Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 251–263 doi:10.7153/mia-2018-21-19

A COMPLEMENT TO DIANANDA'S INEQUALITY

PENG GAO

(*Communicated by I. Perić*)

Abstract. Let $M_{n,r}=(\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$, $r\neq 0$ and $M_{n,0}=\lim_{r\to 0} M_{n,r}$ be the weighted power means of n non-negative numbers x_i with $q_i>0$ satisfying $\sum_{i=1}^n q_i=1$. In particular, $A_n=M_{n,1}$, $G_n=M_{n,0}$ are the arithmetic and geometric means of these numbers, respectively. A result of Diananda shows that

$$M_{n,1/2}-qA_n-(1-q)G_n\geqslant 0,$$
$$M_{n,1/2}-(1-q)A_n-qG_n\leqslant 0,$$

where $q=\min q_i$. In this paper, we prove analogue inequalities in the reversed direction.

1. Introduction

Let $M_{n,r}(\mathbf{x};\mathbf{q})$ be the weighted power means: $M_{n,r}(\mathbf{x};\mathbf{q})=(\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$, where $M_{n,0}(\mathbf{x};\mathbf{q})$ denotes the limit of $M_{n,r}(\mathbf{x};\mathbf{q})$ as $r\to 0$, $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{q}=(q_1,\dots,q_n)$ with $x_i\geqslant 0$, $q_i>0$ for all $1\leqslant i\leqslant n$ and $\sum_{i=1}^n q_i=1$. In this paper, unless otherwise specified, we let $q=\min q_i$ and we assume that $0\leqslant x_1<x_2<\cdots<x_n$.

We define $A_n(\mathbf{x};\mathbf{q})=M_{n,1}(\mathbf{x};\mathbf{q})$, $G_n(\mathbf{x};\mathbf{q})=M_{n,0}(\mathbf{x};\mathbf{q})$, $\sigma_n=\sum_{i=1}^n q_i(x_i-A_n)^2$. We shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x};\mathbf{q})$ and similarly for other means when there is no risk of confusion.

In [5], the following bounds of $M_{n,1/r}$ in terms of A_n, G_n are given:

$$M_{n,\frac{1}{r}}\leqslant (1-q)^{r-1}A_n+(1-(1-q)^{r-1})G_n,\quad 1<r\leqslant 2; \tag{1}$$
$$M_{n,\frac{1}{r}}\geqslant q^{r-1}A_n+(1-q^{r-1})G_n,\quad r\geqslant 2. \tag{2}$$

The reversed inequality of (1) is valid when $0<r<1$ and the above inequalities are generalizations of a result of Diananda ([2], [3]), which corresponds to case $r=2$ of the above inequalities. Note that except for the case $r=2$, the above inequalities only provide one-sided bound for any given $M_{n,1/r}$. It is therefore natural to seek for bounds that are complementary to the above ones. In this paper, we consider one way to achieve this by establishing the following

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Power means.
The author is supported in part by NSFC grant 11371043.

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Paper MIA-21-19

THEOREM 1. *For $r \geqslant 2$, we have*

$$M_{n,\frac{1}{r}} - q^{r-1}A_n - (1-q^{r-1})G_n \leqslant \frac{1/r - q^{r-1}}{2x_1}\sigma_n, \tag{3}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ or $r = n = 2$, $q = 1/2$.

For $1 < r \leqslant 2$, we have

$$M_{n,\frac{1}{r}} - (1-q)^{r-1}A_n - (1-(1-q)^{r-1})G_n \geqslant \frac{1/r - (1-q)^{r-1}}{2x_1}\sigma_n, \tag{4}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ or $r = n = 2$, $q = 1/2$. The reversed inequality of (4) holds for $1/2 \leqslant r < 1$ with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

Our result in fact is motivated by the following bounds for the differences of means:

$$\frac{r-s}{2x_n}\sigma_n \leqslant M_{n,r} - M_{n,s} \leqslant \frac{r-s}{2x_1}\sigma_n, \quad r > s. \tag{5}$$

The above inequalities are closely related to the Ky Fan inequalities and are not valid for all $r > s$. When they are valid, then the constant $(r-s)/2$ is best possible (see [4]) and a necessary condition for inequalities (5) to be valid is that $0 \leqslant r+s \leqslant 3$ (see [4, Lemma 3.1]). Moreover, it is shown in [4, Theorem 3.2] that if $r = 1$, then inequalities (5) hold if and only if $-1 \leqslant s < 1$. If $s = 1$, then inequalities (5) hold if and only if $1 < r \leqslant 2$. In particular, the case $r = 1$, $s = 0$ of (5) yields a result of Cartwright and Field [1]:

$$\frac{\sigma_n}{2x_n} \leqslant A_n - G_n \leqslant \frac{\sigma_n}{2x_1}. \tag{6}$$

Using (6) while noting that the constant $1/2$ is best possible, one sees easily that when $r = 2$, the results given in Theorem 1 are not comparable to the bounds given by (1)–(2).

We can recast inequality (3) as

$$M_{n,\frac{1}{r}} - G_n - \frac{1/r}{2x_1}\sigma_n \leqslant q^{r-1}\Big(A_n - G_n - \frac{1}{2x_1}\sigma_n\Big), \tag{7}$$

from which we see that inequality (3) can be interpreted as a comparison between different inequalities in (5). We can deduce a similar inequality from (4). This combined with our discussions above allows us to prove the right-hand side inequality of (5) for $s = 0$, $0 < r \leqslant 1/2$ and $1 < r \leqslant 2$. It is then interesting to determine all the values of r such that inequalities (5) hold for r and $s = 0$. We shall do this in Section 3 as we prove the following

THEOREM 2. *Let $r \neq 0$, $x_1 = \min\{x_i\}$, $x_n = \max\{x_i\}$, then the right-hand side inequality of (5) holds with $s = 0$ if and only if $0 < r \leqslant 2$, the left-hand side inequality of (5) holds with $s = 0$ if and only if $1 \leqslant r \leqslant 3$. Moreover, in all these cases we have equality holding if and only if $x_1 = x_2 = \cdots = x_n$.*

We note that Theorem 2 implies that the reversed inequality of (4) does not hold for $0<r<1/2$ in general. For otherwise we can recast it in a form similar to inequality (7) to deduce the validity of the right-hand side inequality of (5) for $s=0$, $r>2$.

We note that the following inequality

$$M_{n,\frac{1}{r}}-q^{r-1}A_n-(1-q^{r-1})G_n \geqslant \frac{1/r-q^{r-1}}{2x_n}\sigma_n,$$

is not valid in general as one checks easily that when $n=2$, $q_1=1-q$, $q_2=q$, $x_1=0$, $x_2=1$, the left-hand side expression above is 0 while the right-hand side expression is not 0 in general. Therefore, it is not possible to have a similar lower bound for the left-hand side expression in (3). Similar discussions apply to (4) as well.

2. Proof of Theorem 1

Throughout this section, we assume $n\geqslant 2$, $x_1=1$ and $1<x_2<\ldots<x_n$. We will omit the discussion on the conditions for equality in each inequality as one checks easily that the desired conditions hold by going through our arguments in what follows. We first prove inequality (3) and we define

$$f_n(\mathbf{x};\mathbf{q},q)=M_{n,\frac{1}{r}}-q^{r-1}A_n-(1-q^{r-1})G_n-\frac{1/r-q^{r-1}}{2x_1}\sigma_n.$$

It suffices to show $f_n(\mathbf{x};\mathbf{q},q)\leqslant 0$ and we have

$$\frac{1}{q_n}\cdot\frac{\partial f_n}{\partial x_n}=M_{n,\frac{1}{r}}^{1-\frac{1}{r}}x_n^{\frac{1}{r}-1}-q^{r-1}-(1-q^{r-1})G_nx_n^{-1}-\left(\frac{1}{r}-q^{r-1}\right)(x_n-A_n):=g_n(\mathbf{x};\mathbf{q},q).$$

It suffices to show $g_n(\mathbf{x};\mathbf{q},q)\leqslant 0$ as it implies $f_n(\mathbf{x};\mathbf{q},q)\leqslant \lim_{x_n\to x_{n-1}} f_n(\mathbf{x};\mathbf{q},q)$. By adjusting the value of q in the expression of $\lim_{x_n\to x_{n-1}} f_n(\mathbf{x};\mathbf{q},q)$ (note that it follows from (6) that $\frac{\partial f_n}{\partial q}\geqslant 0$) and repeating the process, it follows easily that $f_n(\mathbf{x};\mathbf{q},q)\leqslant 0$.

Similarly, in order to show $g_n(\mathbf{x};\mathbf{q},q)\leqslant 0$, it suffices to show $\frac{\partial g_n}{\partial x_n}\leqslant 0$. Calculation shows that

$$\frac{1}{1-q_n}\cdot\frac{\partial g_n}{\partial x_n}=\frac{1-1/r}{1-q_n}M_{n,\frac{1}{r}}^{1-\frac{2}{r}}x_n^{\frac{1}{r}-2}\left(q_nx_n^{\frac{1}{r}}-M_{n,\frac{1}{r}}^{\frac{1}{r}}\right)+(1-q^{r-1})G_nx_n^{-2}-\left(\frac{1}{r}-q^{r-1}\right).$$

We make a change of variable $x_i\to y_i^r$ to recast the right-hand side expression above as

$$\begin{aligned}&-\left(1-\frac{1}{r}\right)(q_ny_n+(1-q_n)A'_{n-1})^{r-2}A'_{n-1}y_n^{1-2r}+(1-q^{r-1})G'^{(1-q_n)r}_{n-1}y_n^{q_nr-2r}\\&-\left(\frac{1}{r}-q^{r-1}\right)\\\leqslant&-\left(1-\frac{1}{r}\right)(q_ny_n+(1-q_n)A'_{n-1})^{r-2}A'_{n-1}y_n^{1-2r}+(1-q^{r-1})A'^{(1-q_n)r}_{n-1}y_n^{q_nr-2r}\\&-\left(\frac{1}{r}-q^{r-1}\right),\end{aligned}\tag{8}$$

where $A'_{n-1} = A_{n-1}(\mathbf{y}';\mathbf{q}'), G'_{n-1} = G_{n-1}(\mathbf{y}';\mathbf{q}')$, and

$$\mathbf{y}' = (y_1,\dots,y_{n-1}), \quad \mathbf{q}' = \Big(\frac{q_1}{1-q_n},\dots,\frac{q_{n-1}}{1-q_n}\Big).$$

We further denote $z = y_n/A'_{n-1}$ to see that the right-hand side expression of (8) is

$$\begin{aligned}
&\leqslant y_n^{-r}\Big(-\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-2}z^{1-r}+(1-q^{r-1})z^{q_nr-r}-\Big(\frac{1}{r}-q^{r-1}\Big)z^rA'^r_{n-1}\Big)\\
&\leqslant y_n^{-r}\Big(-\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-2}z^{1-r}+(1-q^{r-1})z^{q_nr-r}-\Big(\frac{1}{r}-q^{r-1}\Big)z^r\Big).
\end{aligned}$$

It suffices to show that the last expression above is non-positive for $z \geqslant 1$. Note first that when $q_nr \leqslant 1$, the last expression above equals

$$\begin{aligned}
&y_n^{-r}z^{1-r}\left(-\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-2}+(1-q^{r-1})z^{q_nr-1}-\Big(\frac{1}{r}-q^{r-1}\Big)z^{2r-1}\right)\\
\leqslant &y_n^{-r}z^{1-r}\left(-\Big(1-\frac{1}{r}\Big)(q_n+1-q_n)^{r-2}+(1-q^{r-1})-\Big(\frac{1}{r}-q^{r-1}\Big)\right)=0.
\end{aligned}$$

Thus we may assume $q_nr > 1$ and in this case, it suffices to show that

$$u(z;q_n,q) = -\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-2}z^{1-q_nr}+(1-q^{r-1})-\Big(\frac{1}{r}-q^{r-1}\Big)z^{(2-q_n)r} \leqslant 0.$$

Now we have

$$\begin{aligned}
&z^{1-(2-q_n)r}\frac{\partial u}{\partial z}\\
=&\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-3}z^{2-2r}\left((rq_n-1)(1-q_n)z^{-1}+(1-(1-q_n)r)q_n\right)\\
&-r(2-q_n)\Big(\frac{1}{r}-q^{r-1}\Big)\\
\leqslant&\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-3}z^{2-2r}\left((rq_n-1)(1-q_n)+(1-(1-q_n)r)q_n\right)\\
&-r(2-q_n)\Big(\frac{1}{r}-q^{r-1}\Big)\\
=&\Big(1-\frac{1}{r}\Big)(q_nz+1-q_n)^{r-3}z^{2-2r}(2q_n-1)-r(2-q_n)\Big(\frac{1}{r}-q^{r-1}\Big).
\end{aligned}$$

If $2q_n-1 \leqslant 0$, then we have $\partial u/\partial z \leqslant 0$, as it follows from [5, (2.1)] that $1/r-q^{r-1} \geqslant 0$ when $r \geqslant 2$. Otherwise, note that

$$(q_nz+1-q_n)^{r-3}z^{2-2r} \leqslant \max\{z^{2-2r}, z^{r-3}z^{2-2r}\} \leqslant 1.$$

Thus we have

$$\begin{aligned}
z^{1-(2-q_n)r}\frac{\partial u}{\partial z} &\leqslant \Big(1-\frac{1}{r}\Big)(2q_n-1)-r(2-q_n)\Big(\frac{1}{r}-q^{r-1}\Big)\\
&\leqslant \Big(1-\frac{1}{r}\Big)(2(1-q)-1)-r(2-(1-q))\Big(\frac{1}{r}-q^{r-1}\Big)\\
&= \Big(1-\frac{1}{r}\Big)(1-2q)-(qr+r)\Big(\frac{1}{r}-q^{r-1}\Big).
\end{aligned}$$

It is easy to see that the last expression above is a convex function of q for fixed r and it is $\leqslant 0$ when $q = 0, 1/2$. Thus $\partial u/\partial z \leqslant 0$ so that $u(z; q_n, q) \leqslant u(1; q_n, q) = 0$. This proves inequality (3).

Now, to prove inequality (4), we use the same notations as above to see that in this case, it suffices to show $f_n(\mathbf{x}; \mathbf{q}, 1-q) \geqslant 0$. Again, this follows from $\frac{\partial g_n(\mathbf{x};\mathbf{q},1-q)}{\partial x_n} \geqslant 0$. Similar to our arguments above, it is easy to see that in this case the left-hand side expression of (8) becomes

$$
\begin{aligned}
&-\left(1-\frac{1}{r}\right)(q_n y_n + (1-q_n)A'_{n-1})^{r-2} A'_{n-1} y_n^{1-2r} + (1-(1-q)^{r-1}) G'^{(1-q_n)r}_{n-1} y_n^{q_n r-2r} \\
&-\left(\frac{1}{r} - (1-q)^{r-1}\right) := h(y_n).
\end{aligned}
$$

It therefore remains to show that $h(y_n) \geqslant 0$ for $y_n \geqslant A'_{n-1}$. Note first that

$$
\begin{aligned}
h(y_n) &\geqslant -\left(1-\frac{1}{r}\right) A'^{r-1}_{n-1} y_n^{1-2r} + (1-(1-q)^{r-1}) G'^{(1-q_n)r}_{n-1} y_n^{q_n r-2r} - \left(\frac{1}{r} - (1-q)^{r-1}\right) \\
&:= y^{1-2r} \tilde{h}(y_n),
\end{aligned}
$$

where

$$
\tilde{h}(y_n) = -\left(1-\frac{1}{r}\right) A'^{r-1}_{n-1} + (1-(1-q)^{r-1}) G'^{(1-q_n)r}_{n-1} y_n^{q_n r-1} - \left(\frac{1}{r} - (1-q)^{r-1}\right) y_n^{2r-1}.
$$

When $q_n r - 1 \geqslant 0$, one checks that $\tilde{h}$ is an increasing function of y_n (note that in our case $(1-q)^{r-1} \geqslant 1/r$), hence is minimized at $y_n = A'_{n-1}$ and it is easy to see that in this case $h(y_n) \geqslant 0$ is equivalent to $h(A'_{n-1}) \geqslant 0$.

We now consider the case $1 - rq_n > 0$. Note first that $\lim_{y_n \to \infty} h(y_n) \geqslant 0$. If $h(y_n)$ is minimized as some $y_n = y > A'_{n-1}$, then we must have $h'(y) = 0$, which yields

$$
\begin{aligned}
&\frac{1-\frac{1}{r}}{(2-q_n)r} (q_n y + (1-q_n)A'_{n-1})^{r-3} A'_{n-1} y^{1-2r} \left((r+1)\, q_n y + (2r-1)\,(1-q_n)\, A'_{n-1}\right) \\
=&(1-(1-q)^{r-1}) G'^{(1-q_n)r}_{n-1} y^{q_n r-2r}.
\end{aligned}
$$

This allows us to rewrite the expression for $h(y)$ as

$$
\begin{aligned}
h(y) =& -\left(1-\frac{1}{r}\right)(q_n y + (1-q_n)A'_{n-1})^{r-2} A'_{n-1} y^{1-2r} \\
&+ (1-(1-q)^{r-1}) G'^{(1-q_n)r}_{n-1} y^{q_n r-2r} - \left(\frac{1}{r} - (1-q)^{r-1}\right) \\
=& -\left(1-\frac{1}{r}\right)(q_n y + (1-q_n)A'_{n-1})^{r-2} A'_{n-1} y^{1-2r} \\
&+ \frac{1}{(2-q_n)r}\left(1-\frac{1}{r}\right)(q_n y + (1-q_n)A'_{n-1})^{r-3} A'_{n-1} y^{1-2r} \\
&\times \left((r+1)\, q_n y + (2r-1)\,(1-q_n)\, A'_{n-1}\right) - \left(\frac{1}{r} - (1-q)^{r-1}\right)
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&= \frac{1}{(2-q_n)r}\left(1-\frac{1}{r}\right)(q_n y+(1-q_n)A'_{n-1})^{r-3}A'_{n-1}y^{1-2r} \\
&\quad \times \left((1-r+rq_n)\,q_n y-(1-rq_n)\,(1-q_n)\,A'_{n-1}\right)-\left(\frac{1}{r}-(1-q)^{r-1}\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
h(y) &\geqslant \frac{1}{(2-q_n)r}\left(1-\frac{1}{r}\right)(q_n y+(1-q_n)A'_{n-1})^{r-3}A'_{n-1}y^{1-2r} \\
&\quad \times \left((1-r+rq_n)\,q_n y-(1-rq_n)\,(1-q_n)\,y\right)-\left(\frac{1}{r}-(1-q)^{r-1}\right) \\
&= -\frac{1-2q_n}{(2-q_n)r}\left(1-\frac{1}{r}\right)(q_n y+(1-q_n)A'_{n-1})^{r-3}A'_{n-1}y^{2-2r}-\left(\frac{1}{r}-(1-q)^{r-1}\right).
\end{aligned}
$$

If $1-2q_n \leqslant 0$, then $h(y) \geqslant 0$. When $1-2q_n > 0$, we see that

$$
\begin{aligned}
h(y) &\geqslant -\frac{1-2q_n}{(2-q_n)r}\left(1-\frac{1}{r}\right)A'^{\,r-3}_{n-1}\cdot A'_{n-1}\cdot A'^{\,2-2r}_{n-1}-\left(\frac{1}{r}-(1-q)^{r-1}\right) \\
&= -\frac{1-2q_n}{(2-q_n)r}\left(1-\frac{1}{r}\right)A'^{\,-r}_{n-1}-\left(\frac{1}{r}-(1-q)^{r-1}\right) \\
&\geqslant -\frac{1-2q_n}{(2-q_n)r}\left(1-\frac{1}{r}\right)-\left(\frac{1}{r}-(1-q)^{r-1}\right) \\
&\geqslant -\frac{1-2q}{(2-q)r}\left(1-\frac{1}{r}\right)-\left(\frac{1}{r}-(1-q)^{r-1}\right).
\end{aligned}
$$

We want to show the last expression above is non-negative. By setting $x = 1-q$, we see that this is equivalent to showing that for $1/2 \leqslant x \leqslant 1$,

$$
m(x) := x^r + x^{r-1} - \left(\frac{3}{r}-\frac{2}{r^2}\right)x - \frac{1}{r^2} \geqslant 0. \tag{10}
$$

We have

$$
\begin{aligned}
m'(x) &= rx^{r-1}+(r-1)x^{r-2}-\left(\frac{3}{r}-\frac{2}{r^2}\right), \\
m''(x) &= (r-1)x^{r-3}(rx+r-2).
\end{aligned}
$$

It is easy to see that $m'(x) \geqslant 0$ for $1/2 \leqslant x \leqslant 1$ when $r = 2$. For $1 < r < 2$, we see that

$$
m'\left(\frac{2-r}{r}\right) = \left(\frac{2-r}{r}\right)^{r-2} - \left(\frac{3}{r}-\frac{2}{r^2}\right).
$$

We want to show the above expression is non-negative and we recast it as

$$
\left(\frac{2-r}{r}\right)^{\frac{2-r}{3-r}}\left(\frac{3}{r}-\frac{2}{r^2}\right)^{\frac{1}{3-r}} \leqslant 1.
$$

Applying the arithmetic-geometric mean inequality, we see that

$$\left(\frac{2-r}{r}\right)^{\frac{2-r}{3-r}}\left(\frac{3}{r}-\frac{2}{r^2}\right)^{\frac{1}{3-r}} \leqslant \left(\frac{2-r}{r}\right)\cdot\frac{2-r}{3-r}+\left(\frac{3}{r}-\frac{2}{r^2}\right)\cdot\frac{1}{3-r}.$$

It therefore suffices to show the right-hand side expression above is $\leqslant 1$, which is equivalent to

$$(r-1)(2r-1)(r-2)\leqslant 0.$$

As $1<r<2$, we see that the above inequality holds, hence it follows that $m'((2-r)/r)\geqslant 0$. As it is also easy to see that $m'(1)\geqslant 0$, $m'(1/2)\geqslant 0$, we see that $m'(x)\geqslant 0$ for $1/2\leqslant x\leqslant 1$ when $1<r<2$. Thus, we conclude that when $1<r\leqslant 2$, $m(x)\geqslant m(1/2)\geqslant 0$.

We then conclude that when $1-rq_n>0$, it also suffices to show that $h(A'_{n-1})\geqslant 0$, which is

$$-\left(1-\frac{1}{r}\right)A'^{-r}_{n-1}+(1-(1-q)^{r-1})G'^{(1-q_n)r}_{n-1}A'^{q_nr-2r}_{n-1}-\left(\frac{1}{r}-(1-q)^{r-1}\right)\geqslant 0.$$

We recast the above inequality as

$$\frac{1-(1-q)^{r-1}}{1-1/r}\left(\frac{G'_{n-1}}{A'_{n-1}}\right)^{(1-q_n)r}+\frac{(1-q)^{r-1}-\frac{1}{r}}{1-1/r}A'^{r}_{n-1}\geqslant 1. \tag{11}$$

Applying the arithmetic-geometric mean inequality, we see that

$$\begin{aligned}&\frac{1-(1-q)^{r-1}}{1-1/r}\left(\frac{G'_{n-1}}{A'_{n-1}}\right)^{(1-q_n)r}+\frac{(1-q)^{r-1}-\frac{1}{r}}{1-1/r}A'^{r}_{n-1}\\ &\geqslant\left(\frac{G'_{n-1}}{A'_{n-1}}\right)^{(1-q_n)r^2(1-(1-q)^{r-1})/(r-1)}\cdot A'^{r^2((1-q)^{r-1}-1/r)/(r-1)}_{n-1}.\end{aligned}$$

It follows that in order for (11) to be valid, it suffices to show that

$$G'_{n-1}\geqslant A'^{1-((1-q)^{r-1}-1/r)/((1-q_n)(1-(1-q)^{r-1}))}_{n-1}. \tag{12}$$

Note first that the above inequality holds trivially when

$$1-\frac{(1-q)^{r-1}-1/r}{(1-q_n)(1-(1-q)^{r-1})}\leqslant 0.$$

Thus, we may assume the left-hand side expression above is >0. For any given q_i, $1\leqslant i\leqslant n$ (and hence q), we recast inequality (12) as

$$F(y_1,y_2,\ldots,y_{n-1}):=G'^{\frac{1-q_n}{2-q_n-(1-1/r)/(1-(1-q)^{r-1})}}_{n-1}-A'_{n-1}\geqslant 0. \tag{13}$$

Inequality (13) holds trivially when $n=2$. Assuming $n \geqslant 3$ and let $\mathbf{a}=(a_1,\ldots,a_{n-1}) \in [1,\infty)^{n-1}$ be the point in which the absolute minimum of F is reached. We may assume that $1=a_1<a_2<\ldots<a_{n-1}$. As the function $x-x^r$ is decreasing for $1/2 \leqslant x \leqslant 1$, it follows easily that

$$2(1-q)(1-(1-q)^{r-1}) \leqslant 1-2^{1-r} \leqslant 1-\frac{1}{r}.$$

We then deduce that

$$\frac{q_{n-1}}{2-q_n-(1-1/r)/(1-(1-q)^{r-1})} \geqslant 1. \tag{14}$$

It follows that

$$\lim_{y_{n-1}\to\infty} F=\infty.$$

Thus, $a_2,\ldots,a_{n-1}$ must solve the equation

$$\nabla F=0.$$

As it is easy to see that the above equation has only one root, it is enough to prove the case $n=3$. We write $y=y_2 \geqslant y_1=1$ to recast inequality (13) in this case as

$$v(y):=y^{\frac{q_2}{2-q_3-(1-1/r)/(1-(1-q)^{r-1})}}-\frac{q_2}{1-q_3}y-\frac{q_1}{1-q_3} \geqslant 0.$$

Again, by (14), we see that for $y \geqslant 1$,

$$v'(y) \geqslant \frac{q_2}{2-q_3-(1-1/r)/(1-(1-q)^{r-1})}-\frac{q_2}{1-q_3} \geqslant 0.$$

Thus, we have for $y \geqslant 1$,

$$v(y) \geqslant v(1)=0.$$

This completes the proof of inequality (4).

Lastly, for the reversed inequality of (4) for $1/2 \leqslant r<1$, the proof is similar to that of (4). Here it suffices to show that $h(y_n) \leqslant 0$ for $y_n \geqslant A'_{n-1}$. Note first that in this case we always have $1-rq_n>0$. As $\lim_{y_n\to\infty} h(y_n) \leqslant 0$, we see that if $h(y_n)$ is minimized at some $y_n=y>A'_{n-1}$, then we must have $h'(y)=0$, which again allows us to recast $h(y)$ as the last expression in (9). As $1-r+rq_n \geqslant 0$, we see that

$$\begin{aligned}
h(y) &\leqslant \frac{1}{(2-q_n)r}\Big(1-\frac{1}{r}\Big)(q_ny+(1-q_n)A'_{n-1})^{r-3}A'_{n-1}y^{1-2r} \\
&\quad\times\big((1-r+rq_n)\,q_nA'_{n-1}-(1-rq_n)\,(1-q_n)\,A'_{n-1}\big)-\Big(\frac{1}{r}-(1-q)^{r-1}\Big) \\
&=-\frac{1-2q_n}{(2-q_n)r}\Big(1-\frac{1}{r}\Big)(q_ny+(1-q_n)A'_{n-1})^{r-3}A'^2_{n-1}y^{1-2r}-\Big(\frac{1}{r}-(1-q)^{r-1}\Big).
\end{aligned}$$

It follows that $h(y) \leqslant 0$ when $1-2q_n \leqslant 0$. When $1-2q_n > 0$, we have (note that $r \geqslant 1/2$)

$$\begin{aligned} h(y) &\leqslant -\frac{1-2q_n}{(2-q_n)r}\Big(1-\frac{1}{r}\Big)A'^{r-3}_{n-1}\cdot A'^{2}_{n-1}\cdot A'^{1-2r}_{n-1} - \Big(\frac{1}{r}-(1-q)^{r-1}\Big) \\ &= -\frac{1-2q_n}{(2-q_n)r}\Big(1-\frac{1}{r}\Big)A'^{-r}_{n-1} - \Big(\frac{1}{r}-(1-q)^{r-1}\Big) \\ &\leqslant -\frac{1-2q_n}{(2-q_n)r}\Big(1-\frac{1}{r}\Big) - \Big(\frac{1}{r}-(1-q)^{r-1}\Big) \\ &\leqslant -\frac{1-2q}{(2-q)r}\Big(1-\frac{1}{r}\Big) - \Big(\frac{1}{r}-(1-q)^{r-1}\Big). \end{aligned}$$

It is easy to show that the last expression above is $\leqslant 0$ as the function $m(x)$ defined in (10) is convex for $1/2 \leqslant x \leqslant 1$ and $m(1/2) \leqslant 0$, $m(1) \leqslant 0$. Thus, it remains to prove $h(A'_{n-1}) \leqslant 0$ and we omit the argument here as it is analogue to that of the case $1 < r \leqslant 2$.

3. Proof of Theorem 2

We consider inequalities (5) with $s = 0$ being fixed throughout this section. Once again we omit the discussions on the conditions for equality in each inequality we shall prove. First note that as the right-hand side inequality of (5) for $0 < r \leqslant 1/2$ and $1 \leqslant r \leqslant 2$ follows from Theorem 1 and (6), we only need to prove it for $1/2 < r < 1$. We may assume that $x_1 = 1 < x_2 < \cdots < x_n$, $q_i > 0$, $1 \leqslant i \leqslant n$. Let

$$D(x_1,\cdots,x_n,q_1,\cdots,q_n) = M_{n,r} - G_n - \frac{r}{2}\sigma_n.$$

To show $D \leqslant 0$, it suffices to show that

$$D_1(x_1,\cdots,x_n,q_1,\cdots,q_n) := \frac{\partial D}{q_n \partial x_n} = M_{n,r}^{1-r}x_n^{r-1} - \frac{G_n}{x_n} - r(x_n - A_n) \leqslant 0.$$

When $n \geqslant 3$, we regard $x_1 = 1$, x_n as fixed and assume that D_1 is maximized at some point $(\mathbf{x}';\mathbf{q}') = (x'_1,\cdots,x'_n,q'_1,\cdots,q'_n)$ with $x'_1 = x_1$, $x'_n = x_n$. Then at this point we must have

$$\frac{\partial D_1}{\partial x_i}\Big|_{(\mathbf{x}';\mathbf{q}')} = 0, \quad 2 \leqslant i \leqslant n-1.$$

Thus, the x'_i, $2 \leqslant i \leqslant n-1$ are solutions of the equation:

$$d_1(x) := (1-r)(M_{n,r}(\mathbf{x}';\mathbf{q}'))^{1-2r}x_n^{r-1}x^{r-1} - \frac{G_n(\mathbf{x}';\mathbf{q}')}{x_n x} + r = 0.$$

As $d'_1(x) = 0$ has only one positive root, it follows from the mean value theorem that the above equation can have at most two different positive solutions.

On the other hand, by applying the method of Lagrange multipliers, we let

$$\tilde{D}_1(x_1,\cdots,x_n,q_1,\cdots,q_n) = D_1(x_1,\cdots,x_n,q_1,\cdots,q_n) - \lambda(\sum_{i=1}^{n} q_i - 1),$$

where λ is a constant. Then at $(\mathbf{x}';\mathbf{q}')$ we must have

$$\frac{\partial \tilde{D}_1}{\partial q_i}\Big|_{(\mathbf{x}';\mathbf{q}')} = 0, \quad 1 \leqslant i \leqslant n.$$

Thus, the x_i', $1 \leqslant i \leqslant n$ are solutions of the equation:

$$d_2(x) := \frac{1-r}{r}(M_{n,r}(\mathbf{x}';\mathbf{q}'))^{1-2r} x_n^{r-1} x^r - \frac{G_n(\mathbf{x}';\mathbf{q}')}{x_n}\ln x + rx - \lambda = 0.$$

As $d_2'(x) = d_1(x)$, it follows from the mean value theorem that there is a solution of $d_1(x) = 0$ between any two adjacent x_i', x_{i+1}', $1 \leqslant i \leqslant n-1$, as they are solutions of $d_2(x) = 0$. But when $n \geqslant 3$, we have at least x_2' as a solution of $d_1(x) = 0$. This would imply that $d_1(x) = 0$ has at least three different positive solutions (for example, one in between x_1' and x_2', one in between x_2' and x_3', and x_2' itself), a contradiction.

Thus, it suffices to show $D_1 \leqslant 0$ for $n = 2$. In this case, we let $0 < q_1 = q < 1$, $q_2 = 1-q$, $x_1 = x > x_2 = 1$ (note that we no longer assume $q = \min\{q_1, q_2\}$ from now on) to recast D_1 as

$$D_1(x,q) = (qx^r + 1 - q)^{(1-r)/r} x^{r-1} - x^{q-1} - r(1-q)(x-1).$$

Note that

$$D_2(x,q) := (1-q)^{-1} D_1'(x,q) = (r-1)(q + (1-q)x^{-r})^{(1-2r)/r} x^{-r-1} + x^{q-2} - r.$$

As $r - 1 < 0$, it follows from the arithmetic-geometric mean inequality with non-positive weights that

$$\begin{aligned}
&\frac{r-1}{r}(q + (1-q)x^{-r})^{(1-2r)/r} x^{-r-1} + \frac{1}{r}x^{q-2} \\
\leqslant &(q + (1-q)x^{-r})^{(1-2r)(r-1)/r^2} x^{(q-1-r^2)/r} \\
\leqslant &1.
\end{aligned}$$

This implies that $D_1'(x,q) \leqslant 0$ and hence $D_1(x,q) \leqslant 0$ for $x \geqslant 1$ and this completes the proof for the right-hand side inequality of (5) for $1/2 < r < 1$.

Next, note that

$$\lim_{q \to 0^+} \frac{D(x,1,q,1-q)}{q} = \frac{x^r - 1}{r} - \ln x - \frac{r}{2}(x-1)^2.$$

As the right-hand side expression above is positive when $r > 2$ and $x \to +\infty$, we then conclude that in order for the right-hand side inequality of (5) for $s = 0$ to hold, it is

necessary to have $r \leqslant 2$ and this completes the proof for the assertion on the right-hand side inequality of (5) for $s = 0$.

Note we also have

$$\lim_{q \to 1^-} \frac{D(x,1,q,1-q)}{1-q} = x \ln x - \frac{x - x^{1-r}}{r} - \frac{r}{2}(x-1)^2.$$

As the right-hand side expression above is negative when $0 < r < 1$ and $x \to 0^+$, we conclude that in order for the left-hand side inequality of (5) for $s = 0$ to hold, we must have $r \geqslant 1$. As [4, Lemma 3.1] implies that we also need to have $r \leqslant 3$ in this case, this shows that it is necessary to have $1 \leqslant r \leqslant 3$ in order for the left-hand side inequality of (5) for $s = 0$ to hold.

It remains to prove the left-hand side inequality of (5) for $s = 0$, $1 \leqslant r \leqslant 3$. Note that the case $1 \leqslant r \leqslant 2$ is a consequence of the left-hand side inequality of (6) and the left-hand side inequality of (5) for $s = 1$, $1 < r \leqslant 2$, valid according to [4, Theorem 3.2]. Thus, we may assume that $2 < r \leqslant 3$. In this case, it suffices to show $D \geqslant 0$ provided that we assume $0 < x_1 < x_2 < \cdots < x_n = 1$. Similar to our discussions above, one shows easily that this follows from $\partial D / \partial x_1 \leqslant 0$ for $n = 2$, which is equivalent to $D_1(x,q) \leqslant 0$ for $0 < x \leqslant 1$. As $D_1(1,q) = 0$, it suffices to show that $D_2(x,q) \geqslant 0$ for $0 < x \leqslant 1$. As $\lim_{x \to 0^+} D_2(x,q) > 0$, $D_2(1,q) = 0$, we only need to show the values of D_2 at points satisfying:

$$\frac{\partial D_2}{\partial x} = 0,$$

are non-negative.

Calculation shows that at these points, we have

$$(r-1)(qx^r + 1 - q)^{(1-2r)/r} x^{r-2} = \frac{(qx^r + 1 - q)(2-q)x^{q-2}}{-q(r+1)x^r + (r-2)(1-q)}.$$

We may assume the denominator of the right-hand side expression above is positive. Substituting this back to the expression for $D_2(x,q)$, we see that it remains to show that for $0 < x \leqslant 1$,

$$\frac{(qx^r + 1 - q)(2-q)x^{q-2}}{-q(r+1)x^r + (r-2)(1-q)} + x^{q-2} - r \geqslant 0.$$

As $x^{q-2} \geqslant 1$ for $0 < x \leqslant 1$, it suffices to show for $0 < x \leqslant 1$,

$$\frac{(qx^r + 1 - q)(2-q)x^{q-2}}{-q(r+1)x^r + (r-2)(1-q)} \geqslant r - 1.$$

We recast the above inequality as

$$\begin{aligned} j(x,q) &:= q(2-q)x^{r+q-2} + (1-q)(2-q)x^{q-2} + q(r^2-1)x^r - (1-q)(r-1)(r-2) \\ &\geqslant 0. \end{aligned}$$

Again as $\lim_{x\to 0^+} j(x,q) > 0$, $j(1,q) \geqslant 0$ when $2 < r \leqslant 3$, we only need to show the values of $j(x,q)$ at points satisfying:

$$\frac{\partial j}{\partial x} = 0,$$

are non-negative.

Calculation shows that at these points, we have

$$q(r^2-1)x^r = \frac{(1-q)(2-q)^2 x^{q-2}}{r} - \frac{q(2-q)(r+q-2)x^{r+q-2}}{r}.$$

Substituting this back to the expression for $j(x,q)$, we see that it suffices to show that

$$\begin{aligned} j_1(x,q) &:= (1-q)(2-q)(r+2-q)x^{q-2} + q(2-q)^2 x^{r+q-2} - (1-q)r(r-1)(r-2) \\ &\geqslant 0. \end{aligned}$$

One checks easily that $\lim_{x\to 0^+} j_1(x,q) \geqslant 0$ and that on setting $y = 1-q$, we have

$$\begin{aligned} j_1(1,q) &= (1-q)(2-q)(r+2-q) + q(2-q)^2 - (1-q)r(r-1)(r-2) \\ &= 1 + (r+2-r(r-1)(r-2))y + (r+1)y^2 \geqslant 1 - y + (r+1)y^2 \geqslant 0, \end{aligned}$$

as one checks that $r+2-r(r-1)(r-2)$ is a decreasing function of $2 < r \leqslant 3$, hence is minimized at $r = 3$.

Thus, we only need to show the values of $j_1(x,q)$ at points satisfying:

$$\frac{\partial j_1}{\partial x} = 0,$$

are non-negative.

Calculation shows that at these points, we have

$$q(2-q)^2 x^{r+q-2} = \frac{(1-q)(2-q)^2(r+2-q)x^{q-2}}{r+q-2}.$$

Substituting this back to the expression for $j_1(x,q)$, we see that it suffices to show that for $0 < x \leqslant 1$,

$$\frac{(2-q)(r+2-q)x^{q-2}}{r+q-2} \geqslant (r-1)(r-2).$$

The above inequality is valid as one checks easily that when $2 < r \leqslant 3$,

$$\frac{(2-q)(r+2-q)x^{q-2}}{r+q-2} \geqslant \frac{(2-q)(r+2-q)}{r+q-2} \geqslant \frac{r+1}{r-1} \geqslant (r-1)(r-2).$$

We now conclude that the left-hand side inequality of (5) is valid for $s = 0$, $2 < r \leqslant 3$ and this completes the proof of the theorem.

Acknowledgement. The author is very grateful to the referee for many valuable comments and suggestions.

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(Received November 29, 2016)

Peng Gao
Department of Mathematics
School of Mathematics and Systems Science
Beijing University of Aeronautics and Astronautics
P. R. China
e-mail: penggao@buaa.edu.cn