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CHARACTERIZATION OF THE HARDY PROPERTY OF MEANS AND THE BEST HARDY CONSTANTS

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Dedicated to the 60th birthdays of Professors
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Abstract. The aim of this paper is to characterize in broad classes of means the so-called Hardy means, i.e., those means $M\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n \to \mathbb{R}_+$ that satisfy the inequality

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant C\sum_{n=1}^{\infty} x_n$$

for all positive sequences (x_n) with some finite positive constant C. One of the main results offers a characterization of Hardy means in the class of symmetric, increasing, Jensen concave and repetition invariant means and also a formula for the best constant C satisfying the above inequality.

1. Introduction

Hardy's celebrated inequality (cf. [23], [24]) states that, for $p>1$,

$$\sum_{n=1}^{\infty}\left(\frac{x_1+\cdots+x_n}{n}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} x_n^p, \tag{1.1}$$

for all nonnegative sequences (x_n).

This inequality, in integral form was stated and proved in [23] but it was also pointed out that this discrete form follows from the integral version. Hardy's original motivation was to get a simple proof of Hilbert's celebrated inequality. About the enormous literature concerning the history, generalizations and extensions of this inequality, we recommend four recent books [30], [31], [44], and [46] for the interested readers.

In this paper, we follow the approach in generalizing Hardy's inequality of the paper [62]. The main idea is to rewrite (1.1) in terms of means.

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First, replacing x_n by $x_n^{1/p}$ and p by $1/p$, we get that

$$\sum_{n=1}^{\infty}\left(\frac{x_1^p+\cdots+x_n^p}{n}\right)^{1/p} \leqslant \left(\frac{1}{1-p}\right)^{1/p}\sum_{n=1}^{\infty} x_n \tag{1.2}$$

for $0<p<1$. This inequality was also established for $p<0$ by Knopp [28]. Taking the limit $p\to 0$, the so-called Carleman inequality (cf. [10]) can also be derived:

$$\sum_{n=1}^{\infty}\sqrt[n]{x_1\cdots x_n} \leqslant e\sum_{n=1}^{\infty} x_n. \tag{1.3}$$

It is also important to note that the constants of the right hand sides of the above inequalities are the smallest possible. For further developments and historical remarks concerning inequality (1.3), we refer to the paper Pečarić–Stolarsky [49].

Now define for $p\in\mathbb{R}$ the pth power (or Hölder) mean of the positive numbers $x_1,\dots,x_n$ by

$$\mathcal{P}_p(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p+\cdots+x_n^p}{n}\right)^{\frac{1}{p}} & \text{if } p\neq 0,\\ \sqrt[n]{x_1\cdots x_n} & \text{if } p=0. \end{cases} \tag{1.4}$$

The power mean $\mathcal{P}_1$ is the arithmetic mean which will also be denoted by $\mathcal{A}$ in the sequel.

Observe that all of the above inequalities are particular cases of the following one

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant C\sum_{n=1}^{\infty} x_n, \tag{1.5}$$

where M is a mean on $\mathbb{R}_+$, that is, M is a real valued function defined on the set $\bigcup_{n=1}^{\infty}\mathbb{R}_+^n$ such that, for all $n\in\mathbb{N}$, $x_1,\dots,x_n>0$,

$$\min(x_1,\dots,x_n) \leqslant M(x_1,\dots,x_n) \leqslant \max(x_1,\dots,x_n).$$

In the sequel, a mean M will be called a *Hardy mean* if there exists a positive real constant C such that (1.5) holds for all positive sequences $x=(x_n)$. The smallest possible extended real value C such that (1.5) is valid will be called the *Hardy constant of* M and denoted by $\mathcal{H}_\infty(M)$. Due to the Hardy, Carleman, and Knopp inequalities, the pth power mean is a Hardy mean if $p<1$. One can easily see that the arithmetic mean is not a Hardy mean, therefore the following result holds.

THEOREM 1.1. *Let $p\in\mathbb{R}$. Then, the power mean $\mathcal{P}_p$ is a Hardy mean if and only if $p<1$. In addition, for $p<1$,*

$$\mathcal{H}_\infty(\mathcal{P}_p) = \begin{cases} (1-p)^{-\frac{1}{p}} & \text{if } p\neq 0,\\ e & \text{if } p=0. \end{cases}$$

The notion of power means is generalized by the concept of *quasi-arithmetic means* (cf. [24]): If $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a continuous strictly monotonic function then the quasi-arithmetic mean $\mathcal{M}_f : \bigcup_{n=1}^{\infty} I^n \to \mathbb{R}$ is defined by

$$\mathcal{M}_f(x_1,\dots,x_n) := f^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right), \qquad x_1,\dots,x_n \in I. \tag{1.6}$$

By taking f as a power function or a logarithmic function on $I = \mathbb{R}_+$, the resulting quasi-arithmetic mean is a power mean. It is well-known that *Hölder means are the only homogeneous quasi-arithmetic means* (cf. [24], [61], [48]).

The following result which completely characterizes the Hardy means among quasi-arithmetic means is due to Mulholland [45].

THEOREM 1.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous strictly monotonic function. Then, the quasi-arithmetic mean $\mathcal{M}_f$ is a Hardy mean if and only if there exist constants $p < 1$ and $C > 0$ such that, for all $n \in \mathbb{N}$ and $x_1,\dots,x_n > 0$,*

$$\mathcal{M}_f(x_1,\dots,x_n) \leqslant C\mathcal{P}_p(x_1,\dots,x_n).$$

In 1938 Gini introduced another extension of power means: For $p,q \in \mathbb{R}$, the *Gini mean* $\mathcal{G}_{p,q}$ of the variables $x_1,\dots,x_n > 0$ is defined as follows:

$$\mathcal{G}_{p,q}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p+\cdots+x_n^p}{x_1^q+\cdots+x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{x_1^p\ln(x_1)+\cdots+x_n^p\ln(x_n)}{x_1^p+\cdots+x_n^p}\right) & \text{if } p = q. \end{cases} \tag{1.7}$$

Clearly, in the particular case $q = 0$, the mean $\mathcal{G}_{p,q}$ reduces to the pth Hölder mean $\mathcal{P}_p$. It is also obvious that $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$. A common generalization of quasi-arithmetic means and Gini means can be obtained in terms of two arbitrary real functions. These means were introduced by Bajraktarević [2], [3] in 1958. Let $I \subseteq \mathbb{R}$ be an interval and let $f,g : I \to \mathbb{R}$ be continuous functions such that g is positive and f/g is strictly monotone. Define the *Bajraktarević mean* $\mathcal{B}_{f,g} : \bigcup_{n=1}^{\infty} I^n \to \mathbb{R}$ by

$$\mathcal{B}_{f,g}(x_1,\dots,x_n) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{g(x_1)+\cdots+g(x_n)}\right), \qquad x_1,\dots,x_n \in I.$$

One can check that $\mathcal{B}_{f,g}$ is a mean on I. In the particular case $g \equiv 1$, the mean $\mathcal{B}_{f,g}$ reduces to $\mathcal{M}_f$, that is, the class of Bajraktarević means is more general than that of the quasi-arithmetic means. By taking power functions, we can see that the Gini means also belong to this class. It is a remarkable result of Aczél and Daróczy [1] that the homogeneous means among the Bajraktarević means defined on $I = \mathbb{R}_+$ are exactly the Gini means.

Finally, we recall the concept of the most general means considered in this paper, the notion of the *deviation means* introduced by Daróczy [12] in 1972. A function $E : I \times I \to \mathbb{R}$ is called a *deviation function* on I if $E(x,x) = 0$ for all $x \in I$ and the

function $y \mapsto E(x,y)$ is continuous and strictly decreasing on I for each fixed $x \in I$. The *E-deviation mean* or *Daróczy mean* of some values $x_1,\dots,x_n \in I$ is now defined as the unique solution y of the equation

$$E(x_1,y)+\cdots+E(x_n,y)=0$$

and is denoted by $\mathcal{D}_E(x_1,\dots,x_n)$. It is immediate to see that the arithmetic deviation $A(x,y)=x-y$ generates the arithmetic mean. More generally, if $E: I\times I \to \mathbb{R}$ is of the form $E(x,y):=f(x)-g(x)\left(\frac{f}{g}\right)(y)$ for some continuous function $f,g: I\to\mathbb{R}$ such that g is positive and f/g is strictly monotone then $\mathcal{D}_E=\mathcal{B}_{f,g}$. Thus, Hölder means, quasi-arithmetic means, Gini means and Bajraktarević means are particular Daróczy means. The class of deviation means was slightly generalized to the class of quasi-deviation means and this class was completely characterized by Páles in [51].

The following result, which gives necessary and also sufficient conditions for the Hardy property of deviation means was established by Páles and Persson [62] in 2004.

THEOREM 1.3. *Let $E:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a deviation on $\mathbb{R}_+$. If $\mathcal{D}_E$ is a Hardy mean, then there exists a positive constant C such that*

$$\mathcal{D}_E(x_1,\dots,x_n)\leqslant C\mathcal{A}(x_1,\dots,x_n)$$

holds for all $n\in\mathbb{N}$ and $x_1,\dots,x_n>0$ and there is no positive constant C^* such that

$$C^*\mathcal{A}(x_1,\dots,x_n)\leqslant \mathcal{D}_E(x_1,\dots,x_n)$$

be valid on the same domain. Conversely, if

$$\mathcal{D}_E(x_1,\dots,x_n)\leqslant C\mathcal{P}_p(x_1,\dots,x_n)$$

is satisfied with a parameter $p<1$ and a positive constant C, then $\mathcal{D}_E$ is a Hardy mean.

As a corollary of the previous result, necessary and also sufficient conditions for the Hardy property were established in the class of Gini means in the same paper.

THEOREM 1.4. *Let $p,q\in\mathbb{R}$. If $\mathcal{G}_{p,q}$ is a Hardy mean, then*

$$\min(p,q)\leqslant 0 \qquad \textit{and} \qquad \max(p,q)\leqslant 1.$$

Conversely, if

$$\min(p,q)\leqslant 0 \qquad \textit{and} \qquad \max(p,q)< 1,$$

then $\mathcal{G}_{p,q}$ is a Hardy mean.

It has been an open problem since 2004 whether the second condition was a necessary and sufficient condition for the Hardy property and also the best Hardy constant was to be determined.

The necessary and sufficient condition for the Hardy property of Gini means was finally found by Pasteczka [47] in 2015. The key was the following general necessary condition for the Hardy property.

LEMMA 1.5. *Assume that* $M\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n \to \mathbb{R}_+$ *is a Hardy mean. Then, for all positive non-*ℓ_1 *sequences* (x_n),

$$\liminf_{n\to\infty} x_n^{-1} M(x_1,\dots,x_n) < \infty.$$

Applying this necessary condition in the class of Gini means with the harmonic sequence $x_n := \frac{1}{n}$, Pasteczka [47] obtained the following characterization of the Hardy property for Gini means.

THEOREM 1.6. *Let* $p,q \in \mathbb{R}$. *Then* $\mathcal{G}_{p,q}$ *is a Hardy mean if and only if*

$$\min(p,q) \leqslant 0 \qquad \textit{and} \qquad \max(p,q) < 1.$$

There was no progress, however, in establishing the Hardy constant of the Gini means. There was only an upper estimate obtained by Páles and Persson in [62].

Motivated by all these preliminaries, the purpose of this paper is twofold:

— To find (in terms of easy-to-check properties) a large subclass of Hardy means.

— To obtain a formula for the Hardy constant in that subclass of means.

2. Means and their basic properties

For investigating the Hardy property of means, we recall several relevant notions. Let $I \subseteq \mathbb{R}$ be an interval and let $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ be an arbitrary mean.

We say that M is *symmetric*, *(strictly) increasing*, and *Jensen convex (concave)* if, for all $n \in \mathbb{N}$, the n-variable restriction $M|_{I^n}$ is a symmetric, (strictly) increasing in each of its variables, and Jensen convex (concave) on I^n, respectively. If $I = \mathbb{R}_+$, we can analogously define the notion of homogeneity of M.

The mean M is called *repetition invariant* if, for all $n,m \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$, the following identity is satisfied

$$M(\underbrace{x_1,\dots,x_1}_{m\text{-times}},\dots,\underbrace{x_n,\dots,x_n}_{m\text{-times}}) = M(x_1,\dots,x_n).$$

The mean M is *strict* if for any $n \geqslant 2$ and any non-constant vector $(x_1,\dots,x_n) \in I^n$,

$$\min(x_1,\dots,x_n) < M(x_1,\dots,x_n) < \max(x_1,\dots,x_n).$$

The mean M is said to be $\min$*-diminishing* if, for any $n \geqslant 2$ and any non-nonstant vector $(x_1,\dots,x_n) \in I^n$,

$$M(x_1,\dots,x_n,\min(x_1,\dots,x_n)) < M(x_1,\dots,x_n).$$

It is easy to check that quasi-arithmetic means are symmetric, strictly increasing, repetition invariant, strict and $\min$-diminishing. More generally, deviation means are symmetric, repetition invariant, strict and $\min$-diminishing (cf. [51]). The increasingness of a deviation mean $\mathcal{D}_E$ is equivalent to the increasingness of the deviation E in its first variable. The Jensen concavity/convexity of quasi-arithmetic and also of deviation

means can be characterized by the concavity/convexity conditions on the generating functions. All these characterizations are consequences of the general results obtained in a series of papers by Losonczi [33, 34, 36, 35, 37, 38] (for Bajraktarević means) and by Daróczy [14, 11, 12, 15, 16] and Páles [50, 52, 53, 54, 55, 56, 57, 58, 59, 60] (for (quasi-)deviation means).

2.1. Kedlaya means

The notion of a Kedlaya mean that we introduce below turns out to be indispensable for the investigation of Hardy means. A mean $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ is called a *Kedlaya mean* if, for all $n \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$,

$$\frac{M(x_1)+M(x_1,x_2)+\cdots+M(x_1,\dots,x_n)}{n} \leqslant M\left(x_1, \frac{x_1+x_2}{2}, \dots, \frac{x_1+\cdots+x_n}{n}\right). \tag{2.1}$$

The motivation for the above terminology comes from the papers [26, 27] by Kedlaya, where he proved that the geometric mean satisfies the inequality (2.1), i.e., it is a Kedlaya mean. The next result provides a sufficient condition in order that a mean be a Kedlaya mean.

THEOREM 2.1. *Every symmetric, Jensen concave and repetition invariant mean is a Kedlaya mean.*

Proof. Let $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ be a symmetric, Jensen concave and repetition invariant mean. Fix $n \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$. Adopting Kedlaya's original proof, for $(i,j,k) \in \{1,\dots,n\}^3$, we define

$$\begin{aligned} a_k(i,j) &:= (n-1)! \cdot \binom{n-i}{j-k}\binom{i-1}{k-1} \Big/ \binom{n-1}{j-1} \\ &= \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-i-j+k)!(i-k)!(j-k)!(k-1)!}. \end{aligned}$$

To provide the corectness of this definition we assume that $m! = \infty$ for negative integers m (it is a natural extension of gamma function). Then, according to [26], we have the following properties:

(1) $a_k(i,j) \geqslant 0$ for all i, j, k;

(2) $a_k(i,j) \in \mathbb{N} \cup \{0\}$ for all i, j, k;

(3) $a_k(i,j) = 0$ for $k > \min(i,j)$;

(4) $a_k(i,j) = a_k(j,i)$ for all i, j, k;

(5) $\sum_{k=1}^{n} a_k(i,j) = (n-1)!$ for all i, j;

(6) $\sum_{i=1}^{n} a_k(i,j) = \begin{cases} n!/j & \text{for } k \leqslant j, \\ 0 & \text{for } k > j. \end{cases}$

Let us construct a matrix A of size $n! \times n!$ divided into n^2 blocks $(A_{i,j})_{i,j\in\{1,\dots,n\}}$ of size $(n-1)! \times (n-1)!$.

The first row of each block $A_{i,j}$ contains the number k exactly $a_k(i,j)$ times for $k \in \{1,\dots,n\}$; this could be done by (5). The subsequent rows are all cyclic permutations of the first one. In this way each row and each column of $A_{i,j}$ contains the number k exactly $a_k(i,j)$ times.

Now, let $c_p(k)$ denote the occurrence of the number k appearing in the pth row of A. Then, by (4), $c_p(k)$ is equal to the number of occurrences of k in the pth column of A.

We are going to calculate $c_p(k)$. The pth row has a nonempty intersection with the block $A_{i,j}$ if

$$i = \left\lfloor \frac{p-1}{(n-1)!} \right\rfloor + 1 =: b(p).$$

Whence, applying property (6), we get

$$c_p(k) = \sum_{i=1}^{n} a_k(i,b(p)) = \begin{cases} n!/b(p) & \text{for } k \leqslant b(p), \\ 0 & \text{for } k > b(p). \end{cases}$$

Now, let us consider the matrix A' obtained from A by replacing $k \mapsto x_k$ for $k \in \{1,\dots,n\}$. We will calculate the mean value of the elements of A' in two different ways. First, we calculate the mean M of each column of A'. By the Jensen concavity of M, the arithmetic mean of the results so obtained does not exceed the result of calculating arithmetic mean of each row of A' and then taking the M mean of the resulting vector of length $n!$. Whence, using the symmetry and the repetition invariance of M, we obtain

$$\frac{1}{n!}\Big((n-1)!M(x_1) + (n-1)!M(x_1,x_2) + \cdots + (n-1)!M(x_1,x_2,\dots,x_n)\Big) \leqslant M\Big(x_1, \frac{x_1+x_2}{2}, \dots, \frac{x_1+x_2+\cdots+x_n}{n}\Big)$$

which simplifies to the inequality (2.1) to be proved. □

COROLLARY 2.2. *If, in addition to the assumptions of Theorem 2.1, M is also increasing and $I = \mathbb{R}_+$, then*

$$M(x_1) + M(x_1,x_2) + \cdots + M(x_1,\dots,x_n) \leqslant n \cdot M\left(x_1 + \cdots + x_n, \frac{x_1+\cdots+x_n}{2}, \dots, \frac{x_1+\cdots+x_n}{n}\right).$$

2.2. Gaussian product

The Gaussian product of means is a broad extension of Gauss' idea of the arithmetic-geometric mean. In 1800 (this year is due to [64]) he proposed the following two-term recursion:

$$x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_n y_n}, \qquad n = 0,1,\dots,$$

where x_0 and y_0 are positive numbers. Gauss [20, p. 370] proved that both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge to a common limit, which is called arithmetic-geometric mean of the initial values x_0 and y_0. J. M. Borwein and P. B. Borwein [9] extended some earlier ideas [19, 32, 63] and generalized this iteration to a vector of continuous, strict means of an arbitrary length. For several recent results about Gaussian product of means see the papers by Baják and Páles [4, 5, 6, 7], by Daróczy and Páles [13, 17, 18], by Głazowska [21, 22], by Matkowski [39, 40, 41, 42], and by Matkowski and Páles [43].

Given $N \in \mathbb{N}$ and a vector $(M_1, \dots, M_N)$ of means defined on a common interval I and having values in I (i.e. $M_i \colon \bigcup_{n=1}^{\infty} I^n \to I$ for every $i \in \{1, \dots, N\}$), let us introduce the mapping $\mathbf{M} \colon \bigcup_{n=1}^{\infty} I^n \to I^N$ by

$$\mathbf{M}(v) := (M_1(v), M_2(v), \dots, M_N(v)), \qquad v \in \bigcup_{n=1}^{\infty} I^n.$$

Whenever, for every $i \in \{1, \dots, N\}$ and every $v \in \bigcup_{n=1}^{\infty} I^n$, the limit $\lim_{k\to\infty} [\mathbf{M}^k(v)]_i$ exists and does not depend on i, then the value of this limit will be called the *Gaussian product* of $(M_1, \dots, M_N)$ evaluated at v. We will denote this limit by $M_{\otimes}(v)$. It is well-known that the Gaussian product can equivalently be defined as a unique function satisfying the following two properties:

(i) $M_{\otimes} \circ \mathbf{M}(v) = M_{\otimes}(v)$ for all $v \in \bigcup_{n=1}^{\infty} I^n$,

(ii) $\min(v) \leqslant M_{\otimes}(v) \leqslant \max(v)$ for all $v \in \bigcup_{n=1}^{\infty} I^n$.

Frequently, whenever each of the means M_i, $i \in \{1, \dots, N\}$ has a certain property, then $M_{\otimes}$ inherits this property. The lemma below (in view of Theorem 2.1) is its very useful exemplification.

LEMMA 2.3. *Let I be an interval, $N \in \mathbb{N}$, and let $(M_1, \dots, M_N) \colon \bigcup_{n=1}^{\infty} I^n \to I^N$. If, for each $i \in \{1, \dots, N\}$, M_i is symmetric/homogeneous/repetition invariant/increasing and Jensen concave/convex, then so is their Gaussian product $M_{\otimes}$.*

Proof. The first four properties are naturally inherited by all of the functions $[\mathbf{M}^k]_i$, for $k \in \mathbb{N}$, $i \in \{1, \dots, N\}$ and, finally, by their pointwise limit. The verification of the statement about the Jensen concavity is just a little bit more sophisticated. In fact, the idea presented below could also be adapted to the remaining properties.

Assume that $M_1, \dots, M_N$ are increasing and Jensen concave. We will prove that $M_{\otimes}$ is Jensen concave. Let $x^{(0)}$, $y^{(0)}$ be the equidimensional vectors and $m^{(0)} = \frac{1}{2}(x^{(0)} + y^{(0)})$. Let

$$x^{(k+1)} = \mathbf{M}(x^{(k)}), \quad y^{(k+1)} = \mathbf{M}(y^{(k)}), \quad m^{(k+1)} = \mathbf{M}(m^{(k)}), \qquad k \in \mathbb{N}.$$

We are going to prove that

$$[m^{(k)}]_i \geqslant \tfrac{1}{2}[x^{(k)} + y^{(k)}]_i \quad \text{for any } i \in \{1, \dots, N\} \text{ and } k \in \mathbb{N}. \tag{2.2}$$

Obviously, this holds for $k=0$. Let us assume that (2.2) holds for some $k\in\mathbb{N}$ and any i. Then, by the increasingness and Jensen concavity of M_i,

$$\begin{aligned}[m^{(k+1)}]_i = M_i(m^{(k)}) &\geqslant M_i\left(\tfrac{1}{2}(x^{(k)}+y^{(k)})\right) \geqslant \tfrac{1}{2}\left(M_i(x^{(k)})+M_i(y^{(k)})\right)\\ &= \tfrac{1}{2}\left([x^{(k+1)}]_i+[y^{(k+1)}]_i\right) = \tfrac{1}{2}[x^{(k+1)}+y^{(k+1)}]_i.\end{aligned}$$

Upon taking the limit $k\to\infty$, one gets

$$M_\otimes\left(\frac{x^{(0)}+y^{(0)}}{2}\right) = M_\otimes(m^{(0)}) \geqslant \tfrac{1}{2}\left(M_\otimes(x^{(0)})+M_\otimes(y^{(0)})\right),$$

which proves that $M_\otimes$ is Jensen concave, indeed. $\square$

3. Main Results

In the sequel, let $I\subseteq\mathbb{R}$ be a nondegenerate interval such that $\inf I=0$. We will denote by $\ell_1(I)$ the collection of all sequences $x=(x_n)_{n=1}^\infty$ such that, for all $n\in\mathbb{N}$, $x_n\in I$ and $\|x\|_1:=\sum_{n=1}^\infty x_n$ is convergent, i.e., $x\in\ell_1$.

Recall (slightly extending the definition) that, for a given mean $M\colon \bigcup_{n=1}^\infty I^n\to I$, the constant $\mathcal{H}_\infty(M)$ is the smallest nonnegative extended real number, called the *Hardy constant of* M, such that

$$\sum_{n=1}^\infty M(x_1,\dots,x_n)\leqslant \mathcal{H}_\infty(M)\sum_{n=1}^\infty x_n,\qquad (x_n)_{n=1}^\infty\in\ell_1(I). \tag{3.1}$$

If $\mathcal{H}_\infty(M)$ is finite, then we say that M is a *Hardy mean*. Given also $n\in\mathbb{N}$, we define $\mathcal{H}_n(M)$ to be the smallest nonnegative number such that

$$M(x_1)+\ldots+M(x_1,\dots,x_n)\leqslant \mathcal{H}_n(M)(x_1+\cdots+x_n),\qquad (x_1,\dots,x_n)\in I^n. \tag{3.2}$$

Due to the mean value property of M, for $n\in\mathbb{N}$, we easily obtain that $1\leqslant\mathcal{H}_n(M)\leqslant n$. The sequence $\big(\mathcal{H}_n(M)\big)_{n=1}^\infty$ will be called the *Hardy sequence of* M.

Several estimates of the Hardy sequences for power means were given during the years. For example Kaluza and Szegő [25] proved $\mathcal{H}_n(\mathcal{P}_p)\leqslant \frac{1}{n(\exp(1/n)-1)}\cdot\mathcal{H}_\infty(\mathcal{P}_p)$ for $p\in[0,1)$ and $n\in\mathbb{N}$. Moreover it is known [24, p.267] that $\mathcal{H}_n(\mathcal{P}_0)\leqslant(1+\frac{1}{n})^n$ for all $n\in\mathbb{N}$.

The basic properties of the Hardy sequence are established in the following

PROPOSITION 3.1. *For every mean $M\colon \bigcup_{n=1}^\infty I^n\to I$, its Hardy sequence is nondecreasing and*

$$\lim_{n\to\infty}\mathcal{H}_n(M)=\mathcal{H}_\infty(M). \tag{3.3}$$

Proof. To verify the nondecreasingness of the Hardy sequence of M, let $(x_1,\dots,x_n)$ in I^n and $\varepsilon\in I$ be arbitrary. Applying inequality (3.2) to the sequence $(x_1,\dots,x_n,\varepsilon)\in I^{n+1}$, we obtain

$$\begin{aligned}M(x_1)+\cdots+M(x_1,\dots,x_n) &\leqslant M(x_1)+\cdots+M(x_1,\dots,x_n)+M(x_1,\dots,x_n,\varepsilon)\\ &\leqslant \mathcal{H}_{n+1}(M)(x_1+\cdots+x_n+\varepsilon).\end{aligned}$$

Upon taking the limit $\varepsilon \to 0$, it follows that

$$M(x_1)+\cdots+M(x_1,\dots,x_n) \leqslant \mathcal{H}_{n+1}(M)(x_1+\cdots+x_n)$$

for all $(x_1,\dots,x_n) \in I^n$. Hence $\mathcal{H}_n(M) \leqslant \mathcal{H}_{n+1}(M)$.

To prove (3.3), we will show first that $\mathcal{H}_n(M) \leqslant \mathcal{H}_\infty(M)$ for all $n \in \mathbb{N}$. If $\mathcal{H}_\infty(M) = \infty$ then this inequality is obvious, hence we may assume that M is a Hardy mean. Fix $n \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$ and choose $\varepsilon \in I$ arbitrarily. Applying (3.1) to the sequence $(x_1,\dots,x_n,\frac{\varepsilon}{2},\frac{\varepsilon}{4},\frac{\varepsilon}{8},\dots) \in \ell_1(I)$, one gets

$$\begin{aligned}
&M(x_1)+\cdots+M(x_1,\dots,x_n)\\
&\qquad \leqslant M(x_1)+\cdots+M(x_1,\dots,x_n)+M(x_1,\dots,x_n,\tfrac{\varepsilon}{2})+M(x_1,\dots,x_n,\tfrac{\varepsilon}{2},\tfrac{\varepsilon}{4})+\cdots\\
&\qquad \leqslant \mathcal{H}_\infty(M)(x_1+\cdots+x_n+\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{4}+\cdots)\\
&\qquad = \mathcal{H}_\infty(M)(x_1+\cdots+x_n+\varepsilon).
\end{aligned}$$

Upon passing the limit $\varepsilon \to 0$, we get

$$M(x_1)+\cdots+M(x_1,\dots,x_n) \leqslant \mathcal{H}_\infty(M)(x_1+\cdots+x_n),$$

which implies $\mathcal{H}_n(M) \leqslant \mathcal{H}_\infty(M)$. Using this inequality, we have also proved that in (3.3) $\leqslant$ holds instead of equality.

To prove the reversed inequality in (3.3), let $(x_n)_{n=1}^\infty \in \ell_1(I)$ be arbitrary. Then, for all $n \leqslant k$, we have that

$$M(x_1)+\cdots+M(x_1,\dots,x_n) \leqslant \mathcal{H}_n(M)(x_1+\cdots+x_n) \leqslant \mathcal{H}_k(M)(x_1+\cdots+x_n)$$

Now taking the limit as $k \to \infty$, we obtain that

$$M(x_1)+\cdots+M(x_1,\dots,x_n) \leqslant \lim_{k\to\infty} \mathcal{H}_k(M)\cdot(x_1+\cdots+x_n)$$

holds for all $n \in \mathbb{N}$. Finally taking the limit as $n \to \infty$, it follows that M satisfies

$$\sum_{n=1}^\infty M(x_1,\dots,x_n) \leqslant \lim_{k\to\infty} \mathcal{H}_k(M) \sum_{n=1}^\infty x_n$$

which yields that the reversed inequality in (3.3) is also true. $\square$

In what follows, we show that the inequality (3.1) is strict in a broad class of means.

PROPOSITION 3.2. *Let $I \subseteq \mathbb{R}_+$ and $M\colon \bigcup_{n=1}^\infty I^n \to I$. If M is a* min*-diminishing, increasing and repetition invariant Hardy mean, then*

$$\sum_{n=1}^\infty M(x_1,\dots,x_n) < \mathcal{H}_\infty(M) \sum_{n=1}^\infty x_n, \qquad (x_n)_{n=1}^\infty \in \ell_1(I).$$

Proof. Let $x=(x_n)_{n=1}^{\infty}\in\ell_1(I)$ be arbitrary. If $x_l<x_k$ for some $l<k$ then, for the sequence

$$x'_n=\begin{cases} x_n & n\notin\{k,l\},\\ x_k & n=l,\\ x_l & n=k,\end{cases}$$

we have

$$\begin{aligned} M(x_1,\dots,x_n)&=M(x'_1,\dots,x'_n) && \text{for } n<l \text{ or } n\geqslant k,\\ M(x_1,\dots,x_n)&\leqslant M(x'_1,\dots,x'_n) && \text{for } n\in\{l,\dots,k-1\}.\end{aligned}$$

Therefore

$$M(x_1)+\dots+M(x_1,\dots,x_n)+\dots\leqslant M(x'_1)+\dots+M(x'_1,\dots,x'_n)+\dots.$$

Whence we may assume that x is non-increasing.

Let $\hat{x}=(x_1,x_1,\dots,x_n,x_n,\dots)$. Then, by the repetition invariance and the min-diminishing property of M, we get

$$\begin{aligned} M(x_1,\dots,x_n)&=M(\hat{x}_1,\dots,\hat{x}_{2n}),\\ M(x_1,\dots,x_n)&=M(\hat{x}_1,\dots,\hat{x}_{2n-1}) && \text{if } x_1=x_n,\\ M(x_1,\dots,x_n)&<M(\hat{x}_1,\dots,\hat{x}_{2n-1}) && \text{if } x_1\neq x_n.\end{aligned}$$

Since $x_n\to 0$ as $n\to\infty$, hence $x_1\neq x_n$ holds for some n. Therefore

$$2\cdot\sum_{n=1}^{\infty}M(x_1,\dots,x_n)<\sum_{n=1}^{\infty}M(\hat{x}_1,\dots,\hat{x}_n)\leqslant\mathcal{H}_\infty(M)\sum_{n=1}^{\infty}\hat{x}_n=2\mathcal{H}_\infty(M)\sum_{n=1}^{\infty}x_n.$$

This completes the proof of the proposition. □

The next result offers a fundamental lower estimate for the Hardy constant of a mean.

THEOREM 3.3. *Let $M\colon\bigcup_{n=1}^{\infty}I^n\to I$ be a mean. Then, for all sequences $(x_n)_{n=1}^{\infty}$ in I that does not belong to ℓ_1,*

$$\liminf_{n\to\infty}x_n^{-1}M(x_1,\dots,x_n)\leqslant\mathcal{H}_\infty(M). \tag{3.4}$$

Proof. Assume, on the contrary, that

$$\mathcal{H}_\infty(M)<\liminf_{n\to\infty}x_n^{-1}M(x_1,\dots,x_n).$$

Then, there exists $\varepsilon>0$ and n_0 such that, for all $n\geqslant n_0$,

$$(1+\varepsilon)\mathcal{H}_\infty(M)x_n<M(x_1,\dots,x_n). \tag{3.5}$$

Choose $n_1>n_0$ such that

$$\sum_{n=1}^{n_0}x_n\leqslant\varepsilon\sum_{n=n_0+1}^{n_1}x_n \tag{3.6}$$

Thus, using (3.5), Proposition 3.1, and finally (3.6), we obtain

$$\sum_{n=n_0+1}^{n_1}(1+\varepsilon)\mathcal{H}_\infty(M)x_n < \sum_{n=n_0+1}^{n_1} M(x_1,\dots,x_n) \leqslant \sum_{n=1}^{n_1} M(x_1,\dots,x_n)$$
$$\leqslant \mathcal{H}_{n_1}(M)\sum_{n=1}^{n_1} x_n \leqslant \mathcal{H}_\infty(M)\sum_{n=1}^{n_1} x_n \leqslant (1+\varepsilon)\mathcal{H}_\infty(M)\sum_{n=n_0+1}^{n_1} x_n.$$

This contradiction validates (3.4). □

The main result of our paper is contained in the following theorem.

THEOREM 3.4. *Let $M\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n \to \mathbb{R}_+$ be an increasing, symmetric, repetition invariant, and Jensen concave mean. Then*

$$\mathcal{H}_\infty(M) = \sup_{y>0}\liminf_{n\to\infty}\frac{n}{y}\cdot M\left(\frac{y}{1},\frac{y}{2},\dots,\frac{y}{n}\right). \tag{3.7}$$

As a trivial consequence of the above result, M is a Hardy mean if and only if the number $\mathcal{H}_\infty(M)$ given in (3.7) is finite.

Proof. For the proof of the theorem, denote

$$C := \sup_{y>0}\liminf_{n\to\infty}\frac{n}{y}\cdot M\left(\frac{y}{1},\frac{y}{2},\dots,\frac{y}{n}\right).$$

The inequality $\mathcal{H}_\infty(M) \geqslant C$ is simply a consequence of Theorem 3.3.

To show the reversed inequality, we may assume that C is finite. Fix $x \in \ell_1(\mathbb{R}_+)$ and denote $y := \|x\|_1$. Then there exists a sequence (n_k), $n_k \to \infty$ such that

$$n_k \cdot M\left(\frac{y}{1},\frac{y}{2},\dots,\frac{y}{n_k}\right) \leqslant (C+\tfrac{1}{k})y, \qquad k\in\mathbb{N}.$$

By the increasingness of M and by the obvious inequality $x_1+\dots+x_{n_k} \leqslant y$, the previous inequality yields

$$n_k \cdot M\left(x_1+\dots+x_{n_k}, \frac{x_1+\dots+x_{n_k}}{2},\dots,\frac{x_1+\dots+x_{n_k}}{n_k}\right) \leqslant \left(C+\tfrac{1}{k}\right)y, \qquad k\in\mathbb{N}.$$

Therefore, in view of Corollary 2.2, we obtain

$$M(x_1)+M(x_1,x_2)+\dots+M(x_1,\dots,x_{n_k}) \leqslant (C+\tfrac{1}{k})y, \qquad k\in\mathbb{N}.$$

Upon passing the limit $k\to\infty$, one gets

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant Cy = C\|x\|_1.$$

This completes the proof of inequality $\mathcal{H}_\infty(M) \leqslant C$. □

COROLLARY 3.5. *If, in addition to the assumptions of Theorem 3.4, M is also homogeneous, then*

$$\mathcal{H}_\infty(M) = \lim_{n\to\infty} n \cdot M\left(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}\right).$$

Proof. In view of the previous theorem we only need to prove that the limit of the sequence (p_n) exists (possible infinite), where

$$p_n := n \cdot M\left(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}\right).$$

For, it suffices to show that this sequence is nondecreasing. Fix $n \in \mathbb{N}$. Let us consider the two vectors u, v of dimension $n(n+1)$ defined by

$$u := (\underbrace{n, \dots, n}_{n+1}, \underbrace{\tfrac{n}{2}, \dots, \tfrac{n}{2}}_{n+1}, \dots, \underbrace{\tfrac{n}{n-1}, \dots, \tfrac{n}{n-1}}_{n+1}, \underbrace{1, \dots, 1}_{n+1});$$
$$v := (\underbrace{n+1, \dots, n+1}_{n}, \underbrace{\tfrac{n+1}{2}, \dots, \tfrac{n+1}{2}}_{n}, \dots, \underbrace{\tfrac{n+1}{n}, \dots, \tfrac{n+1}{n}}_{n}, \underbrace{1, \dots, 1}_{n}).$$

By the homogeneity and repetition invariance of M, we have that $M(u) = p_n$ and $M(v) = p_{n+1}$. Divide vectors u and v into $n+1$ parts of dimension n:

$$u^{(i)} := (\underbrace{\tfrac{n}{i}, \dots, \tfrac{n}{i}}_{i}, \underbrace{\tfrac{n}{i+1}, \dots, \tfrac{n}{i+1}}_{n-i}), \qquad i = 0, \dots, n;$$
$$v^{(i)} := (\underbrace{\tfrac{n+1}{i+1}, \dots, \tfrac{n+1}{i+1}}_{n}), \qquad i = 0, \dots, n.$$

For $i \geqslant 1$, each element $\frac{n}{i}$ appears $(n-i+1)$ times in $u^{(i-1)}$ and i times in $u^{(i)}$, that is, $(n+1)$ times altogether. Therefore, the arithmetic mean of $u^{(i)}$, denoted by $\mathcal{A}(u^{(i)})$, is equal to $\frac{n+1}{i+1}$ for $i = 1, \dots, n$ and $\mathcal{A}(u^{(0)}) = n$.

Let $u_k^{(i)}$, for $k = 1, \dots, n!$ and $i = 0, \dots, n$, denote the vectors that are obtained from all possible permutations of the components of $u^{(i)}$. Observe that

$$(u^{(0)}, v^{(1)}, \dots, v^{(n)}) = \frac{1}{n!} \sum_{k=1}^{n!} (u_k^{(0)}, \dots, u_k^{(n)}).$$

Then, by the increasingness, Jensen concavity and symmetry of the mean M, we obtain

$$\begin{aligned} p_{n+1} = M(v) &= M(v^{(0)}, v^{(1)}, \dots, v^{(n)}) \geqslant M(u^{(0)}, v^{(1)}, \dots, v^{(n)}) \\ &\geqslant \frac{1}{n!} \sum_{k=1}^{n!} M(u_k^{(0)}, \dots, u_k^{(n)}) = M(u^{(0)}, \dots, u^{(n)}) = M(u) = p_n. \end{aligned}$$

This proves that (p_n) is non-deceasing and, therefore it has a (possibly infinite) limit. $\square$

4. Applications

In this section we demonstrate the consequences of our results for Gini means and also for the Gaussian product of symmetric, homogeneous, increasing, Jensen concave and repetition invariant means, in particular, the Gaussian product of Hölder means.

4.1. Gini means

Gini means are symmetric and repetition invariant and min-diminishing (first two properties are simple while the third one was proved in [51]). Moreover, by the results of Losonczi [34, 35], the Gini mean $\mathcal{G}_{p,q}$ is increasing and Jensen concave if and only if $pq \leqslant 0$ and $\min(p,q) \leqslant 0 \leqslant \max(p,q) \leqslant 1$, respectively. In particular it implies that Hölder mean $\mathcal{P}_p$ is Jensen concave if and only if $p \leqslant 1$.

In view of Theorem 1.6, we have the characterization of pairs (p,q) such that $\mathcal{G}_{p,q}$ is a Hardy mean. In order to calculate the Hardy constant of Gini means using Corollary 3.5, we need to establish the following result.

LEMMA 4.1. *Let* $p,q \in (-\infty,1)$. *Then*

$$\lim_{n\to\infty} n \cdot \mathcal{G}_{p,q}\left(1,\tfrac{1}{2},\dots,\tfrac{1}{n}\right) = \begin{cases} \left(\dfrac{1-q}{1-p}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ e^{\frac{1}{1-p}} & \text{if } p = q. \end{cases}$$

Proof. For every $s \in (-1,\infty)$, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^s = \int_0^1 x^s dx = \frac{1}{1+s}.$$

Using this equality, for $p,q<1$, $p \neq q$, we simply obtain

$$\begin{aligned} &\lim_{n\to\infty} n \cdot \mathcal{G}_{p,q}\left(1,\tfrac{1}{2},\dots,\tfrac{1}{n}\right) \\ &\quad = \lim_{n\to\infty} n \cdot \left(\frac{1+2^{-p}+3^{-p}+\cdots+n^{-p}}{1+2^{-q}+3^{-q}+\cdots+n^{-q}}\right)^{\frac{1}{p-q}} \\ &\quad = \lim_{n\to\infty} \left(\frac{\frac{1}{n}\left[\left(\frac{1}{n}\right)^{-p}+\left(\frac{2}{n}\right)^{-p}+\left(\frac{3}{n}\right)^{-p}+\cdots+\left(\frac{n-1}{n}\right)^{-p}+1\right]}{\frac{1}{n}\left[\left(\frac{1}{n}\right)^{-q}+\left(\frac{2}{n}\right)^{-q}+\left(\frac{3}{n}\right)^{-q}+\cdots+\left(\frac{n-1}{n}\right)^{-q}+1\right]}\right)^{\frac{1}{p-q}} \\ &\quad = \left(\frac{1-q}{1-p}\right)^{\frac{1}{p-q}}. \end{aligned}$$

The proof for the case $p=q<1$ is analogous. □

Using this lemma and the properties that are mentioned just before, we obtain the following

COROLLARY 4.2. *Let* $p,q \in \mathbb{R}$, $\min(p,q) \leqslant 0 \leqslant \max(p,q) < 1$. *Then*

$$\mathcal{H}_\infty(\mathcal{G}_{p,q}) = \begin{cases} \left(\dfrac{1-q}{1-p}\right)^{\frac{1}{p-q}} & p \neq q, \\ e & p = q = 0. \end{cases}$$

Proof. Due to the assumption $\min(p,q) \leqslant 0 \leqslant \max(p,q) < 1$ and in view of the results of Losonczi [34, 35], the Gini mean $\mathcal{G}_{p,q}$ is increasing and Jensen concave. Furthermore, $\mathcal{G}_{p,q}$ is symmetric, homogeneous, and repetition invariant. Therefore, by Corollary 3.5 and Lemma 4.1, we have

$$\mathcal{H}_\infty(\mathcal{G}_{p,q}) = \lim_{n\to\infty} n \cdot \mathcal{G}_{p,q}\left(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}\right) = \begin{cases} \left(\dfrac{1-q}{1-p}\right)^{\frac{1}{p-q}} & p \neq q, \\ e & p = q = 0, \end{cases}$$

which was to be proved. □

4.2. Gaussian product

PROPOSITION 4.3. *Let* $N \in \mathbb{N}$ *and let* $M_1, \dots, M_N \colon \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ *be symmetric, homogeneous, increasing, Jensen concave and repetition invariant means. If* M_i *is Hardy for each* $i \in \{1, \dots, N\}$, *then so is their Gaussian product* $M_\otimes$ *and*

$$\mathcal{H}_\infty(M_\otimes) = M_\otimes\big(\mathcal{H}_\infty(M_1), \dots, \mathcal{H}_\infty(M_N)\big). \tag{4.1}$$

Proof. In view of Lemma 2.3, the Gaussian product $M_\otimes$ is a symmetric, homogeneous, increasing, Jensen concave and repetition invariant mean. The Jensen concavity and the local boundedness by the Bernstein–Doetsch Theorem implies that $M_\otimes$ is concave and therefore it is also continuous (see [8], [29]). Thus, by Corollary 3.5, we have

$$\begin{aligned} \mathcal{H}_\infty(M_\otimes) &= \lim_{n\to\infty} n \cdot M_\otimes\left(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}\right) \\ &= \lim_{n\to\infty} n \cdot M_\otimes\left(M_1(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}), \dots, M_N(1, \tfrac{1}{2}, \dots, \tfrac{1}{n})\right) \\ &= \lim_{n\to\infty} M_\otimes\left(nM_1(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}), \dots, nM_N(1, \tfrac{1}{2}, \dots, \tfrac{1}{n})\right) \\ &= M_\otimes\left(\lim_{n\to\infty} nM_1(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}), \dots, \lim_{n\to\infty} nM_N(1, \tfrac{1}{2}, \dots, \tfrac{1}{n})\right) \\ &= M_\otimes\left(\mathcal{H}_\infty(M_1), \dots, \mathcal{H}_\infty(M_N)\right), \end{aligned}$$

which proves formula (4.1). □

COROLLARY 4.4. *Let* $N \in \mathbb{N}$ *and* $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ *then the Gaussian product* $\mathcal{P}_\otimes$ *of the Hölder means* $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_N}$ *is a Hardy mean if and only if* $\max_{1\leqslant k\leqslant N} \lambda_k < 1$. *Furthermore, in this case,*

$$\mathcal{H}_\infty(\mathcal{P}_\otimes) = \mathcal{P}_\otimes\big(\mathcal{H}_\infty(\mathcal{P}_{\lambda_1}), \dots, \mathcal{H}_\infty(\mathcal{P}_{\lambda_N})\big). \tag{4.2}$$

Proof. The first part of the statement of the above Corollary was proved in [47] by Pasteczka. If $\lambda_k < 1$, then $\mathcal{P}_{\lambda_k}$ is a Jensen concave mean, therefore (4.2) is a particular case of (4.1). $\square$

For example, for the *geometric-harmonic mean* $\mathcal{P}_{-1} \otimes \mathcal{P}_0$, i.e., for the Gaussian product of the harmonic mean $\mathcal{P}_{-1}$ and the geometric mean $\mathcal{P}_0$, we get

$$\mathcal{H}_\infty(\mathcal{P}_{-1} \otimes \mathcal{P}_0) = (\mathcal{P}_{-1} \otimes \mathcal{P}_0)(\mathcal{H}_\infty(\mathcal{P}_{-1}), \mathcal{H}_\infty(\mathcal{P}_0)) = (\mathcal{P}_{-1} \otimes \mathcal{P}_0)(2, e) \approx 2,318.$$

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