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APPLICATIONS OF THE HERMITE–HADAMARD INEQUALITY

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Abstract. We show how the recent improvement of the Hermite-Hadamard inequality can be applied to some (not necessarily convex) planar figures and three-dimensional bodies satisfying some kind of regularity.

1. Introduction

The classical Hermite-Hadamard inequality [4] states that for a convex function $f\colon [a,b] \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Due to its simple and elegant form it became a natural object of investigations. Neuman and Bessenyei [6, 1] proved the version for simplices saying that if $\Delta \subset \mathbb{R}^n$ is a simplex with barycenter $\mathbf{b}$ and vertices $\mathbf{x}_0,\ldots,\mathbf{x}_n$ and $f\colon \Delta \to \mathbb{R}$ is convex, then

$$f(\mathbf{b}) \leqslant \frac{1}{\mathrm{Vol}(\Delta)}\int_\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{f(\mathbf{x}_0)+\ldots+f(\mathbf{x}_n)}{n+1}. \tag{2}$$

The following generalizations for convex function on disk and ball can be found in [3]: If $D(O,R) \subset \mathbb{R}^2$ is a disk and $f\colon D \to \mathbb{R}$ is convex and continuous, then

$$f(O) \leqslant \frac{1}{\pi R^2}\iint\limits_{D(O,R)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{2\pi R}\int\limits_{\partial D(O,R)} f(x,y)\,\mathrm{d}s$$

and If $B(O,R) \subset \mathbb{R}^3$ is a ball and $f\colon B \to \mathbb{R}$ is convex and continuous, then

$$f(O) \leqslant \frac{3}{4\pi R^3}\iiint\limits_{B(O,R)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leqslant \frac{1}{4\pi R^2}\iint\limits_{\partial B(O,R)} f(x,y,z)\,\mathrm{d}S.$$

The stronger version of the right-hand side of (1) ([9, page 140])

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right)$$

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also received generalizations for simplices [10], disks, 3-balls and regular n-gons P [2]:

$$\frac{1}{\mathrm{Vol}(\Delta)}\int_{\Delta} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{1}{n+1}f(\mathbf{b}) + \frac{n}{n+1}\,\frac{f(\mathbf{x}_0)+\ldots+f(\mathbf{x}_n)}{n+1}, \tag{3}$$

$$\frac{1}{\pi R^2}\iint\limits_{D(O,R)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{3}f(O) + \frac{2}{3}\cdot\frac{1}{2\pi R}\int\limits_{\partial D(O,R)} f(x,y)\,\mathrm{d}s,$$

$$\frac{3}{4\pi R^3}\iiint\limits_{B(O,R)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leqslant \frac{1}{4}f(O) + \frac{3}{4}\cdot\frac{1}{4\pi R^2}\iint\limits_{\partial B(O,R)} f(x,y,z)\,\mathrm{d}S,$$

$$\frac{1}{\mathrm{Area}(P)}\iint_P f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{3}f(O) + \frac{2}{3\,\mathrm{Perim}(P)}\int_{\partial P} f(x,y)\,\mathrm{d}s.$$

In this paper we use the lower and upper estimates for the average of a convex function over a simplex obtained by the authors in [7, 8]. We provide the alternate proof of the above results and then generalize them to figures and bodies satisfying some regularity conditions and to a broader class of functions.

2. Definitions and lemmas

Suppose $\mathbf{x}_0,\ldots,\mathbf{x}_n \in \mathbb{R}^n$ are the vertices of a simplex $\Delta \subset \mathbb{R}^n$.
For a nonempty set $K \subset \{0,\ldots,n\}$ we denote by Δ_K the simplex $\mathrm{conv}\{\mathbf{x}_i : i \in K\}$.
For every set $K \subsetneq \{0,\ldots,n\}$ we denote by $\Delta^{[K]}$ the simplex with vertices

$$\mathbf{x}_j^{[K]} = \frac{1}{n+1}\sum_{i\in K}\mathbf{x}_i + \frac{n+1-k}{n+1}\mathbf{x}_j, \quad j \in \{0,\ldots,n\}\setminus K.$$

We shall denote by $h_{\mathbf{a}}^{\lambda}$ the homothety with center $\mathbf{a}$ and scale λ, i.e. the mapping defined by the formula

$$h_{\mathbf{a}}^{\lambda}(\mathbf{x}) = \mathbf{a} + \lambda(\mathbf{x}-\mathbf{a}).$$

By ∂B we shall denote the boundary of the set B.

REMARK 2.1. In the plane the simplices $\Delta^{[K]}$ are: the triangle (if $K=\emptyset$), intersection of the triangle and a line parallel to one of its sides and passing through its barycenter (if K has one element) and the barycenter itself if K has two elements.

In case of three dimensions we have respectively: the tetrahedron, triangles parallel to its faces, segments parallel to its edges; all of them having the same barycenter.

Note that the simplices $\Delta^{[K]}$ can be obtained by applying homotheties to the faces of Δ. The details are explained in [7].

If $U \subset \mathbb{R}^k$ and $f\colon U \to \mathbb{R}$ is a Riemann integrable function, then by

$$\mathrm{Avg}(f,U) = \frac{1}{\mathrm{Vol}(U)}\int_U f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

we shall denote its average value over U. For simplicity of notation if $A,B,\dots,Z$ are points and $U=\operatorname{conv}\{A,B,\dots,Z\}$ we shall write $\operatorname{Avg}(f,AB\dots Z)$.

The following results provide the main tools for our investigations:

THEOREM 2.1. ([8]) *Suppose $f\colon \Delta\to\mathbb{R}$ is a convex function and $K,L\subset\{0,\dots,n\}$ are disjoint, nonempty sets. Then*

$$\operatorname{Avg}(f,\Delta_{K\cup L})\leqslant\frac{\operatorname{card}K}{\operatorname{card}K\cup L}\cdot\operatorname{Avg}(f,\Delta_K)+\frac{\operatorname{card}L}{\operatorname{card}K\cup L}\cdot\operatorname{Avg}(f,\Delta_L).$$

THEOREM 2.2. ([7]) *If $K\subset L$ are proper subsets of $\{0,\dots,n\}$ and $f\colon\Delta\to\mathbb{R}$ is a convex function, then*

$$f(\mathbf{b})\leqslant\operatorname{Avg}(f,\Delta^{[L]})\leqslant\operatorname{Avg}(f,\Delta^{[K]})\leqslant\operatorname{Avg}(f,\Delta).$$

The above theorems were proven by Chen [2] in case Δ is a triangle.

In the sequel we shall apply both theorems to some planar and 3-dimensional bodies.

3. Quadrilateral

Bessenyei in [1] proved that if $ABCD$ is a parallelogram and f is a convex function, then $\operatorname{Avg}(f,ABCD)\leqslant\frac{1}{4}(f(A)+f(B)+f(C)+f(D))$.

We will try to generalize and improve this result.

Consider a quadrilateral $ABCD$ such that the segment AC divides its area evenly (see Figure 1a).

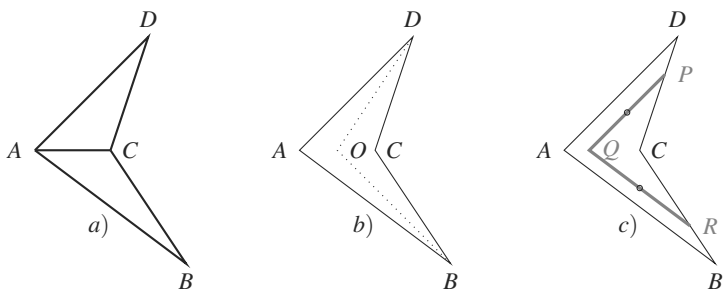


Figure 1: *Quadrilateral with equal halves*

We can apply Theorem 2.1 to both triangles ABC and ADC to obtain

$$\operatorname{Avg}(f,ABC)\leqslant\frac{1}{3}f(B)+\frac{2}{3}\operatorname{Avg}(f,AC),\tag{4}$$

$$\operatorname{Avg}(f,ACD)\leqslant\frac{1}{3}f(D)+\frac{2}{3}\operatorname{Avg}(f,AC),\tag{5}$$

which yields

$$\begin{aligned}\operatorname{Avg}(f,ABCD) &\leqslant \frac{1}{3}\left(\frac{f(B)+f(D)}{2}+2\operatorname{Avg}(f,AC)\right) \\ &\leqslant \frac{1}{3}\left(\frac{f(B)+f(D)}{2}+f(A)+f(C)\right).\end{aligned} \tag{6}$$

By adding a midpoint O of the segment AC we can get another upper bound (see Figure 1b):

$$\begin{aligned}\operatorname{Avg}(f,AOB) &\leqslant \frac{1}{3}f(O)+\frac{2}{3}\operatorname{Avg}(f,AB)\\ \operatorname{Avg}(f,BOC) &\leqslant \frac{1}{3}f(O)+\frac{2}{3}\operatorname{Avg}(f,BC)\\ \operatorname{Avg}(f,COD) &\leqslant \frac{1}{3}f(O)+\frac{2}{3}\operatorname{Avg}(f,CD)\\ \operatorname{Avg}(f,DOA) &\leqslant \frac{1}{3}f(O)+\frac{2}{3}\operatorname{Avg}(f,DA)\end{aligned}$$

and since the four triangles have the same area this produces

$$\operatorname{Avg}(f,ABCD) \leqslant \frac{f(O)}{3}+\frac{2}{3}\frac{\operatorname{Avg}(f,AB)+\operatorname{Avg}(f,BC)+\operatorname{Avg}(f,CD)+\operatorname{Avg}(f,DA)}{4} \tag{7}$$

$$\leqslant \frac{f(O)}{3}+\frac{2}{3}\frac{f(A)+f(B)+f(C)+f(D)}{4}. \tag{8}$$

Thus we have proven the following

THEOREM 3.1. *Let $ABCD$ be a quadrilateral such that the segment AC divides it into two triangles of equal area and O be the midpoint of AC. If $f\colon ABCD \to \mathbb{R}$ is such that its restrictions to triangles ABC and ACD are convex, then the inequalities* (6), (7) *and* (8) *hold.*

To obtain the lower bound we apply Theorem 2.2 to both triangles ABC and ADC. By Remark 2.1 we have four reasonable choices for each triangle, so we can produce 16 inequalities. An example is shown on the Figure 1c: the segments $PQ = h_C^{2/3}(DA)$ and $QR = h_C^{2/3}(AB)$ pass through the barycenters of both triangles and therefore $\operatorname{Avg}(f,PQ) \leqslant \operatorname{Avg}(f,ACD)$ and $\operatorname{Avg}(f,QR) \leqslant \operatorname{Avg}(f,ABC)$, which leads to

$$\frac{\operatorname{Avg}(f,PQ)+\operatorname{Avg}(f,QR)}{2} \leqslant \operatorname{Avg}(f,ABCD).$$

The reader will easily find two other pairs of segments for which Theorem 2.2 can be applied.

A parallelogram offers more opportunities: firstly, we obtain inequalities (4) and (5) with BD and AC swapped thus obtaining an improvement of Bessenyei's result

THEOREM 3.2. *Let $ABCD$ be a parallelogram with center O and $f\colon ABCD \to \mathbb{R}$ be such that its restrictions to triangles AOB, BOC, COD and DOA are convex, then the inequalities (7) and (8) hold and additionally*

$$\begin{aligned}\operatorname{Avg}(f,ABCD) &\leqslant \frac{1}{3}\min\left\{\tfrac{f(B)+f(D)}{2}+2\operatorname{Avg}(f,AC),\tfrac{f(A)+f(C)}{2}+2\operatorname{Avg}(f,BD)\right\}\\ &\leqslant \frac{1}{3}\min\left\{\tfrac{f(B)+f(D)}{2}+f(A)+f(C),\tfrac{f(A)+f(C)}{2}+f(B)+f(C)\right\}\\ &\leqslant \tfrac{f(A)+f(B)+f(C)+f(D)}{4}.\end{aligned}$$

The fact that the parallelogram can be divided into four triangles of equal area opens new opportunities. For example we can apply Theorem 2.1 to AOB (and then cyclically to others) as follows:

$$\operatorname{Avg}(f,AOB) \leqslant \frac{1}{3}f(A)+\frac{2}{3}\operatorname{Avg}(f,OB).$$

Summing and taking into account that the point O halves both diagonals we get

THEOREM 3.3. *Under the assumption of Theorem 3.2*

$$\operatorname{Avg}(f,ABCD) \leqslant \frac{1}{3}\frac{f(A)+f(B)+f(C)+f(D)}{4}+\frac{2}{3}\frac{\operatorname{Avg}(f,AC)+\operatorname{Avg}(f,BD)}{2}.$$

The reader will find more estimates applicable to parallelograms and rhombus in Section 4 devoted to polygons.

As above using Theorem 2.2 we can produce 64 different lower bounds. Figure 2 illustrates two, probably the most spectacular, inequalities:

THEOREM 3.4. *Under the assumption of Theorem 3.2 let $A'B'C'D' = h_O^{2/3}(ABCD)$, $KL = h_D^{2/3}(AO)$, $LM = h_D^{2/3}(OC)$, $PQ = h_B^{2/3}(AO)$, $QR = h_B^{2/3}(OC)$. Then the following inequalities hold (see Figure 2)*

$$\frac{\operatorname{Avg}(f,KL)+\operatorname{Avg}(f,LM)+\operatorname{Avg}(f,PQ)+\operatorname{Avg}(f,QR)}{4} \leqslant \operatorname{Avg}(f,ABCD),$$

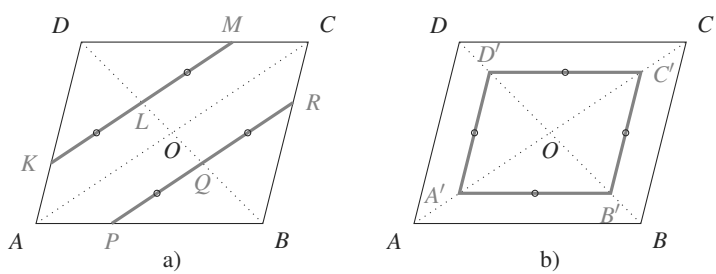


Figure 2: *Lower bounds for parallelogram*

$$\frac{\mathrm{Avg}(f,A'B')+\mathrm{Avg}(f,B'C')+\mathrm{Avg}(f,C'D')+\mathrm{Avg}(f,D'A')}{4}\leqslant \mathrm{Avg}(f,ABCD).$$

REMARK 3.1. If f is convex on the parallelogram, then obviously f is convex on all four triangles. The converse does not hold. In both cases one can easily conclude the inequality

$$\frac{1}{4}\sum_{k=1}^{4} f(O_k)\leqslant \mathrm{Avg}(f,ABCD),$$

where O_k's are the barycenters of the triangles. This inequality in case of convex f yields $f(O)\leqslant \mathrm{Avg}(f,ABCD)$. In the case of convexity on triangles only this may not be true (consider the quadrilateral $\{(x,y):|x|+|y|=1\}$ and $f(x,y)=-|y|$).

4. Fans and n-gons

We shall call *nice* a star-shaped polygon P with vertices $A_0,\dots,A_{n-1}$ satisfying the following condition: there exists a point O called *center* in the kernel of P such that all triangles OA_kA_{k+1}, $k=0,\dots,n-1$ are of the same area. If additionally all segments A_kA_{k+1} are of the same length, then we shall call it *very nice*.

Regular n-gons are very nice, but the class of nice and very nice polygons is much broader.

THEOREM 4.1. *If P is a nice polygon and $f:P\to\mathbb{R}$ is convex on every triangle formed by its center O and two consecutive vertices, then*

$$\mathrm{Avg}(f,P)\leqslant \frac{1}{3}f(O)+\frac{2}{3}\cdot\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Avg}(f,A_kA_{k+1}), \tag{9}$$

$$\mathrm{Avg}(f,P)\leqslant \frac{1}{3}\cdot\frac{1}{n}\sum_{k=0}^{n-1}f(A_k)+\frac{2}{3}\cdot\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Avg}(f,OA_k), \tag{10}$$

$$\mathrm{Avg}(f,P)\leqslant \frac{1}{3}f(O)+\frac{2}{3}\cdot\frac{1}{n}\sum_{k=0}^{n-1}f(A_k). \tag{11}$$

If additionally $A'_k=h_O^{2/3}(A_k)$ for $k=0,\dots,n-1$, then

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Avg}(f,A'_kA'_{k+1})\leqslant \mathrm{Avg}(f,P) \tag{12}$$

(see Figure 3).

Proof. To obtain (9) apply Theorem 2.1 to vertex O and side A_kA_{k+1}, then add up the inequalities. Similarly, for (10) use vertex A_k and side OA_{k+1}. The inequality (11) can be obtained from (9) or (10) by applying standard Hermite-Hadamard inequalities.

Finally (12) is consequence of Theorem 2.2. $\square$

REMARK 4.1. Every triangle is a nice polygon with its barycenter as O.

REMARK 4.2. In case of a very nice polygon, the inequalities (9) and (12) can be written as

$$\mathrm{Avg}(f,\partial h_O^{2/3}(P)) \leqslant \mathrm{Avg}(f,P) \leqslant \frac{1}{3}f(O)+\frac{2}{3}\mathrm{Avg}(f,\partial P).$$

Suppose n is even. Then we can group the triangles in pairs to get

$$\mathrm{Avg}(f,OA_kA_{k+1}) \leqslant \frac{1}{3}f(A_{k+1})+\frac{2}{3}\mathrm{Avg}(f,OA_k)$$
$$\mathrm{Avg}(f,OA_{k-1}A_k) \leqslant \frac{1}{3}f(A_{k-1})+\frac{2}{3}\mathrm{Avg}(f,OA_k).$$

This shows that the following result holds true.

THEOREM 4.2. *Under assumptions of Theorem 4.1 if the number of vertices of P is even, then*

$$\mathrm{Avg}(f,P) \leqslant \frac{1}{3}\frac{1}{n/2}\sum_{k\ odd} f(A_k)+\frac{2}{3}\frac{1}{n/2}\sum_{k\ even}\mathrm{Avg}(f,OA_k),$$
$$\mathrm{Avg}(f,P) \leqslant \frac{1}{3}\frac{1}{n/2}\sum_{k\ even} f(A_k)+\frac{2}{3}\frac{1}{n/2}\sum_{k\ odd}\mathrm{Avg}(f,OA_k),$$

(see Figure 3).

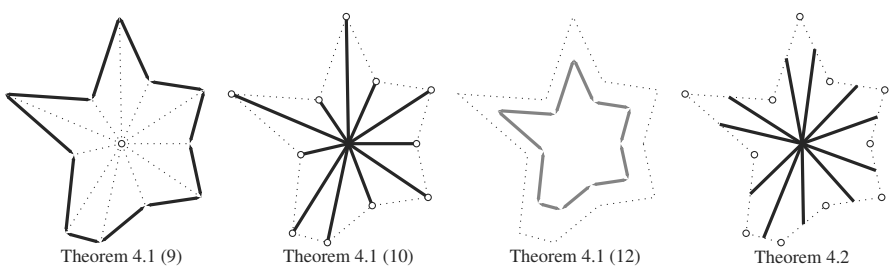


Figure 3: *Hermite-Hadamard inequalities for n-gon*

REMARK 4.3. Note that every nice n-gon can be considered a nice $2n$-gon by adding the midpoints of its sides to the set of vertices (see Figure 3).

Remark 3.1 remains valid also in this case.

A fan is a polygon with vertices $O,A_1,\dots,A_n$ such that all triangles OA_kA_{k+1}, $k=1,\dots,n-1$ are of the same orientation and $\sum_{k=1}^{n-1}\angle A_kOA_{k+1}<2\pi$. For nice fans we obtain similar results as for nice n-gons. We encourage the reader to formulate an equivalent of Theorems 4.1 and 4.2.

5. Annulus

In [2] the following version of Hermite-Hadamard inequality can be found

THEOREM 5.1. ([2], Th. 2.2) *Let U be a convex subset of a plane, and $D \subset U$ be an annulus with radii $r < R$ and $C(s)$ denotes a co-centric circle with radius s, then for a convex function $f\colon U \to \mathbb{R}$ hold*

$$\operatorname{Avg}\left(f, C\left(\tfrac{2(r^2+rR+R^2)}{3(r+R)}\right)\right) \leqslant \operatorname{Avg}(f,D)$$

and

$$\operatorname{Avg}(f,D) \leqslant \frac{2r+R}{3(r+R)}\operatorname{Avg}(f,C(r)) + \frac{r+2R}{3(r+R)}\operatorname{Avg}(f,C(R)).$$

We shall improve this result. For $n > 4$ let A_k^n, $k = 0,\dots,n-1$ be the vertices of a regular n-gon inscribed in $C(R)$ and B_k^n, $k = 0,\dots,n-1$ be the vertices of a regular n-gon inscribed in $C(r)$ and rotated anticlockwise by $\frac{\pi}{n}$. The two polygons bound the area D_n, and divide it into $2n$ isosceles triangles $\mathscr{K}_k^n = A_k^n A_{k+1}^n B_k^n$ and $\mathscr{L}_k^n = B_k^n B_{k+1}^n A_{k+1}^n$. We have

$$\operatorname{Area}\mathscr{K}_k^n = R\sin\frac{\pi}{n}\left(R\cos\frac{\pi}{n} - r\right), \quad \operatorname{Area}\mathscr{L}_k^n = r\sin\frac{\pi}{n}\left(R - r\cos\frac{\pi}{n}\right), \tag{13}$$

$$\operatorname{Area}D_n = \sum_{k=0}^{n-1}(\operatorname{Area}\mathscr{K}_k^n + \operatorname{Area}\mathscr{L}_k^n) = n\sin\frac{\pi}{n}\cos\frac{\pi}{n}(R^2 - r^2). \tag{14}$$

Denote by K_k^n, L_k^n the barycenters of $\mathscr{K}_k^n$ and $\mathscr{L}_k^n$. Applying the Hermite-Hadamard inequality, equations (13) and (14) we obtain

$$\begin{aligned}
\operatorname{Avg}(f,D_n) &= \frac{1}{\operatorname{Area}D_n}\sum_{k=0}^{n-1}\left(\int_{\mathscr{K}_k^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathscr{L}_k^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\\
&= \sum_{k=0}^{n-1}\left(\frac{\operatorname{Area}\mathscr{K}_k^n}{\operatorname{Area}D_n}\operatorname{Avg}(f,\mathscr{K}_k^n) + \frac{\operatorname{Area}\mathscr{L}_k^n}{\operatorname{Area}D_n}\operatorname{Avg}(f,\mathscr{L}_k^n)\right)\\
&\geqslant \frac{R\left(R\cos\frac{\pi}{n} - r\right)}{\cos\frac{\pi}{n}(R^2-r^2)}\frac{1}{n}\sum_{k=0}^{n-1} f(K_k^n) + \frac{r\left(R - r\cos\frac{\pi}{n}\right)}{\cos\frac{\pi}{n}(R^2-r^2)}\frac{1}{n}\sum_{k=0}^{n-1} f(L_k^n).
\end{aligned} \tag{15}$$

As n tends to infinity the two regular polygons with vertices K_k^n and L_k^n respectively approach the circles of radii $\frac{1}{3}(2R+r)$ and $\frac{1}{3}(R+2r)$, and the arithmetic means in (15) tend to averages of f over these circles, so we have proven the following fact.

THEOREM 5.2. *Under the assumptions of Theorem 5.1 the inequality*

$$\frac{R}{r+R}\operatorname{Avg}\left(f, C\left(\frac{r+2R}{3}\right)\right) + \frac{r}{r+R}\operatorname{Avg}\left(f, C\left(\frac{2r+R}{3}\right)\right) \leqslant \operatorname{Avg}(f,D)$$

holds.

Similar reasoning and the strengthened version of the Hermite-Hadamard inequality (3) applied to $\mathscr{K}_k^n$ and $\mathscr{L}_k^n$ produce a better right bound.

THEOREM 5.3. *Under the assumptions of Theorem 5.1 the inequality*

$$\begin{aligned}\operatorname{Avg}(f,D) \leqslant \frac{1}{3}\left[\frac{R}{r+R}\operatorname{Avg}\left(f,C\left(\frac{r+2R}{3}\right)\right)+\frac{r}{r+R}\operatorname{Avg}\left(f,C\left(\frac{2r+R}{3}\right)\right)\right]\\ +\frac{2}{3}\left[\frac{r+2R}{3(r+R)}\operatorname{Avg}(f,C(R))+\frac{2r+R}{3(r+R)}\operatorname{Avg}(f,C(r))\right]\end{aligned}$$

holds.

REMARK 5.1. Suppose the function f is such there exist a point $O \in U$ and half-lines starting from O such that f is convex in each sector bounded by them. Then, if O is the center of D, the inequalities in (15) are valid for all triangles except these intersecting with the sectors' boundaries. Thus they can be neglected as n tends to infinity, and the Theorems 5.2 and 5.3 remain valid for f.

6. Platonic bodies and related polytopes

Let $B \subset \mathbb{R}^3$ be a platonic body inscribed in a sphere with center O. Define the following sets:

- $\mathscr{S}$ – set of segments joining O with vertices of B
- $\mathscr{O}$ – set of segments joining O with centers of faces
- $\mathscr{E}$ – set of edges of B
- $\mathscr{D}$ – set of segments joining centers of faces with their vertices.

THEOREM 6.1. *Let $f\colon B \to \mathbb{R}$ be a function such that its restriction to every pyramid formed by a face as a base and O as its apex is convex. Then with the above notation the following inequalities hold:*

$$\operatorname{Avg}(f,B) \leqslant \frac{1}{4}f(O)+\frac{3}{4}\operatorname{Avg}(f,\partial B), \tag{16}$$

$$\operatorname{Avg}(f,B) \leqslant \frac{1}{2}\operatorname{Avg}(f,\mathscr{O})+\frac{1}{2}\operatorname{Avg}(f,\mathscr{E}), \tag{17}$$

$$\operatorname{Avg}(f,B) \leqslant \frac{1}{2}\operatorname{Avg}(f,\mathscr{S})+\frac{1}{2}\operatorname{Avg}(f,\mathscr{D}), \tag{18}$$

and

$$\operatorname{Avg}(f,\partial h_O^{3/4}(B)) \leqslant \operatorname{Avg}(f,B). \tag{19}$$

Proof. Let F be a face of B with vertices $A_0,\dots,A_{n-1}$ and O' be its circumcenter. Split the pyramid FO into simplices $OO'A_kA_{k+1}$. Applying Theorem 2.1 we obtain

$$\operatorname{Avg}(f,OO'A_kA_{k+1}) \leqslant \frac{1}{4}f(O)+\frac{3}{4}\operatorname{Avg}(f,O'A_kA_{k+1}).$$

Summing over k and F we obtain (16). The inequalities

$$\operatorname{Avg}(f,OO'A_kA_{k+1}) \leqslant \frac{1}{2}\operatorname{Avg}(f,OO')+\frac{1}{2}\operatorname{Avg}(f,A_kA_{k+1})$$

lead to (17), while

$$\operatorname{Avg}(f,OO'A_kA_{k+1}) \leqslant \frac{1}{2}\operatorname{Avg}(f,OA_k)+\frac{1}{2}\operatorname{Avg}(f,O'A_{k+1})$$

give (18). Finally Theorem 2.2 leads to the inequalities

$$\operatorname{Avg}(f,h_O^{3/4}(O'A_kA_{k+1})) \leqslant \operatorname{Avg}(f,OO'A_kA_{k+1})$$

that finally yield (19). $\square$

Based on a platonic body B we can build a new polytope B^* in the following way: on every face F of B we build or excavate a regular pyramid of the same height with apex O_F. If we denote by

- $\mathscr{S}$ – set of segments joining O with vertices of B,
- $\mathscr{O}^*$ – set of segments joining O with O_F's,
- $\mathscr{E}$ – set of edges of B,
- $\mathscr{D}^*$ – set of segments joining O_F's of the pyramids with vertices of F,

then the same reasoning as above shows that the next theorem is valid.

THEOREM 6.2. *Let $f\colon B^*\to\mathbb{R}$ be a function such that its restriction to every tetrahedron formed by O, O_F and two adjacent vertices of F. Then with the above notation the following inequalities hold:*

$$\begin{aligned}
\operatorname{Avg}(f,B^*) &\leqslant \frac{1}{4}f(O)+\frac{3}{4}\operatorname{Avg}(f,\partial B^*),\\
\operatorname{Avg}(f,B^*) &\leqslant \frac{1}{2}\operatorname{Avg}(f,\mathscr{O}^*)+\frac{1}{2}\operatorname{Avg}(f,\mathscr{E}),\\
\operatorname{Avg}(f,B^*) &\leqslant \frac{1}{2}\operatorname{Avg}(f,\mathscr{S})+\frac{1}{2}\operatorname{Avg}(f,\mathscr{D}^*),
\end{aligned}$$

and

$$\operatorname{Avg}(f,\partial h_O^{3/4}(B^*)) \leqslant \operatorname{Avg}(f,B^*).$$

7. Dipyramid and dicone

Let P be a regular, convex n-gon with vertices $A_0,\dots,A_{n-1}$. Suppose X is a point on the line l perpendicular to the plane containing P and passing through its center. The set $\mathscr{U}_X=\bigcup_{k=0}^{n-1}XA_kA_{k+1}$ will be called a *Chinese umbrella with vertex* X. The umbrella's scaffold will be denoted by $\mathscr{S}_X=\bigcup_{k=0}^{n-1}XA_k$.

By a *dipyramid* with vertices O_0,O_1 we mean the body $\mathscr{D}$ bounded by two Chinese umbrellas $\mathscr{U}_{O_0}$ and $\mathscr{U}_{O_1}$. The dipyramid may be convex or not, depending on the position of its vertices with respect to the plane of the polygon.

The following result is a consequence of Theorems 2.1 and 2.2.

THEOREM 7.1. *Let $\mathscr{D}$ be a dipyramid with vertices O_0 and O_1, and let $f:\mathscr{D}\to\mathbb{R}$ be a function that is convex on every simplex $O_0O_1A_kA_{k+1}$, $k=0,\dots,n-1$. Then*

$$\operatorname{Avg}(f,\mathscr{D})\leqslant\tfrac{1}{4}f(O_0)+\tfrac{3}{4}\operatorname{Avg}(f,\mathscr{U}_{O_1}),\tag{20}$$

$$\operatorname{Avg}(f,\mathscr{D})\leqslant\tfrac{1}{4}f(O_1)+\tfrac{3}{4}\operatorname{Avg}(f,\mathscr{U}_{O_0}),\tag{21}$$

$$\operatorname{Avg}(f,h_{O_0}^{3/4}(\mathscr{U}_{O_1}))\leqslant\operatorname{Avg}(f,\mathscr{D}),\tag{22}$$

$$\operatorname{Avg}(f,h_{O_1}^{3/4}(\mathscr{U}_{O_0}))\leqslant\operatorname{Avg}(f,\mathscr{D}),\tag{23}$$

$$\operatorname{Avg}(f,\mathscr{D})\leqslant\tfrac{1}{2}\left(\operatorname{Avg}(f,O_0O_1)+\operatorname{Avg}(f,\partial P)\right),\tag{24}$$

$$\operatorname{Avg}(f,\mathscr{D})\leqslant\tfrac{1}{2}\left(\operatorname{Avg}(f,\mathscr{S}_{O_0})+\operatorname{Avg}(f,\mathscr{S}_{O_1})\right),\tag{25}$$

(see Figure 4).

Proof. Grouping the vertices of the simplex $O_0O_1A_kA_{k+1}$ into $\{O_0\},\{O_1A_kA_{k+1}\}$ one gets the inequalities (20) and (22). Similar split $\{O_1\},\{O_0A_kA_{k+1}\}$ gives (21) and (23).

Inequalities (24) and (25) follow by grouping them into $\{O_0O_1\}$, $\{A_kA_{k+1}\}$ and $\{O_0A_k\}$, $\{O_1A_{k+1}\}$ respectively. □

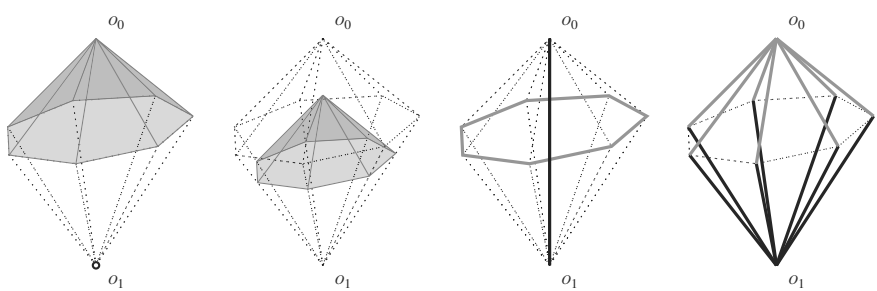


Figure 4: *Hermite-Hadamard inequalities for dipyramid*

Denote by η_i, $i=0,1$ the angle between the line l and the plane of a side of $\mathscr{U}_{O_i}$ and for $0<s<1$ let $O_s=(1-s)O_0+sO_1$. The umbrella $\mathscr{U}_{O_s}$ splits the dipyramid $\mathscr{D}$ into two dipyramids $\mathscr{D}_s^0$ and $\mathscr{D}_s^1$ with vertices O_0,O_s and O_1,O_s respectively. It is clear, that

$$\mathrm{Vol}(\mathscr{D}_s^0)=s\mathrm{Vol}(\mathscr{D}) \quad \text{and} \quad \mathrm{Vol}(\mathscr{D}_s^1)=(1-s)\mathrm{Vol}(\mathscr{D}). \tag{26}$$

Note also that

$$\frac{\mathrm{Area}(\mathscr{U}_{O_0})}{\mathrm{Area}(\mathscr{U}_{O_1})}=\frac{\sin\eta_1}{\sin\eta_0}$$

which gives

$$\mathrm{Area}(\partial\mathscr{D})=\frac{\sin\eta_0+\sin\eta_1}{\sin\eta_1}\,\mathrm{Area}(\mathscr{U}_{O_0})=\frac{\sin\eta_0+\sin\eta_1}{\sin\eta_0}\,\mathrm{Area}(\mathscr{U}_{O_1}). \tag{27}$$

Now we are ready to generalize the results of Theorem 7.1.

THEOREM 7.2. *Under the assumptions of Theorem 7.1 for all* $0<s<1$ *the inequalities*

$$\mathrm{Avg}(f,\mathscr{D})\leqslant \tfrac{1}{4}f(O_s)+\tfrac{3}{4}\left(s\,\mathrm{Avg}(f,\mathscr{U}_{O_0})+(1-s)\,\mathrm{Avg}(f,\mathscr{U}_{O_1})\right), \tag{28}$$

$$s\,\mathrm{Avg}(f,h_{O_s}^{3/4}(\mathscr{U}_{O_0}))+(1-s)\,\mathrm{Avg}(f,h_{O_s}^{3/4}(\mathscr{U}_{O_1}))\leqslant \mathrm{Avg}(f,\mathscr{D}), \tag{29}$$

$$\mathrm{Avg}(f,\partial h_{O_s}^{3/4}(\mathscr{D}))\leqslant \frac{\sin\eta_1\,\mathrm{Avg}(f,\mathscr{D}_s^0)}{\sin\eta_0+\sin\eta_1}+\frac{\sin\eta_0\,\mathrm{Avg}(f,\mathscr{D}_s^1)}{\sin\eta_0+\sin\eta_1} \tag{30}$$

are valid.

Proof. To prove (28) we apply inequality (20) to $\mathscr{D}_s^0$ and to $\mathscr{D}_s^1$ and obtain

$$\begin{aligned}\mathrm{Avg}(f,\mathscr{D}_s^0)&\leqslant \tfrac{1}{4}f(O_s)+\tfrac{3}{4}\,\mathrm{Avg}(f,\mathscr{U}_{O_0}),\\ \mathrm{Avg}(f,\mathscr{D}_s^1)&\leqslant \tfrac{1}{4}f(O_s)+\tfrac{3}{4}\,\mathrm{Avg}(f,\mathscr{U}_{O_1}).\end{aligned}$$

Now we multiply the first inequality by s, the second by $1-s$ and add up both inequalities taking into account equalities (26).

The proof of (29) is similar but uses (22) and (23).

And finally from (22), (23) and (27) (which remains valid for homothetic images also) it follows that

$$\frac{\sin\eta_0+\sin\eta_1}{\sin\eta_{1-i}}\,\frac{1}{\mathrm{Area}(\partial h_{O_s}^{3/4}(\mathscr{D}))}\int_{h_{O_s}^{3/4}(\mathscr{U}_{O_i})} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\leqslant \mathrm{Avg}(f,\mathscr{D}_s^i),\quad i=0,1.$$

We complete the proof by dividing these inequalities by the first term and adding side by side, because $\partial h_{O_s}^{3/4}(\mathscr{D})=h_{O_s}^{3/4}(\mathscr{U}_{O_0})\cup h_{O_s}^{3/4}(\mathscr{U}_{O_1})$. $\square$

COROLLARY 7.3. *Let* $s^* = \frac{\sin\eta_0}{\sin\eta_0+\sin\eta_1}$. *The equations* (27), (26) *and Theorem* 7.2 *imply that*

$$\operatorname{Avg}(f,\mathscr{D}) \leqslant \tfrac{1}{4}f(O_{s^*}) + \tfrac{3}{4}\operatorname{Avg}(f,\partial\mathscr{D}),$$
$$\operatorname{Avg}(f,\partial h^{3/4}_{O_{s^*}}(\mathscr{D})) \leqslant \operatorname{Avg}(f,\mathscr{D}).$$

With n growing to infinity our dipyramids approximate a *dicone*, where the n-gon P gets replaced by a circle of radius R. All formulas (20)–(24), (28)–(30) remain valid, while the formula (25) needs a modification.

Let us introduce a coordinate system in the most natural way (center O at the center of P, z-axis along the line l and x-axis along OA_0). Then we have

$$\operatorname{Avg}(f,\mathscr{S}_{O_0}) = \frac{1}{n|O_0A_0|}\sum_{k=0}^{n-1}\int_{O_0A_k} f(x,y,z)\,\mathrm{d}l$$

with $x = r\cos\varphi,\ y = r\sin\varphi,\ z = (R-r)\cot\eta_0,\ 0 \leqslant r \leqslant R$

$$\begin{aligned}
&= \frac{1}{nR}\sum_{k=0}^{n-1}\int_0^R f\left(r\cos\tfrac{2\pi k}{n}, r\sin\tfrac{2\pi k}{n}, (R-r)\cot\eta_0\right)\mathrm{d}r\\
&\to \frac{1}{2\pi R}\int_0^R\int_0^{2\pi} f\left(r\cos\varphi, r\sin\varphi, (R-r)\cot\eta_0\right)\mathrm{d}\varphi\,\mathrm{d}r\\
&= \frac{1}{2\pi R}\iint\limits_{x^2+y^2\leqslant R^2} f\left(x,y,(R-\sqrt{x^2+y^2})\cot\eta_0\right)\frac{1}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{\sin\eta_0}{2\pi R}\iint\limits_{\mathscr{U}_{O_0}} \frac{f(x,y,z)}{\sqrt{x^2+y^2}}\,\mathrm{d}S.
\end{aligned}$$

Therefore the following result holds.

THEOREM 7.4. *If* f *is a convex function defined on a dicone* $\mathscr{D}$, *and* $g(\mathbf{x}) = \frac{f(\mathbf{x})}{\operatorname{dist}(\mathbf{x},l)}$ *(*$\operatorname{dist}(\mathbf{x},l)$ *denotes the distance from* $\mathbf{x}$ *to the line* l*), then*

$$\operatorname{Avg}(f,\mathscr{D}) \leqslant \frac{R}{4}\left(\operatorname{Avg}(g,\mathscr{U}_{O_0}) + \operatorname{Avg}(g,\mathscr{U}_{O_1})\right).$$

8. Cube

A cube being a Platonic body enjoys all properties discussed in Section 6. In this section we present a handful of other applications of Theorems 2.1 and 2.2.

THEOREM 8.1. *Let* $\mathscr{C}$ *be a cube, and* $f\colon \mathscr{C} \to \mathbb{R}$ *be a function convex on every pyramid formed by a face and the center* O *of the cube. Fix two opposite vertices of a cube and let* P *be a set being the sum of six diagonals of faces meeting at these vertices. Let* Q *be the set of three main diagonals joining the remaining six vertices. Then*

$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \frac{1}{2}\operatorname{Avg}(f,P) + \frac{1}{2}\operatorname{Avg}(f,Q).$$

Proof. Let $ABCD$ be the face containing diagonal AC in P. The pyramid $ABCDO$ is the sum of two simplices $ABCO$ and $ACDO$. Splitting vertices of each of them into groups $\{AC\},\{BO\}$ and $\{AC\},\{DO\}$ respectively one gets

$$\operatorname{Avg}(f,ABCO) \leqslant \frac{1}{2}\left(\operatorname{Avg}(f,AC)+\operatorname{Avg}(f,BO)\right)$$
$$\operatorname{Avg}(f,ACDO) \leqslant \frac{1}{2}\left(\operatorname{Avg}(f,AC)+\operatorname{Avg}(f,DO)\right)$$

which gives

$$2\operatorname{Avg}(f,ABCDO) \leqslant \operatorname{Avg}(f,AC)+\frac{1}{2}(\operatorname{Avg}(f,BO)+\operatorname{Avg}(f,DO)).$$

We complete the proof in usual way, applying the same process to all diagonals in P. $\square$

THEOREM 8.2. *Let $\mathscr{C}$ be a cube, and $f\colon \mathscr{C}\to\mathbb{R}$ be a function convex on every pyramid formed by a face and the center O of the cube. Fix two opposite vertices of $\mathscr{C}$ and let S be a set being the sum of edges meeting at these vertices. Let Q be the set of three main diagonals joining the remaining six vertices. Then*

$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \frac{1}{2}\operatorname{Avg}(f,S)+\frac{1}{2}\operatorname{Avg}(f,Q).$$

Proof. The proof goes exactly the same way as the previous one, but this time we split the vertices of simplices into groups $\{CO\},\{AB\}$ and $\{CO\},\{AD\}$ respectively which leads to

$$2\operatorname{Avg}(f,ABCDO) \leqslant \operatorname{Avg}(f,OC)+\frac{1}{2}(\operatorname{Avg}(f,AB)+\operatorname{Avg}(f,AD)). \quad \square$$

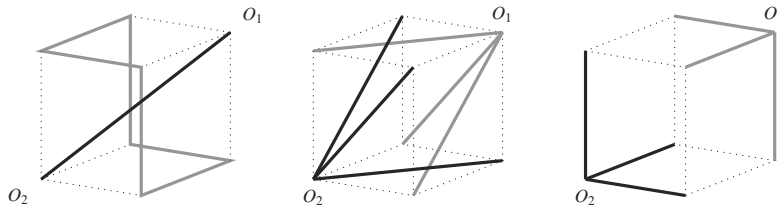
The cube can be split into six simplices of equal volumes in different ways. One of them is particularly interesting – we shall call it a *diagonal split*. Select two opposite vertices, say O_1 and O_2. The remaining vertices can be connected by edges of the cube so that they form a closed polygonal line $L=V_0\ldots V_5V_0$. The diagonal split consists of six simplices $O_1O_2V_kV_{k+1}$. Note that O_1V_k and O_2V_{k+1} are of the same length – they are both edges of the cube or diagonals of its faces. The next theorem shows how the diagonal split can be explored.

THEOREM 8.3. *Consider a diagonal split of a cube $\mathscr{C}$. Denote by S the set of six edges adjacent to O_1 and O_2 and by P the set of six diagonals of faces adjacent to O_1 and O_2. If $f\colon\mathscr{C}\to\mathbb{R}$ is convex on each simplex of the split, then*

$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \frac{1}{2}\operatorname{Avg}(f,O_1O_2)+\frac{1}{2}\operatorname{Avg}(f,L), \tag{31}$$
$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \operatorname{Avg}(f,P), \tag{32}$$
$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \operatorname{Avg}(f,S), \tag{33}$$

(see Figure 5).

Proof. To prove (31) use Theorem 2.1 dividing the vertices of $O_1O_2V_kV_{k+1}$ into groups $\{O_1O_2\}$ and $\{V_kV_{k+1}\}$. Two other splits lead to (32) and (33). $\square$



Theorem 8.3 (31) Theorem 8.3 (32) Theorem 8.3 (33)

Figure 5: *Hermite-Hadamard inequalities for cube*

Next theorem presents an interesing asymmetric case:

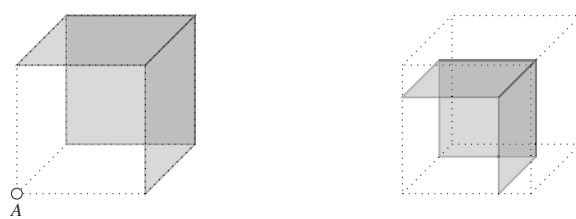
THEOREM 8.4. *Let A be a vertex of the cube $\mathscr{C}$ and let S be the set consisting of its faces nonadjacent to A. If $f : \mathscr{C} \to \mathbb{R}$ is convex, then*

$$\operatorname{Avg}(f,\mathscr{C}) \leqslant \frac{1}{4}f(A) + \frac{3}{4}\operatorname{Avg}(f,S), \tag{34}$$

$$\operatorname{Avg}(f,h_A^{3/4}(S)) \leqslant \operatorname{Avg}(f,\mathscr{C}), \tag{35}$$

(see Figure 6).

We leave the obvious proof to the reader.



Theorem 8.4 (34) Theorem 8.4 (35)

Figure 6: *Hermite-Hadamard inequalities for cube ctd.*

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