad p_j''(\varphi(z_j)) = 0, \quad |p_j(\varphi(z_j))| = \frac{2}{2 + (\alpha+1)(\alpha+4)} \frac{1}{(1 - |\varphi(z_j)|^2)^\alpha},$$

$$q_j(\varphi(z_j)) = 0, \quad q_j'(\varphi(z_j)) = 0, \quad |q_j''(\varphi(z_j))| = \frac{2|\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^{\alpha+2}}.$$

Then for any compact operator $K : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$, by Lemma 3.1 we obtain

$$\begin{aligned} \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi(k_j)\|_{\mathcal{Z}^\beta} - \limsup_{j \rightarrow \infty} \|K(k_j)\|_{\mathcal{Z}^\beta} \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)| |\varphi(z_j)|}{(1 - |\varphi(z_j)|^2)^{1+\alpha}}, \end{aligned}$$

$$\begin{aligned} \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi(p_j)\|_{\mathcal{Z}^\beta} - \limsup_{j \rightarrow \infty} \|K(p_j)\|_{\mathcal{Z}^\beta} \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |u''(z_j)|}{(1 - |\varphi(z_j)|^2)^\alpha} \end{aligned}$$

and

$$\begin{aligned} \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim \limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi(q_j)\|_{\mathcal{Z}^\beta} - \limsup_{j \rightarrow \infty} \|K(q_j)\|_{\mathcal{Z}^\beta} \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |u(z_j)| |\varphi'(z_j)|^2 |\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^{\alpha+2}}. \end{aligned}$$

From the definition of the essential norm, we obtain

$$\begin{aligned} \|u\mathcal{C}_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &= \inf_K \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)| |\varphi(z_j)|}{(1 - |\varphi(z_j)|^2)^{1+\alpha}} \\ &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\alpha}} = E, \end{aligned}$$

$$\begin{aligned} \|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &= \inf_K \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |u''(z_j)|}{(1 - |\varphi(z_j)|^2)^\alpha} \\ &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} = F \end{aligned}$$

and

$$\begin{aligned} \|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &= \inf_K \|u\mathcal{C}_\varphi - K\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{(1 - |z_j|^2)^\beta |u(z_j)| |\varphi'(z_j)|^2 |\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^{\alpha+2}} \\ &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} = G. \end{aligned}$$

Hence

$$\|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \gtrsim \max \{E, F, G\}.$$

Finally, we prove that

$$\|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{A, B, C\} \quad \text{and} \quad \|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{E, F, G\}.$$

For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ by $(K_r f)(z) = f_r(z) = f(rz)$, $f \in H(\mathbb{D})$. It is obvious that $f_r \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $r \rightarrow 1$. Moreover, the operator K_r is compact on H_α^∞ and $\|K_r\|_{H_\alpha^\infty \rightarrow H_\alpha^\infty} \leq 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \rightarrow 1$ as $j \rightarrow \infty$. Then for every positive integer j , the operator $u\mathcal{C}_\varphi K_{r_j} : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is compact. By the definition of the essential norm, we get

$$\|u\mathcal{C}_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \leq \limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi - u\mathcal{C}_\varphi K_{r_j}\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta}. \tag{28}$$

Therefore, we only need to prove that

$$\limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi - u\mathcal{C}_\varphi K_{r_j}\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{A, B, C\}$$

and

$$\limsup_{j \rightarrow \infty} \|u\mathcal{C}_\varphi - u\mathcal{C}_\varphi K_{r_j}\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{E, F, G\}.$$

For any $f \in H_\alpha^\infty$ such that $\|f\|_{H_\alpha^\infty} \leq 1$, we consider

$$\begin{aligned} &\|(u\mathcal{C}_\varphi - u\mathcal{C}_\varphi K_{r_j})f\|_{\mathcal{Z}^\beta} \\ &= |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| \\ &\quad + |u'(0)(f - f_{r_j})(\varphi(0)) + u(0)(f - f_{r_j})'(\varphi(0))\varphi'(0)| \\ &\quad + \|u(f - f_{r_j}) \circ \varphi\|_{**}, \end{aligned} \tag{29}$$

here $\|g\|_{**} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g''(z)|$.

It is obvious that

$$\lim_{j \rightarrow \infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0 \tag{30}$$

and

$$\lim_{j \rightarrow \infty} |u'(0)(f - f_{r_j})(\varphi(0)) + u(0)(f - f_{r_j})'(\varphi(0))\varphi'(0)| = 0. \tag{31}$$

Now, we consider

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \|u \cdot (f - f_{r_j}) \circ \varphi\|_{**} \\ & \leq \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})(\varphi(z))| |u''(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})(\varphi(z))| |u''(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})''(\varphi(z))| |\varphi'(z)|^2 |u(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})''(\varphi(z))| |\varphi'(z)|^2 |u(z)| \\ & = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6, \end{aligned} \tag{32}$$

where $N \in \mathbb{N}$ is large enough such that $r_j \geq \frac{1}{2}$ for all $j \geq N$,

$$Q_1 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_2 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_3 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})(\varphi(z))| |u''(z)|,$$

$$Q_4 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})(\varphi(z))| |u''(z)|,$$

$$Q_5 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2)^\beta |(f - f_{r_j})''(\varphi(z))| |\varphi'(z)|^2 |u(z)|$$

and

$$Q_6 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |(f - f_{r_j})''(\varphi(z))| |\varphi'(z)|^2 |u(z)|.$$

Since $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{F}^\beta$ is bounded, applying the operator uC_φ to $1, z$ and z^2 , we obtain $(uC_\varphi 1)''(z) = u''(z)$, $(uC_\varphi z)''(z) = u''(z)\varphi(z) + 2u'(z)\varphi'(z) + u(z)\varphi''(z)$ and

$$(uC_\varphi z^2)''(z) = u''(z)\varphi^2(z) + 4u'(z)\varphi(z)\varphi'(z) + 2u(z)\varphi(z)\varphi''(z) + 2u(z)\varphi'^2(z).$$

Thus $u \in \mathcal{F}^\beta$. Using the boundedness of φ , we also get

$$\tilde{J}_1 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty$$

and $\tilde{J}_2 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'(z)|^2 |u(z)| < \infty$. Since $r_j f'_{r_j} \rightarrow f'$, as well as $r_j^2 f''_{r_j} \rightarrow f''$ uniformly on compact subsets of \mathbb{D} as $j \rightarrow \infty$, we have

$$Q_1 \leq \tilde{J}_1 \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f'(w) - r_j f'(r_j w)| = 0 \quad (33)$$

and

$$Q_5 \leq \tilde{J}_2 \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f''(w) - r_j^2 f''(r_j w)| = 0. \quad (34)$$

Similarly, from the fact that $u \in \mathcal{F}^\beta$ we have

$$Q_3 \leq \|u\|_{\mathcal{F}^\beta} \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f(w) - f(r_j w)| = 0. \quad (35)$$

Next we consider Q_2 . We have $Q_2 \leq \limsup_{j \rightarrow \infty} (S_1 + S_2)$, where

$$S_1 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

and

$$S_2 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta r_j |f'(r_j \varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)|.$$

First we estimate S_1 . Using the fact that $\|f\|_{H_\alpha^\infty} \leq 1$ and Lemma 2.1, we have

$$\begin{aligned} S_1 &= \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f'(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\ &\quad \times \frac{(1 - |\varphi(z)|^2)^{\alpha+1} (\alpha + 3)}{|\varphi(z)|} \frac{|\varphi(z)|}{(\alpha + 3)(1 - |\varphi(z)|^2)^{\alpha+1}} \\ &\lesssim \frac{\|f\|_{H_\alpha^\infty}}{r_N} \sup_{|\varphi(z)| > r_N} \frac{|2u'(z)\varphi'(z) + u(z)\varphi''(z)| (1 - |z|^2)^\beta |\varphi(z)|}{(\alpha + 3)(1 - |\varphi(z)|^2)^{\alpha+1}} \quad (36) \\ &\lesssim \sup_{|\varphi(z)| > r_N} \frac{|2u'(z)\varphi'(z) + u(z)\varphi''(z)| (1 - |z|^2)^\beta |\varphi(z)|}{(\alpha + 3)(1 - |\varphi(z)|^2)^{\alpha+1}} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a - \frac{2\alpha + 5}{\alpha + 3}g_a + \frac{\alpha + 2}{\alpha + 3}h_a)\|_{\mathcal{F}^\beta} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a)\|_{\mathcal{F}^\beta} + \frac{(2\alpha + 5)}{(\alpha + 3)} \sup_{|a| > r_N} \|uC_\varphi(g_a)\|_{\mathcal{F}^\beta} \\ &\quad + \frac{\alpha + 2}{(\alpha + 3)} \sup_{|a| > r_N} \|uC_\varphi(h_a)\|_{\mathcal{F}^\beta}. \end{aligned}$$

Taking limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_1 &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi(f_a)\|_{\mathcal{Z}^\beta} + \limsup_{|a| \rightarrow 1} \|uC_\varphi(g_a)\|_{\mathcal{Z}^\beta} \\ &\quad + \limsup_{|a| \rightarrow 1} \|D_{\varphi,u}^n(h_a)\|_{\mathcal{Z}^\beta} \\ &= A + B + C. \end{aligned}$$

Similarly, we have $\limsup_{j \rightarrow \infty} S_2 \lesssim A + B + C$, i.e., we get that

$$Q_2 \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{37}$$

From (36), we see that

$$\limsup_{j \rightarrow \infty} S_1 \lesssim \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\alpha}} = E.$$

Similarly we have $\limsup_{j \rightarrow \infty} S_2 \lesssim E$. Therefore

$$Q_2 \lesssim E. \tag{38}$$

Next we consider Q_4 . We have $Q_4 \leq \limsup_{j \rightarrow \infty} (S_3 + S_4)$, where

$$S_3 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f(\varphi(z))| |u''(z)|$$

and

$$S_4 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f(r_j \varphi(z))| |u''(z)|.$$

After a calculation, we have

$$\begin{aligned} S_3 &= \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f(\varphi(z))| |u''(z)| \frac{(1 - |\varphi(z)|^2)^\alpha (2 + (\alpha + 1)(\alpha + 4))}{2} \\ &\quad \times \frac{2}{2 + (\alpha + 1)(\alpha + 4)} \frac{1}{(1 - |\varphi(z)|^2)^\alpha} \\ &\lesssim \|f\|_{H_\alpha^\infty} \sup_{|\varphi(z)| > r_N} \frac{2}{2 + (\alpha + 1)(\alpha + 4)} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} \\ &\lesssim \sup_{|\varphi(z)| > r_N} \frac{2}{2 + (\alpha + 1)(\alpha + 4)} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a)\|_{\mathcal{Z}^\beta} + \frac{2(\alpha + 1)(\alpha + 3)}{2 + (\alpha + 1)(\alpha + 4)} \sup_{|a| > r_N} \|uC_\varphi(g_a)\|_{\mathcal{Z}^\beta} \\ &\quad + \frac{(\alpha + 1)(\alpha + 2)}{2 + (\alpha + 1)(\alpha + 4)} \sup_{|a| > r_N} \|uC_\varphi(h_a)\|_{\mathcal{Z}^\beta} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a)\|_{\mathcal{Z}^\beta} + \sup_{|a| > r_N} \|uC_\varphi(g_a)\|_{\mathcal{Z}^\beta} + \sup_{|a| > r_N} \|uC_\varphi(h_a)\|_{\mathcal{Z}^\beta}. \end{aligned} \tag{39}$$

Taking limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_3 &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi(f_a)\|_{\mathcal{U}^\beta} + \limsup_{|a| \rightarrow 1} \|uC_\varphi(g_a)\|_{\mathcal{U}^\beta} \\ &\quad + \limsup_{|a| \rightarrow 1} \|uC_\varphi(h_a)\|_{\mathcal{U}^\beta} \\ &= A + B + C. \end{aligned}$$

Similarly, we have $\limsup_{j \rightarrow \infty} S_4 \lesssim A + B + C$, i.e., we get that

$$Q_4 \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{40}$$

From (39), we see that

$$\limsup_{j \rightarrow \infty} S_3 \lesssim \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} = F.$$

Similarly we have that $\limsup_{j \rightarrow \infty} S_4 \lesssim F$. Therefore

$$Q_4 \lesssim F. \tag{41}$$

Finally we consider Q_6 . We have $Q_6 \leq \limsup_{j \rightarrow \infty} (S_5 + S_6)$, where

$$S_5 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |f''(\varphi(z))| |\varphi'(z)|^2 |u(z)|$$

and

$$S_6 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta r_j^2 |f''(r_j \varphi(z))| |\varphi'(z)|^2 |u(z)|.$$

After a calculation, we have

$$\begin{aligned} S_5 &\lesssim \|f\|_{H_\alpha^\infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |\varphi'(z)|^2 |u(z)| \frac{2|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} \\ &\lesssim \sup_{|\varphi(z)| > r_N} (1 - |z|^2)^\beta |\varphi'(z)|^2 |u(z)| \frac{2|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a - 2g_a + h_a)\|_{\mathcal{U}^\beta} \\ &\lesssim \sup_{|a| > r_N} \left(\|uC_\varphi(f_a)\|_{\mathcal{U}^\beta} + \|uC_\varphi(g_a)\|_{\mathcal{U}^\beta} + \|uC_\varphi(h_a)\|_{\mathcal{U}^\beta} \right). \end{aligned} \tag{42}$$

Taking limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_5 &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi(f_a)\|_{\mathcal{U}^\beta} + \limsup_{|a| \rightarrow 1} \|uC_\varphi(g_a)\|_{\mathcal{U}^\beta} \\ &\quad + \limsup_{|a| \rightarrow 1} \|uC_\varphi(h_a)\|_{\mathcal{U}^\beta} \\ &= A + B + C. \end{aligned}$$

Similarly, we have $\limsup_{j \rightarrow \infty} S_6 \lesssim A + B + C$, i.e., we get that

$$Q_6 \lesssim A + B + C \lesssim \max \{A, B, C\}. \tag{43}$$

From (42), we see that

$$\limsup_{j \rightarrow \infty} S_5 \lesssim \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|^2 |u(z)|}{(1 - |\varphi(z)|^2)^{\alpha+2}} = G.$$

Similarly we have that $\limsup_{j \rightarrow \infty} S_6 \lesssim G$. Therefore

$$Q_6 \lesssim G. \tag{44}$$

Hence, by (29), (30), (31), (32), (33), (34), (35), (37), (40) and (43) we get

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \|u C_\varphi - u C_\varphi K_{r_j}\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \\ &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{H_\alpha^\infty} \leq 1} \|(u C_\varphi - u C_\varphi K_{r_j})f\|_{\mathcal{Z}^\beta} \\ &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{H_\alpha^\infty} \leq 1} \|u \cdot (f - f_{r_j}) \circ \varphi\|_{**} \lesssim \max \{A, B, C\}. \end{aligned} \tag{45}$$

Similarly, by (29), (30), (31), (32), (33), (34), (35), (38), (41) and (44) we get

$$\limsup_{j \rightarrow \infty} \|u C_\varphi - u C_\varphi K_{r_j}\|_{H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{E, F, G\}. \tag{46}$$

Therefore, by (28), (45) and (46), we obtain

$$\|u C_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{A, B, C\} \quad \text{and} \quad \|u C_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \lesssim \max \{E, F, G\}.$$

This completes the proof of Theorem 3.1. \square

Next, we give another characterization for the essential norm of weighted composition operator $u C_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$.

THEOREM 3.2. *Let $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} such that $u C_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded. Then*

$$\|u C_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \approx \max \{N_1, N_2, N_3\},$$

where

$$\begin{aligned} N_1 &:= \limsup_{j \rightarrow \infty} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}, \\ N_2 &:= \limsup_{j \rightarrow \infty} j^\alpha \|u''\varphi^{j-1}\|_{v_\beta}, \quad N_3 := \limsup_{j \rightarrow \infty} j^{\alpha+2} \|u(\varphi')^2\varphi^{j-1}\|_{v_\beta}. \end{aligned}$$

Proof. From Theorem 2.1 we know that the boundedness of $u C_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is equivalent to the boundedness of the operators $(2u'\varphi' + u\varphi'')C_\varphi : H_{v_{\alpha+1}}^\infty \rightarrow H_{v_\beta}^\infty$, $u''C_\varphi : H_{v_\alpha}^\infty \rightarrow H_{v_\beta}^\infty$ and $u\varphi'^2C_\varphi : H_{v_{\alpha+2}}^\infty \rightarrow H_{v_\beta}^\infty$.

The upper estimate. By Lemmas 2.2, 3.2 and 3.3, we get

$$\begin{aligned} \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{v_{\alpha+1}}^\infty \rightarrow H_{v_\beta}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_{\alpha+1}}} \\ &= \limsup_{j \rightarrow \infty} \frac{j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}}{j^{\alpha+1} \|z^{j-1}\|_{v_{\alpha+1}}} \\ &\approx \limsup_{j \rightarrow \infty} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}, \end{aligned}$$

$$\begin{aligned} \|u''C_\varphi\|_{e, H_{v_\alpha}^\infty \rightarrow H_{v_\beta}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u''\varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_\alpha}} = \limsup_{j \rightarrow \infty} \frac{j^\alpha \|u''\varphi^{j-1}\|_{v_\beta}}{j^\alpha \|z^{j-1}\|_{v_\alpha}} \\ &\approx \limsup_{j \rightarrow \infty} j^\alpha \|u''\varphi^{j-1}\|_{v_\beta} \end{aligned}$$

and

$$\begin{aligned} \|u\varphi'^2 C_\varphi\|_{e, H_{v_{\alpha+2}}^\infty \rightarrow H_{v_\beta}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u\varphi'^2 \varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_{\alpha+2}}} = \limsup_{j \rightarrow \infty} \frac{j^{\alpha+2} \|u\varphi'^2 \varphi^{j-1}\|_{v_\beta}}{j^{\alpha+2} \|z^{j-1}\|_{v_{\alpha+2}}} \\ &\approx \limsup_{j \rightarrow \infty} j^{\alpha+2} \|u\varphi'^2 \varphi^{j-1}\|_{v_\beta}. \end{aligned}$$

It follows that

$$\begin{aligned} \|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{L}^\beta} &\lesssim \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{v_{\alpha+2}}^\infty \rightarrow H_{v_\beta}^\infty} + \|u''C_\varphi\|_{e, H_{v_\alpha}^\infty \rightarrow H_{v_\beta}^\infty} \\ &\quad + \|u\varphi'^2 C_\varphi\|_{e, H_{v_{\alpha+2}}^\infty \rightarrow H_{v_\beta}^\infty} \\ &\lesssim \max\{N_1, N_2, N_3\}. \end{aligned}$$

The lower estimate. From Theorem 2.1, Lemmas 2.2, 3.2 and 3.3, we have

$$\begin{aligned} \|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{L}^\beta} &\gtrsim E = \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{v_{\alpha+1}}^\infty \rightarrow H_{v_\beta}^\infty} \\ &= \limsup_{j \rightarrow \infty} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_{\alpha+1}}} \\ &\approx \limsup_{j \rightarrow \infty} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta}, \end{aligned}$$

$$\begin{aligned} \|uC_\varphi\|_{e, H_\alpha^\infty \rightarrow \mathcal{L}^\beta} &\gtrsim F = \|u''C_\varphi\|_{e, H_{v_\alpha}^\infty \rightarrow H_{v_\beta}^\infty} = \limsup_{j \rightarrow \infty} \frac{\|u''\varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_\alpha}} \\ &\approx \limsup_{j \rightarrow \infty} j^\alpha \|u''\varphi^{j-1}\|_{v_\beta} \end{aligned}$$

and

$$\begin{aligned} \|uC_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} &\gtrsim G = \|u\varphi'^2 C_\varphi\|_{e,H_{\alpha+2}^\infty \rightarrow H_\beta^\infty} = \limsup_{j \rightarrow \infty} \frac{\|u\varphi'^2 \varphi^{j-1}\|_{v_\beta}}{\|z^{j-1}\|_{v_{\alpha+2}}} \\ &\approx \limsup_{j \rightarrow \infty} j^{\alpha+2} \|u\varphi'^2 \varphi^{j-1}\|_{v_\beta}. \end{aligned}$$

Therefore

$$\|uC_\varphi\|_{e,H_\alpha^\infty \rightarrow \mathcal{Z}^\beta} \gtrsim \max \{N_1, N_2, N_3\}.$$

This completes the proof of this theorem. \square

From Theorems 3.1 and 3.2, we immediately get the following characterization for the compactness of $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$.

COROLLARY 3.1. *Let $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} such that the operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is bounded, then the following conditions are equivalent:*

- (a) *The operator $uC_\varphi : H_\alpha^\infty \rightarrow \mathcal{Z}^\beta$ is compact.*
 (b)

$$\limsup_{|\varphi(w)| \rightarrow 1} \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{Z}^\beta} = \limsup_{|\varphi(w)| \rightarrow 1} \|uC_\varphi g_{\varphi(w)}\|_{\mathcal{Z}^\beta} = \limsup_{|\varphi(w)| \rightarrow 1} \|uC_\varphi h_{\varphi(w)}\|_{\mathcal{Z}^\beta} = 0.$$

- (c)

$$\begin{aligned} \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^\alpha} &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}} \\ &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}} = 0. \end{aligned}$$

- (d)

$$\limsup_{j \rightarrow \infty} j^{\alpha+1} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\beta} = 0, \quad \limsup_{j \rightarrow \infty} j^\alpha \|u''\varphi^{j-1}\|_{v_\beta} = 0$$

and

$$\limsup_{j \rightarrow \infty} j^{\alpha+2} \|u(\varphi')^2 \varphi^{j-1}\|_{v_\beta} = 0.$$

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