

A REAL COUNTEREXAMPLE TO TWO INEQUALITIES INVOLVING PERMANENTS

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Abstract. The objective of this article is to provide a counterexample to the permanent version of Oppenheim’s inequality using real symmetric matrices. This also provides a counterexample to the permanent on top conjecture for real symmetric positive semidefinite matrices.

1. Introduction

In [2], Bapat and Sunder raised the question of whether the inequality

$$\text{per}(A \circ B) \leq \text{per}(A) \prod_{j=1}^n b_{jj} \tag{1}$$

holds for hermitian positive semidefinite $n \times n$ matrices A and B . The quantity $\text{per}(A)$ denotes the permanent of A and the notation $A \circ B$ is for the Hadamard (entrywise) product of A and B . This is the permanental version of Oppenheim’s inequality. It was disproved in [4]. In the example presented there $B = A'$ and A is a complex 7×7 correlation matrix. It is the objective of the current article to provide a counterexample with $B = A$ and A a real correlation matrix. The reader may consult [1], [3] and [5] for additional information about the permanental Oppenheim inequality.

PERMANENT ON TOP CONJECTURE. For a positive semidefinite $n \times n$ matrix A , define the convolution operator $\Pi(A)$ on the symmetric group S_n by its matrix

$$\Pi(A)_{\sigma, \rho} = \prod_{j=1}^n a_{\sigma(j), \rho(j)}.$$

Then $\text{per}(A)$ is the largest eigenvalue of $\Pi(A)$.

The permanent on top conjecture was recently disproved by Shchesnovich [6] using a matrix with complex entries. The matrix presented below gives a real matrix counterexample since the permanent on top conjecture for a real positive semidefinite matrix A implies (1) for all real symmetric positive semidefinite matrices A and B .

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To understand why, suppose that the permanent on top conjecture holds for a real positive semidefinite matrix A and that B is a real correlation matrix of the same size. Then we may write $B = X'X$ where X is a real matrix. Then, denoting M_n the space of all mappings from $\{1, \dots, n\}$ to itself and S_n the symmetric group on $\{1, \dots, n\}$ we have

$$\prod_{j=1}^n b_{\sigma(j),\rho(j)} = \sum_{\lambda \in M_n} \prod_{j=1}^n x_{\lambda(j),\sigma(j)} \prod_{j=1}^n x_{\lambda(j),\rho(j)}$$

and therefore

$$\begin{aligned} n! \text{per}(A \circ B) &= \sum_{\sigma, \rho \in S_n} \sum_{\lambda \in M_n} \prod_{j=1}^n x_{\lambda(j),\sigma(j)} \prod_{j=1}^n a_{\sigma(j),\rho(j)} \prod_{j=1}^n x_{\lambda(j),\rho(j)} \\ &= \sum_{\lambda \in M_n} \xi'_\lambda \Pi(A) \xi_\lambda \\ &\leq \sum_{\lambda \in M_n} \|\Pi(A)\| \|\xi_\lambda\|^2 \\ &\leq \text{per}(A) \sum_{\lambda \in M_n} \sum_{\sigma \in S_n} \prod_{j=1}^n x_{\lambda(j),\sigma(j)}^2 \\ &= \text{per}(A) \sum_{\sigma \in S_n} \prod_{j=1}^n b_{\sigma(j),\sigma(j)} \\ &= n! \text{per}(A) \end{aligned}$$

where ξ_λ denotes the function

$$\xi_\lambda(\sigma) = \prod_{j=1}^n x_{\lambda(j),\sigma(j)}$$

written as a column vector and $\|\Pi(A)\|$ denotes the operator norm (largest eigenvalue) of $\Pi(A)$.

2. The Counterexample

With $n = 16$, we will take and $B = A$. We describe the matrix A informally. Take a regular dodecahedron circumscribed in the unit sphere centred at the origin in 3-dimensional Euclidean space. It has 20 vertices on the unit sphere. Now take the central point of each face and project that radially onto the unit sphere. This gives twelve points, the vertices of a regular icosahedron also circumscribed in the unit sphere. Consider all 32 vertex points. Clearly they occur in antipodal pairs. For each of the 16 antipodal pairs, select one of its points. Now let X be the 16×3 matrix in which each row is the position vector of the corresponding selected point. There are many ways of doing this. One yields

$$X = \begin{pmatrix} a_+ & a_- & \frac{1}{\sqrt{3}} \\ a_+ & -b_+ & -d_- \\ a_+ & -b_+ & d_- \\ a_+ & a_- & -\frac{1}{\sqrt{3}} \\ a_+ & 2b_- & 0 \\ a_- & b_- & d_+ \\ a_- & -a_+ & \frac{1}{\sqrt{3}} \\ a_- & -a_+ & -\frac{1}{\sqrt{3}} \\ a_- & b_- & -d_+ \\ a_- & 2b_+ & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{5}} & -c_-^2 & c_+ \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -c_-^2 & -c_+ \\ \frac{1}{\sqrt{5}} & c_+^2 & -c_- \\ \frac{1}{\sqrt{5}} & c_+^2 & c_- \end{pmatrix}$$

where $a_{\pm} = \frac{\sqrt{75 \pm 30\sqrt{5}}}{15}$, $b_{\pm} = \frac{\sqrt{150 \pm 30\sqrt{5}}}{30}$, $c_{\pm} = \frac{\sqrt{50 \pm 10\sqrt{5}}}{10}$ and $d_{\pm} = \frac{1 \pm \sqrt{5}}{2\sqrt{3}}$. We set $A = XX'$ the correlation matrix

$$\begin{pmatrix} 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & a_+ & a_+ & a_- & -a_- & a_- & a_+ \\ \frac{\sqrt{5}}{3} & 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & a_+ & a_+ & a_+ & a_- & -a_- & a_- \\ \frac{1}{3} & \frac{\sqrt{5}}{3} & 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & a_+ & a_- & a_+ & a_+ & a_- & -a_- \\ \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & 1 & \frac{\sqrt{5}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & a_+ & -a_- & a_- & a_+ & a_+ & a_- \\ \frac{\sqrt{5}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & a_+ & a_- & -a_- & a_- & a_+ & a_+ \\ \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & \frac{1}{3} & a_- & a_+ & -a_- & -a_+ & -a_- & a_+ \\ \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & a_- & a_+ & a_+ & -a_- & -a_+ & -a_- \\ -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{5}}{3} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{\sqrt{5}}{3} & a_- & -a_- & a_+ & a_+ & -a_- & -a_+ \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & \frac{1}{3} & 1 & \frac{1}{3} & a_- & -a_+ & -a_- & a_+ & a_+ & -a_- \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & \frac{1}{3} & 1 & a_- & -a_- & -a_+ & -a_- & a_+ & a_+ \\ a_+ & a_+ & a_+ & a_+ & a_+ & a_- & a_- & a_- & a_- & a_- & 1 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ a_+ & a_+ & a_- & -a_- & a_- & a_+ & a_+ & -a_- & -a_+ & -a_- & \frac{1}{\sqrt{5}} & 1 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ a_- & a_+ & a_+ & a_- & -a_- & -a_- & a_+ & a_+ & -a_- & -a_+ & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -a_- & a_- & a_+ & a_+ & a_- & -a_+ & -a_- & a_+ & a_+ & -a_- & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ a_- & -a_- & a_- & a_+ & a_+ & a_- & -a_+ & -a_- & a_+ & a_+ & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & \frac{1}{\sqrt{5}} \\ a_+ & a_- & -a_- & a_- & a_+ & a_+ & -a_- & -a_+ & -a_- & a_+ & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \end{pmatrix}$$

Let $S = A \circ A$. Calculations show that

$$\begin{aligned} \text{per}(A) &= \frac{276791296}{405}, & \text{per}(S) &= \frac{308690026770890752}{403563009375}, \\ \frac{\text{per}(S)}{\text{per}(A)} &= \frac{9420472008389}{8417028988125} > 1. \end{aligned}$$

REFERENCES

- [1] R. B. BAPAT, *Recent developments and open problems in the theory of permanents*, Math. Student **76** (2007), no. 1–4, 55–69 (2008).
- [2] R. B. BAPAT, V. S. SUNDER, *On majorization and Schur products*, Linear Algebra Appl. **72**, (1985) 107–117.
- [3] J. CHOLLET, *Is there a permanent analogue to Oppenheim’s inequality?*, Amer. Math. Monthly **84**, (1) (1982) 57–59.
- [4] S. W. DRURY, *A counterexample to a question of Bapat and Sunder*, Electronic Journal of Linear Algebra, **31**, (2016) 69–70.
- [5] R. J. GREGORAC, I. R. HENTZEL, *A note on the analogue of Oppenheim’s inequality for permanents*, Linear Algebra Appl. **94**, (1987) 109–112.
- [6] V. S. SHCHESNOVICH, *The permanent-on-top conjecture is false*, Linear Algebra Appl. **490**, (2016) 196–201.

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