

## A NEW CHARACTERIZATION OF DIFFERENCES OF GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM THE BLOCH SPACE INTO WEIGHTED-TYPE SPACES

QINGHUA HU AND XIANGLING ZHU

(Communicated by S. Stević)

*Abstract.* In this paper, we give a new characterization for the boundedness and compactness of differences of generalized weighted composition operators from the Bloch space into weighted-type spaces. Moreover, we give some estimates for the essential norm of these operators.

### 1. Introduction

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the class of functions analytic in  $\mathbb{D}$ . We denote by  $S(\mathbb{D})$  the set of analytic self-map of  $\mathbb{D}$ . For  $a \in \mathbb{D}$ , let  $\sigma_a$  be the automorphism of  $\mathbb{D}$  exchanging 0 for  $a$ , i.e.,  $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ . For  $z, w \in \mathbb{D}$ , the pseudo-hyperbolic distance between  $z$  and  $w$  is given by

$$\rho(z, w) = |\sigma_w(z)| = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

It is well known that  $\rho(z, w) \leq 1$ .

Throughout this paper, every self-map  $\varphi$  induces a linear composition operator  $C_\varphi$  which is defined on  $H(\mathbb{D})$  by  $C_\varphi(f)(z) = f(\varphi(z))$ ,  $f \in H(\mathbb{D})$ ,  $z \in \mathbb{D}$ . Let  $\varphi \in S(\mathbb{D})$  and  $u \in H(\mathbb{D})$ . The weighted composition operator, denoted by  $uC_\varphi$ , is defined as follows.

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let  $n$  be a nonnegative integer. Let  $f^{(n)}$  denote the  $n$ -th derivative of  $f$  and  $f^{(0)} = f$ . A linear operator  $D_{\varphi, u}^n$  is defined by

$$(D_{\varphi, u}^n f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The operator  $D_{\varphi, u}^n$  is called the generalized weighted composition operator. In fact, if  $n = 0$  and  $u(z) = 1$ , then  $D_{\varphi, u}^n$  is the operator  $C_\varphi$ . If  $u(z) = 1$ , then  $D_{\varphi, u}^n$  is the

*Mathematics subject classification* (2010): 30D45, 47B38.

*Keywords and phrases:* Bloch space, difference, generalized weighted composition operators, essential norm.

The second author was partially supported by NSF of China (No.11471143). The second author is the corresponding author.

operator  $C_\varphi D^n$ , which was studied, for example, in [3, 17, 25]. If  $n = 0$ , then  $D^n_{\varphi,u}$  is just the operator  $uC_\varphi$ . If  $n = 1$  and  $u(z) = \varphi'(z)$ , then  $D^n_{\varphi,u} = DC_\varphi$ , which was studied in [3, 6, 7, 8, 9, 17, 18, 22]. The operator  $D^n_{\varphi,u}$  was introduced by Zhu in [29], and studied in [5, 10, 19, 20, 21, 24, 29, 30, 31].

Let  $0 < \beta < \infty$ . An  $f \in H(\mathbb{D})$  is said to belong to the  $\beta$ -Bloch space, denoted by  $\mathcal{B}^\beta$ , if

$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f'(z)| < \infty.$$

$\mathcal{B}^\beta$  is a Banach space under the norm  $\|\cdot\|_{\mathcal{B}^\beta}$ . When  $\beta = 1$ , we write  $\mathcal{B}^1$  by  $\mathcal{B}$ , which is called the Bloch space. We say that an  $f \in H(\mathbb{D})$  belongs to the little Bloch space, denoted by  $\mathcal{B}_0$ , if  $\lim_{|z| \rightarrow 1} |f'(z)|(1 - |z|^2) = 0$ .

Let  $0 < \alpha < \infty$ . The weighted-type space, denoted by  $H^\infty_\alpha$ , is defined as follows.

$$H^\infty_\alpha = \{f \in H(\mathbb{D}) : \|f\|_{H^\infty_\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty\}.$$

Let  $X$  and  $Y$  be Banach spaces. The essential norm of a bounded linear operator  $T : X \rightarrow Y$  is its distance to the set of compact operators  $K : X \rightarrow Y$ , that is,  $\|T\|_{e,X \rightarrow Y} = \inf\{\|T - K\| : K \text{ is compact}\}$ .

It is well known that  $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$  is bounded for any  $\varphi$  by Schwarz-Pick Lemma. The compactness and essential norm of the operator  $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$  were studied by many authors (see, e.g., [2, 13, 23, 26, 27]). In particular, Wulan, Zheng and Zhu [26] proved that  $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$  is compact if and only if  $\lim_{j \rightarrow \infty} \|\varphi^j\|_{\mathcal{B}} = 0$ . Zhao in [27] showed that  $\|C_\varphi\|_{e,\mathcal{B} \rightarrow \mathcal{B}} = \frac{\epsilon}{2} \limsup_{j \rightarrow \infty} \|\varphi^j\|_{\mathcal{B}}$ .

Many researchers have studied the differences of two composition operators on various function spaces in recent 20 years. See [1, 15] for more information of this study. It is easy to see that the operator  $C_\varphi - C_\psi : \mathcal{B} \rightarrow \mathcal{B}$  is bounded for any  $\varphi$  and  $\psi$ . See [4] and [14] for the study of compact differences of composition operators on the Bloch space. Recently, Shi and Li obtained some estimates for the essential norm of the operator  $C_\varphi - C_\psi : \mathcal{B} \rightarrow \mathcal{B}$  in [16]. Among others, they showed that

$$\|C_\varphi - C_\psi\|_{e,\mathcal{B} \rightarrow \mathcal{B}} \approx \limsup_{j \rightarrow \infty} \|\varphi^j - \psi^j\|_{\mathcal{B}} = \limsup_{j \rightarrow \infty} \|(C_\varphi - C_\psi)p_j\|_{\mathcal{B}}.$$

Here  $p_j(z) = z^j$ .

In [31], Zhu studied the boundedness and compactness of  $D^n_{\varphi,u} : \mathcal{B} \rightarrow H^\infty_\alpha$ . See [10] for more characterizations of the operator  $D^n_{\varphi,u} : \mathcal{B} \rightarrow H^\infty_\alpha$ . In [11], Liu and Li studied the operator  $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \rightarrow H^\infty_\alpha$ . Among others, they showed that  $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \rightarrow H^\infty_\alpha$  is compact if and only if  $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \rightarrow H^\infty_\alpha$  is bounded and the following equalities hold.

$$\lim_{|\varphi(z)| \rightarrow 1} |M_{u,\varphi}(z)| \rho(\varphi(z), \psi(z)) = 0; \tag{1}$$

$$\lim_{|\psi(z)| \rightarrow 1} |M_{v,\psi}(z)| \rho(\varphi(z), \psi(z)) = 0; \tag{2}$$

$$\lim_{|\varphi(z)| \rightarrow 1, |\psi(z)| \rightarrow 1} |M_{u,\varphi}(z) - M_{v,\psi}(z)| = 0. \quad (3)$$

Here

$$M_{u,\varphi}(z) = \frac{(1 - |z|^2)^\alpha u(z)}{(1 - |\varphi(z)|^2)^n}, \quad M_{v,\psi}(z) = \frac{(1 - |z|^2)^\alpha v(z)}{(1 - |\psi(z)|^2)^n}. \quad (4)$$

The present paper, motivated by [11, 16], gives a new characterization of the boundedness and compactness of the operator  $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$ . Moreover, we give some estimates for the essential norm of the operator  $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$ .

For two quantities  $A$  and  $B$  which may depend on  $\varphi$  and  $\psi$ , we use the abbreviation  $A \lesssim B$  whenever there is a positive constant  $c$  such that  $A \leq cB$ . We write  $A \approx B$ , if  $A \lesssim B \lesssim A$ .

## 2. Boundedness of $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$

In this section, we give a new characterization of the boundedness of the operator  $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$ . Let  $\mathbb{N}$  denote the set of all positive integers. Let  $j \in \mathbb{N}$ . We define  $p_j(z) = z^j$ ,  $z \in \mathbb{D}$ . Let  $n \in \mathbb{N}$ . For any  $a \in \mathbb{D}$ , we define the following two families test functions:

$$E_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^{n+1}}, \quad H_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^{n+1}} \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

From [28], we see that  $f \in \mathcal{B}$  if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{n+1} |f^{(n+1)}(z)| < \infty.$$

It is easy to check that  $E_a, H_a \in \mathcal{B}^{n+1}$ . Thus, there exist  $f_a, g_a \in \mathcal{B}$  such that  $f_a^{(n)} = E_a, g_a^{(n)} = H_a$ .

In order to prove the result in this section, we need the following lemmas.

LEMMA 2.1. [11] *Let  $n \in \mathbb{N}$ . For all  $z, w \in \mathbb{D}$ ,*

$$b_n(z, w) := \sup_{\|f\|_{\mathcal{B}} \leq 1} |(1 - |z|^2)^n f^{(n)}(z) - (1 - |w|^2)^n f^{(n)}(w)| \lesssim \rho(z, w).$$

Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . We denote

$$\mathcal{D}_u^\# \varphi(z) = \frac{(1 - |z|^2)^\alpha u(z)}{(1 - |\varphi(z)|^2)^n}, \quad \mathcal{D}_v^\# \psi(z) = \frac{(1 - |z|^2)^\alpha v(z)}{(1 - |\psi(z)|^2)^n}.$$

LEMMA 2.2. *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Then the following inequalities hold.*

(i)

$$\sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \lesssim \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.$$

(ii)

$$\sup_{z \in \mathbb{D}} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \lesssim \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.$$

(iii)

$$\sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \lesssim \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.$$

*Proof.* (i) For any  $z \in \mathbb{D}$ , we have

$$\begin{aligned} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} &\geq |u(z) f_{\varphi(z)}^{(n)}(\varphi(z)) - v(z) f_{\varphi(z)}^{(n)}(\psi(z))| (1 - |z|^2)^\alpha \\ &= \left| \mathcal{D}_u^\# \varphi(z) - \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)^n}{(1 - \overline{\varphi(z)}\psi(z))^{n+1}} \mathcal{D}_v^\# \psi(z) \right| \\ &\geq \left| \mathcal{D}_u^\# \varphi(z) \right| - \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)^n}{|1 - \overline{\varphi(z)}\psi(z)|^{n+1}} |\mathcal{D}_v^\# \psi(z)| \end{aligned}$$

and

$$\begin{aligned} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} &\geq |u(z) g_{\varphi(z)}^{(n)}(\varphi(z)) - v(z) g_{\varphi(z)}^{(n)}(\psi(z))| (1 - |z|^2)^\alpha \\ &= \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)^n}{|1 - \overline{\varphi(z)}\psi(z)|^{n+1}} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)). \end{aligned}$$

Hence

$$\begin{aligned} &|\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \\ &\leq \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} \rho(\varphi(z), \psi(z)) + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} \\ &\leq \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty}. \end{aligned} \quad (5)$$

Similarly,

$$\begin{aligned} &|\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ &\leq \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\psi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\psi(z)}\|_{H_\alpha^\infty}. \end{aligned} \quad (6)$$

Therefore, from (5) we obtain

$$\begin{aligned} &\sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \\ &\leq \sup_{z \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \sup_{z \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} \\ &\leq \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}. \end{aligned}$$

(ii) From (6) and similarly to the proof of (i) we get the desired result.

(iii) By Lemma 2.1,

$$\begin{aligned}
& \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_{\varphi(z)}\|_{H_\alpha^\infty} \\
& \geq \left| \mathcal{D}_u^\# \varphi(z) - \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)^n}{(1 - \overline{\varphi(z)}\psi(z))^{n+1}} \mathcal{D}_v^\# \psi(z) \right| \\
& \geq \left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| - \left| 1 - \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)^n}{(1 - \overline{\varphi(z)}\psi(z))^{n+1}} \right| |\mathcal{D}_v^\# \psi(z)| \\
& \geq \left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| - \left| (1 - |\varphi(z)|^2)^n f_{\varphi(z)}^{(n)}(\varphi(z)) \right. \\
& \quad \left. - (1 - |\psi(z)|^2)^n f_{\varphi(z)}^{(n)}(\psi(z)) \right| |\mathcal{D}_v^\# \psi(z)| \\
& \geq \left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| - b_n(\varphi(z), \psi(z)) |\mathcal{D}_v^\# \psi(z)| \\
& \gtrsim \left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| - |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)).
\end{aligned}$$

Thus, by (6) we obtain

$$\begin{aligned}
\left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| & \lesssim \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_{\varphi(z)}\|_{H_\alpha^\infty} + |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\
& \lesssim \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_{\varphi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_{\psi(z)}\|_{H_\alpha^\infty} \\
& \quad + \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_{\psi(z)}\|_{H_\alpha^\infty}.
\end{aligned} \tag{7}$$

Therefore,

$$\sup_{z \in \mathbb{D}} \left| \mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z) \right| \lesssim \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty}.$$

The proof is complete.  $\square$

LEMMA 2.3. *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Then the following inequalities hold.*

(i)

$$\sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} \lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

(ii)

$$\sup_{a \in \mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} \lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

*Proof.* (i) When  $a = 0$ , it is clear that  $f_a^{(n)}(z) = 1$ . Thus,

$$\|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} = \|u - v\|_{H_\alpha^\infty} \lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

For any  $a \in \mathbb{D}$  with  $a \neq 0$ , we have

$$f_a^{(n)}(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^{n+1}} = (1 - |a|^2) \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k! \Gamma(n+1)} \overline{a}^k z^k, \quad z \in \mathbb{D}.$$

By Stirling's formula, we have

$$\begin{aligned}
\|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} &\leq (1 - |a|^2) \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k!\Gamma(n+1)} |a|^k \|u\varphi^k - v\psi^k\|_{H_\alpha^\infty} \\
&= (1 - |a|^2) \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k!\Gamma(n+1)} k^{-n} |a|^k k^n \|u\varphi^k - v\psi^k\|_{H_\alpha^\infty} \\
&\lesssim (1 - |a|^2) \sum_{k=0}^{\infty} |a|^k \sup_{j \geq n} (j-n)^n \|u\varphi^{j-n} - v\psi^{j-n}\|_{H_\alpha^\infty} \\
&\lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}. \tag{8}
\end{aligned}$$

Since  $a$  is arbitrary, we see that (i) holds.

(ii) When  $a = 0$ , it is clear that  $g_a^{(n)}(z) = -z$ . Thus,

$$\|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} = \|u\varphi - v\psi\|_{H_\alpha^\infty} \lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

For any  $a \in \mathbb{D}$  with  $a \neq 0$ , we have

$$\begin{aligned}
g_a^{(n)}(z) &= \frac{1 - |a|^2}{(1 - \bar{a}z)^{n+1}} \cdot \frac{a - z}{1 - \bar{a}z} \\
&= (1 - |a|^2) \left( \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k!\Gamma(n+1)} \bar{a}^k z^k \right) \left( a - (1 - |a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^{k+1} \right) \\
&= a f_a^{(n)}(z) - (1 - |a|^2)^2 \left( \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k!\Gamma(n+1)} \bar{a}^k z^k \right) \left( \sum_{k=0}^{\infty} \bar{a}^k z^{k+1} \right) \\
&= a f_a^{(n)}(z) - (1 - |a|^2)^2 \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+n+1)}{l!\Gamma(n+1)} \right) \bar{a}^{k-1} z^k.
\end{aligned}$$

By Stirling's formula, we have

$$\sum_{l=0}^{k-1} \frac{\Gamma(l+n+1)}{l!\Gamma(n+1)} \approx \sum_{l=0}^{k-1} l^n \approx k^{n+1}, \quad k \rightarrow \infty.$$

Therefore,

$$\begin{aligned}
\|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} &\leq \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + (1 - |a|^2)^2 \\
&\quad \times \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+n+1)}{l!\Gamma(n+1)} \right) |a|^{k-1} \|u\varphi^k - v\psi^k\|_{H_\alpha^\infty} \\
&\lesssim \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} \\
&\quad + (1 - |a|^2)^2 \sum_{k=1}^{\infty} \frac{k^{n+1}}{k^n} |a|^{k-1} \sup_{j \geq n} (j-n)^n \|u\varphi^{j-n} - v\psi^{j-n}\|_{H_\alpha^\infty} \\
&\lesssim \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty} \\
&\lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.
\end{aligned}$$

By the arbitrariness of  $a$  we see that (ii) holds. The proof is complete.  $\square$

The following result is the main result in this section.

**THEOREM 2.1.** *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Then  $D_{\varphi, u}^n - D_{\psi, v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  is bounded if and only if*

$$\sup_{j \in \mathbb{N}} \|(D_{\varphi, u}^n - D_{\psi, v}^n)P_j\|_{H_\alpha^\infty} < \infty. \quad (9)$$

*Proof.* First we assume that  $D_{\varphi, u}^n - D_{\psi, v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  is bounded. For any  $j \in \mathbb{N}$ ,  $\|P_j\|_{\mathcal{B}} \approx 1$ . Thus

$$\infty > \|D_{\varphi, u}^n - D_{\psi, v}^n\|_{\mathcal{B} \rightarrow H_\alpha^\infty} \gtrsim \|(D_{\varphi, u}^n - D_{\psi, v}^n)P_j\|_{H_\alpha^\infty},$$

as desired.

Conversely, we assume that (9) holds. Let  $f \in \mathcal{B}$  and  $\|f\|_{\mathcal{B}} \leq 1$ . By Lemmas 2.1–2.3 and the proof of Theorem 1 in [11] we have

$$\begin{aligned} \|(D_{\varphi, u}^n - D_{\psi, v}^n)f\|_{H_\alpha^\infty} &\lesssim \sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + \sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \\ &\lesssim \sup_{a \in \mathbb{D}} \|(D_{\varphi, u}^n - D_{\psi, v}^n)f_a\|_{H_\alpha^\infty} + \sup_{a \in \mathbb{D}} \|(D_{\varphi, u}^n - D_{\psi, v}^n)g_a\|_{H_\alpha^\infty} \\ &\lesssim \sup_{j \in \mathbb{N}} \|(D_{\varphi, u}^n - D_{\psi, v}^n)P_j\|_{H_\alpha^\infty} < \infty. \end{aligned}$$

Therefore,  $D_{\varphi, u}^n - D_{\psi, v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  is bounded.  $\square$

### 3. Essential norm estimates

In this section we give an estimate for the essential norm of  $D_{\varphi, u}^n - D_{\psi, v}^n$  from  $\mathcal{B}$  to  $H_\alpha^\infty$ . For this purpose, we need some auxiliary results as follows.

**LEMMA 3.1.** *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Then the following statements hold.*

(i)

$$\begin{aligned} &\limsup_{s \rightarrow 1} \sup_{|\varphi(z)| > s} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \\ &\lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi, u}^n - D_{\psi, v}^n)f_a\|_{H_\alpha^\infty} + \limsup_{|a| \rightarrow 1} \|(D_{\varphi, u}^n - D_{\psi, v}^n)g_a\|_{H_\alpha^\infty}. \end{aligned}$$

(ii)

$$\begin{aligned} &\limsup_{s \rightarrow 1} \sup_{|\psi(z)| > s} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ &\lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi, u}^n - D_{\psi, v}^n)f_a\|_{H_\alpha^\infty} + \limsup_{|a| \rightarrow 1} \|(D_{\varphi, u}^n - D_{\psi, v}^n)g_a\|_{H_\alpha^\infty}. \end{aligned}$$

(iii)

$$\begin{aligned} & \limsup_{s \rightarrow 1} \sup_{\substack{|\varphi(z)| > s \\ |\psi(z)| > s}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ & \lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}. \end{aligned}$$

*Proof.* For any  $z \in \mathbb{D}$ , from the proof of Lemma 2.2 we have

$$|\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) \leq \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty},$$

$$|\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \leq \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\psi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\psi(z)}\|_{H_\alpha^\infty}$$

and

$$\begin{aligned} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| & \lesssim \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\psi(z)}\|_{H_\alpha^\infty} \\ & \quad + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\psi(z)}\|_{H_\alpha^\infty}. \end{aligned}$$

From the above inequalities the assertion follows easily. The proof is complete.  $\square$

LEMMA 3.2. *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Suppose that  $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  is bounded, then the following inequalities hold.*

(i)

$$\limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} \lesssim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}.$$

(ii)

$$\limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty} \lesssim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}.$$

*Proof.* For each  $N$  and any  $a \in \mathbb{D}$  with  $|a| > 1/2$ , from the proof of Lemma 2.3, we have

$$\begin{aligned} & \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} \\ & \lesssim (1 - |a|^2) \sum_{k=0}^N \frac{\Gamma(k+n+1)}{k! \Gamma(n+1)} k^{-n} |a|^k k^n \|u \varphi^k - v \psi^k\|_{H_\alpha^\infty} \\ & \quad + (1 - |a|^2) \sum_{k=N+1}^{\infty} |a|^k \sup_{j \geq N+n+1} (j-n)^n \|u \varphi^{j-n} - v \psi^{j-n}\|_{H_\alpha^\infty}. \end{aligned} \quad (10)$$

Taking the limit  $|a| \rightarrow 1$  in (10),

$$\limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} \lesssim \sup_{j \geq N+n+1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}$$

for any positive integer  $N$ . Therefore

$$\limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} \lesssim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

Also for each  $N$  and any  $a \in \mathbb{D}$  with  $|a| > 1/2$ , from the proof of Lemma 2.3,

$$\begin{aligned} & \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} \\ & \leq \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + (1 - |a|^2)^2 \\ & \quad \times \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+n+1)}{l!\Gamma(n+1)} \right) |a|^{k-1} \|u\varphi^k - v\psi^k\|_{H_\alpha^\infty} \\ & \leq \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + \sup_{j \geq N+n+1} (j-n)^n \|u\varphi^{j-n} - v\psi^{j-n}\|_{H_\alpha^\infty} \\ & \quad + (1 - |a|^2)^2 \sum_{k=1}^N k|a|^{k-1} k^n \|u\varphi^k - v\psi^k\|_{H_\alpha^\infty}. \end{aligned} \quad (11)$$

Letting  $|a| \rightarrow 1$  in (11). We get

$$\begin{aligned} & \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} \\ & \lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + \sup_{j \geq N+n+1} (j-n)^n \|u\varphi^{j-n} - v\psi^{j-n}\|_{H_\alpha^\infty} \\ & \lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_a\|_{H_\alpha^\infty} + \sup_{j \geq N+n+1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty} \end{aligned}$$

for any positive integer  $N$ . Thus, by (i) we obtain

$$\limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)g_a\|_{H_\alpha^\infty} \lesssim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

The proof is complete.  $\square$

**THEOREM 3.1.** *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Suppose that  $D_{\varphi,u}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  and  $D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  are bounded, then*

$$\|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e, \mathcal{B} \rightarrow H_\alpha^\infty} \approx \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n)p_j\|_{H_\alpha^\infty}.$$

*Proof.* For  $r \in [0, 1)$ , set  $K_r : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that  $f_r - f \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $r \rightarrow 1$ . Moreover,  $K_r$  is compact on  $\mathcal{B}$  and  $\|K_r\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq 1$ . Let  $\{r_j\} \subset (0, 1)$  be a sequence such that  $r_j \rightarrow 1$

as  $j \rightarrow \infty$ . Then for each positive integer  $j$ , the operator  $(D_{\varphi,u}^n - D_{\psi,v}^n)K_{r_j} : \mathcal{B} \rightarrow H_\alpha^\infty$  is compact. By the definition of the essential norm we have

$$\begin{aligned} \|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e,\mathcal{B} \rightarrow H_\alpha^\infty} &\leq \limsup_{j \rightarrow \infty} \|D_{\varphi,u}^n - D_{\psi,v}^n - (D_{\varphi,u}^n - D_{\psi,v}^n)K_{r_j}\|_{\mathcal{B} \rightarrow H_\alpha^\infty} \\ &= \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n)(I - K_{r_j})\|_{\mathcal{B} \rightarrow H_\alpha^\infty} \\ &\leq \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n)(I - K_{r_j})f\|_{H_\alpha^\infty} \\ &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}} \Omega_j^f(z), \end{aligned}$$

where

$$\Omega_j^f(z) := |u(z)(f - f_{r_j})^{(n)}(\varphi(z)) - v(z)(f - f_{r_j})^{(n)}(\psi(z))|(1 - |z|^2)^\alpha.$$

For any  $r \in (0, 1)$ , define

$$\begin{aligned} \mathbb{D}_1 &:= \{z \in \mathbb{D} : |\varphi(z)| \leq r, |\psi(z)| \leq r\}, & \mathbb{D}_2 &:= \{z \in \mathbb{D} : |\varphi(z)| \leq r, |\psi(z)| > r\}, \\ \mathbb{D}_3 &:= \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| \leq r\}, & \mathbb{D}_4 &:= \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| > r\}. \end{aligned}$$

Then

$$\begin{aligned} \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}} \Omega_j^f &= \max_{1 \leq i \leq 4} \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}_i} \Omega_j^f \\ &= \max\{\limsup_{j \rightarrow \infty} J_1, \limsup_{j \rightarrow \infty} J_2, \limsup_{j \rightarrow \infty} J_3, \limsup_{j \rightarrow \infty} J_4\}, \end{aligned}$$

where  $J_i = \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}_i} \Omega_j^f$ . Using the fact that  $u, v \in H_\alpha^\infty$  we get

$$\begin{aligned} \limsup_{j \rightarrow \infty} J_1 &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}_1} \Omega_j^f \\ &\leq \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{|\varphi(z)| \leq r} |u(z)(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |z|^2)^\alpha \\ &\quad + \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{|\psi(z)| \leq r} |v(z)(f - f_{r_j})^{(n)}(\psi(z))|(1 - |z|^2)^\alpha \\ &= 0. \end{aligned}$$

In addition, we have

$$\begin{aligned} \Omega_j^f(z) &\leq |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + |\mathcal{D}_u^\# \varphi(z)| \\ &\quad \times |(f - f_{r_j})^{(n)}(\varphi(z))(1 - |\varphi(z)|^2)^n - (f - f_{r_j})^{(n)}(\psi(z))(1 - |\psi(z)|^2)^n| \\ &\leq |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ &\quad + b_n(\varphi(z), \psi(z)) |\mathcal{D}_u^\# \varphi(z)| \\ &\lesssim |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ &\quad + |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)). \end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_j^f(z) &\lesssim |(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |\varphi(z)|^2)^n |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ &\quad + |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\limsup_{j \rightarrow \infty} J_2 &\lesssim \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}_2} (|\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ &\quad + |(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |\varphi(z)|^2)^n |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)|) \\ &\leq \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{|\varphi(z)| \leq r} |(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |\varphi(z)|^2)^n \\ &\quad \times |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ &= \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)),\end{aligned}$$

where we used  $\sup_{z \in \mathbb{D}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| < \infty$ , since  $D_{\varphi, u}^n - D_{\psi, v}^n$  is bounded (see Theorem 1 of [11]), and  $(f - f_{r_j})^{(n)} \rightarrow 0$  uniformly on compact subset of  $\mathbb{D}$  as  $j \rightarrow \infty$  again in the last inequality. Since  $r$  is arbitrary, we have

$$\limsup_{j \rightarrow \infty} J_2 \lesssim \lim_{r \rightarrow 1} \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)).$$

Similarly,

$$\limsup_{j \rightarrow \infty} J_3 \lesssim \lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)).$$

In addition,

$$\begin{aligned}\limsup_{j \rightarrow \infty} J_4 &\lesssim \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}_4} \left( |(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |\varphi(z)|^2)^n \right. \\ &\quad \left. \times |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \right) \\ &\lesssim \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{\substack{|\varphi(z)| > r \\ |\psi(z)| > r}} \|f - f_{r_j}\|_{\mathcal{B}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ &\quad + \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ &\lesssim \sup_{\substack{|\varphi(z)| > r \\ |\psi(z)| > r}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)),\end{aligned}$$

where we used the fact  $\limsup_{j \rightarrow \infty} \|f - f_{r_j}\|_{\mathcal{B}} \leq 2$  in the last inequality. Thus,

$$\limsup_{j \rightarrow \infty} J_4 \lesssim \lim_{r \rightarrow 1} \sup_{\substack{|\varphi(z)| > r \\ |\psi(z)| > r}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| + \lim_{r \rightarrow 1} \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)).$$

Then we have

$$\begin{aligned}
& \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{z \in \mathbb{D}} \Omega_j^f(z) = \max\{\limsup_{j \rightarrow \infty} J_1, \limsup_{j \rightarrow \infty} J_2, \limsup_{j \rightarrow \infty} J_3, \limsup_{j \rightarrow \infty} J_4\} \\
& \lesssim \lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) + \lim_{r \rightarrow 1} \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\
& \quad + \lim_{r \rightarrow 1} \sup_{\substack{|\varphi(z)| > r \\ |\psi(z)| > r}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)|,
\end{aligned}$$

which together with Lemmas 3.1 and 3.2 imply

$$\begin{aligned}
& \|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e, \mathcal{B} \rightarrow H_\alpha^\infty} \\
& \lesssim \lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) + \lim_{r \rightarrow 1} \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\
& \quad + \lim_{r \rightarrow 1} \sup_{\substack{|\varphi(z)| > r \\ |\psi(z)| > r}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\
& \lesssim \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty} \\
& \lesssim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}. \tag{12}
\end{aligned}$$

Next, we prove that

$$\|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e, \mathcal{B} \rightarrow H_\alpha^\infty} \gtrsim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}.$$

Let  $j$  be any positive integer. Then  $\|p_j\|_{\mathcal{B}} \approx 1$  and  $p_j \rightarrow 0$  weakly in  $\mathcal{B}$ . This follows since a bounded sequence contained in  $\mathcal{B}_0$  which converges uniformly to 0 on compact subsets of  $\mathbb{D}$  converges weakly to 0 in  $\mathcal{B}$  (see [12]). Thus, if  $K$  is any compact operator from  $\mathcal{B}$  to  $H_\alpha^\infty$ , then  $\lim_{j \rightarrow \infty} \|K p_j\|_{H_\alpha^\infty} = 0$ . Hence,

$$\begin{aligned}
\|D_{\varphi,u}^n - D_{\psi,v}^n - K\| & \gtrsim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n - K) p_j\|_{H_\alpha^\infty} \\
& \geq \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}.
\end{aligned}$$

Thus

$$\|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e, \mathcal{B} \rightarrow H_\alpha^\infty} \gtrsim \limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty}. \tag{13}$$

Combining (12) with (13), we immediately get the desired result. The proof is complete.  $\square$

From Theorem 3.1, we immediately get the following result.

**THEOREM 3.2.** *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Suppose that  $D_{\varphi,u}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  and  $D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  are bounded, then*

$$\begin{aligned} & \|D_{\varphi,u}^n - D_{\psi,v}^n\|_{e, \mathcal{B} \rightarrow H_\alpha^\infty} \\ & \approx \limsup_{r \rightarrow 1} \sup_{|\varphi(z)| > r} |\mathcal{D}_u^\# \varphi(z)| \rho(\varphi(z), \psi(z)) + \limsup_{r \rightarrow 1} \sup_{|\psi(z)| > r} |\mathcal{D}_v^\# \psi(z)| \rho(\varphi(z), \psi(z)) \\ & \quad + \limsup_{\substack{r \rightarrow 1 \\ |\varphi(z)| > r \\ |\psi(z)| > r}} |\mathcal{D}_u^\# \varphi(z) - \mathcal{D}_v^\# \psi(z)| \\ & \approx \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}. \end{aligned}$$

From Theorem 3.1, we also immediately get the following corollary.

**COROLLARY 3.1.** *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$ . Let  $\varphi$  and  $\psi$  be analytic self-maps of  $\mathbb{D}$ . Suppose that  $D_{\varphi,u}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  and  $D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  are bounded, then  $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{B} \rightarrow H_\alpha^\infty$  is compact if and only if*

$$\limsup_{j \rightarrow \infty} \|(D_{\varphi,u}^n - D_{\psi,v}^n) p_j\|_{H_\alpha^\infty} = 0.$$

#### REFERENCES

- [1] E. BERKSON, *Composition operators isolated in the uniform operator topology*, Proc. Amer. Math. Soc. **81** (1981), 230–232.
- [2] C. COWEN AND B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.
- [3] R. HIBSCHWEILER AND N. PORTNOY, *Composition followed by differentiation between Bergman and Hardy spaces*, Rocky Mountain J. Math. **35** (2005), 843–855.
- [4] T. HOSOKAWA AND S. OHNO, *Differences of composition operators on the Bloch spaces*, J. Operator Theory **57** (2007), 229–242.
- [5] H. LI AND X. FU, *A new characterization of generalized weighted composition operators from the Bloch space into the Zygmund space*, J. Funct. Spaces Appl. Volume **2013**, Article ID 925901, 12 pages.
- [6] S. LI AND S. STEVIĆ, *Composition followed by differentiation between Bloch type spaces*, J. Comput. Anal. Appl. **9** (2007), 195–205.
- [7] S. LI AND S. STEVIĆ, *Composition followed by differentiation from mixed-norm spaces to  $\alpha$ -Bloch spaces*, Sb. Math. **199** (12) (2008), 1847–1857.
- [8] S. LI AND S. STEVIĆ, *Composition followed by differentiation between  $H^\infty$  and  $\alpha$ -Bloch spaces*, Houston J. Math. **35** (2009), 327–340.
- [9] S. LI AND S. STEVIĆ, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput. **217** (2010), 3144–3154.
- [10] S. LI AND S. STEVIĆ, *Generalized weighted composition operators from  $\alpha$ -Bloch spaces into weighted-type spaces*, J. Ineq. Appl. **2015** (2015), 265, DOI 10.11866/s13660-015-0770-9.
- [11] X. LIU AND S. LI, *Differences of generalized weighted composition operators from the Bloch space into Bers-type spaces*, Filomat, **31** (2017), 1671–1680.
- [12] B. MACCLUER AND R. ZHAO, *Essential norm of weighted composition operators between Bloch-type spaces*, Rocky. Mountain J. Math. **33** (2003), 1437–1458.
- [13] K. MADIGAN AND A. MATHESON, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), 2679–2687.
- [14] P. NIEMINEN, *Compact differences of composition operators on Bloch and Lipschitz spaces*, Comput. Method Funct. Theory **7** (2007), 325–344.

- [15] J. SHAPIRO AND C. SUNDBERG, *Isolation amongst the composition operators*, Pacific J. Math. **145** (1990), 117–152.
- [16] Y. SHI AND S. LI, *Essential norm estimates for differences of composition operators on the Bloch space*, Math. Ineq. Appl., **20** (2017), 543–555.
- [17] S. STEVIĆ, *Norm and essential norm of composition followed by differentiation from  $\alpha$ -Bloch spaces to  $H_{\mu}^{\infty}$* , Appl. Math. Comput. **207** (2009), 225–229.
- [18] S. STEVIĆ, *Products of composition and differentiation operators on the weighted Bergman space*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), 623–635.
- [19] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces*, Appl. Math. Comput. **211** (2009), 222–233.
- [20] S. STEVIĆ, *Weighted differentiation composition operators from the mixed-norm space to the  $n$ th weighted-type space on the unit disk*, Abstr. Appl. Anal. **2010** (2010), Article ID 246287.
- [21] S. STEVIĆ, *Weighted differentiation composition operators from  $H^{\infty}$  and Bloch spaces to  $n$ th weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3634–3641.
- [22] S. STEVIĆ, *Characterizations of composition followed by differentiation between Bloch-type spaces*, Appl. Math. Comput. **218** (2011), 4312–4316.
- [23] M. TJANI, *Compact composition operators on some Möbius invariant Banach space*, PhD dissertation, Michigan State University, 1996.
- [24] W. YANG AND X. ZHU, *Differences of generalized weighted composition operators between growth spaces*, Ann. Polon. Math. **112** (2014), 67–83.
- [25] Y. WU AND H. WULAN, *Products of differentiation and composition operators on the Bloch space*, Collet. Math. **63** (2012), 93–107.
- [26] H. WULAN, D. ZHENG AND K. ZHU, *Compact composition operators on BMOA and the Bloch space*, Proc. Amer. Math. Soc. **137** (2009), 3861–3868.
- [27] R. ZHAO, *Essential norms of composition operators between Bloch type spaces*, Proc. Amer. Math. Soc. **138** (2010), 2537–2546.
- [28] K. ZHU, *Operator Theory in Function Spaces*, American Mathematical Society, Providence, RI, 2007.
- [29] X. ZHU, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type space*, Integ. Tran. Spec. Funct. **18** (2007), 223–231.
- [30] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces*, Numer. Funct. Anal. Opt. **30** (2009), 881–893.
- [31] X. ZHU, *Generalized weighted composition operators from Bloch spaces into Bers-type spaces*, Filomat **26** (2012), 1163–1169.

(Received October 4, 2016)

Qinghua Hu  
College of Mathematics Physics and Information Engineering  
Jiaxing University  
314001 Jiaxing, Zhejiang, P. R. China  
e-mail: hqmath@sina.com

Xiangling Zhu  
Department of Mathematics  
Jiaying University  
514015 Meizhou, Guangdong, P. R. China  
e-mail: jyuzxl@163.com