

## THE MULTI-PARAMETER HAUSDORFF OPERATORS ON $H^1$ AND $L^p$

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(Communicated by I. Perić)

*Abstract.* In the present paper, we characterize the nonnegative functions  $\varphi$  for which the multi-parameter Hausdorff operator  $\mathcal{H}_\varphi$  generated by  $\varphi$  is bounded on either the multi-parameter Hardy space  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$  or  $L^p(\mathbb{R}^n)$ ,  $p \in [1, \infty]$ . The corresponding operator norms are also obtained. Our results improve some recent results in [4, 15, 16, 18] and give an answer to an open question posted by Liflyand [12].

### 1. Introduction and main result

Let  $\varphi$  be a locally integrable function on  $(0, \infty)$ . The classical one-parameter Hausdorff operator  $\mathcal{H}_\varphi$  is defined for suitable functions  $f$  on  $\mathbb{R}$  by

$$\mathcal{H}_\varphi f(x) = \int_0^\infty f\left(\frac{x}{t}\right) \frac{\varphi(t)}{t} dt.$$

The Hausdorff operator  $\mathcal{H}_\varphi$  is an interesting operator in harmonic analysis. There are many classical operators in analysis which are special cases of the Hausdorff operator if one chooses suitable kernel functions  $\varphi$ , such as the classical Hardy operator, its adjoint operator, the Cesàro type operators, the Riemann-Liouville fractional integral operator. See the survey article [13] and the references therein. In the recent years, there is an increasing interest in the study of boundedness of the Hausdorff operator on some function spaces, see for example [1, 2, 4, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19].

When  $\varphi$  is a locally integrable function on  $(0, \infty)^n$ , there are several high-dimensional extensions of  $\mathcal{H}_\varphi$ . One of them is the *multi-parameter Hausdorff operator*  $\mathcal{H}_\varphi$  defined for suitable functions  $f$  on  $\mathbb{R}^n$  by

$$\mathcal{H}_\varphi f(x_1, \dots, x_n) = \int_0^\infty \cdots \int_0^\infty f\left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}\right) \frac{\varphi(t_1, \dots, t_n)}{t_1 \cdots t_n} dt_1 \cdots dt_n.$$

Let  $\Phi^{(1)}, \dots, \Phi^{(n)}$  be  $C^\infty$ -functions with compact support satisfying  $\int_{\mathbb{R}} \Phi^{(1)}(x) dx = \cdots = \int_{\mathbb{R}} \Phi^{(n)}(x) dx = 1$ . Then, for any  $(t_1, \dots, t_n) \in (0, \infty)^n$ , we denote

$$\otimes_{j=1}^n \Phi_{t_j}^{(j)}(\mathbf{x}) := \prod_{j=1}^n \frac{1}{t_j} \Phi^{(j)}\left(\frac{x_j}{t_j}\right), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

*Mathematics subject classification* (2010): 47B38, 42B30.

*Keywords and phrases:* Hausdorff operators, multi-parameter Hardy spaces, Hilbert transforms, maximal functions.

This work is supported by Vietnam National Foundation for Science and Technology Development (Grant No. 101.02-2017.304).

Following Gundy and Stein [6], we define the *multi-parameter Hardy space*  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  as the set of all functions  $f \in L^1(\mathbb{R}^n)$  such that

$$\|f\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} := \|M_\Phi f\|_{L^1(\mathbb{R})} < \infty,$$

where  $M_\Phi f$  is the *multi-parameter smooth maximal function* of  $f$  defined by

$$M_\Phi f(\mathbf{x}) = \sup_{(t_1, \dots, t_n) \in (0, \infty)^n} |f * (\otimes_{j=1}^n \Phi_{t_j}^{(j)})(\mathbf{x})|, \quad \mathbf{x} \in \mathbb{R}^n.$$

REMARK 1.

- (i)  $\|\cdot\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}$  defines a norm on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ , whose size depends on the choice of  $\{\Phi^{(j)}\}_{j=1}^n$ , but the space  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  does not depend on this choice.
- (ii) If  $f$  is in  $H^1(\mathbb{R})$ , then the function

$$f \otimes \dots \otimes f(\mathbf{x}) = \prod_{j=1}^n f(x_j), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

is in  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ . Moreover, there exist two positive constants  $C_1, C_2$  independent of  $f$  such that

$$C_1 \|f\|_{H^1(\mathbb{R})}^n \leq \|f \otimes \dots \otimes f\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} \leq C_2 \|f\|_{H^1(\mathbb{R})}^n.$$

In the setting of two-parameter, Liflyand and Móricz showed in [15] that  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \mathbb{R})$  provided  $\varphi \in L^1((0, \infty)^2)$ . In the setting of  $n$ -parameter, one of Weisz’s important results (see [18, Theorem 7]) showed that  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  provided  $\varphi(t_1, \dots, t_n) = \prod_{i=1}^n \varphi_i(t_i)$  with  $\varphi_i \in L^1(\mathbb{R})$  for all  $1 \leq i \leq n$ . Recently, in the setting of two-parameter, Fan and Zhao showed in [4] that the condition  $\varphi \in L^1((0, \infty)^2)$  is also a necessary condition for  $H^1(\mathbb{R} \times \mathbb{R})$ -boundedness of  $\mathcal{H}_\varphi$  if  $\varphi$  is nonnegative valued. However, it seems that Fan-Zhao’s method can not be used to obtain the exact norm of  $\mathcal{H}_\varphi$  on  $H^1(\mathbb{R} \times \mathbb{R})$ . So, in the setting of  $n$ -parameter, a natural question arises: Can one find the exact norm of  $\mathcal{H}_\varphi$  on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ ? Very recently, in the setting of one-parameter, this question was solved by Hung, Ky and Quang [7].

Motivated by the above question and an open question posted by Liflyand [12, Problem 5], we characterize the nonnegative functions  $\varphi$  for which  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ . More precisely, our main result is the following:

**THEOREM 1.** *Let  $\varphi$  be a nonnegative function in  $L^1_{\text{loc}}((0, \infty)^n)$ . Then  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  if and only if*

$$\int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n < \infty. \tag{1}$$

Moreover, in that case,

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} = \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

Theorem 1 not only gives an affirmative answer to the above question, but also gives an answer to [12, Problem 5]. It should be pointed out that the norm of the Hausdorff operator  $\mathcal{H}_\varphi (\int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n)$  does not depend on the choice of the above functions  $\{\Phi^{(j)}\}_{j=1}^n$ , moreover, it still holds when the above norm  $\|\cdot\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}$  is replaced by

$$\|f\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} := \sum_{\mathbf{e} \in \{0,1\}^n} \|\mathbf{H}_\mathbf{e} f\|_{L^1(\mathbb{R}^n)},$$

where  $\mathbf{H}_\mathbf{e} f$ 's are the *multi-parameter Hilbert transforms* of  $f$ . See Theorem 8 for details.

Also we characterize the nonnegative functions  $\varphi$  for which  $\mathcal{H}_\varphi$  is bounded on  $L^p(\mathbb{R}^n)$ ,  $p \in [1, \infty]$ . Our next result can be stated as follows.

**THEOREM 2.** *Let  $p \in [1, \infty]$  and let  $\varphi$  be a nonnegative function in  $L^1_{\text{loc}}((0, \infty)^n)$ . Then  $\mathcal{H}_\varphi$  is bounded on  $L^p(\mathbb{R}^n)$  if and only if*

$$\int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \dots dt_n < \infty. \tag{2}$$

Moreover, in that case,

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} = \int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \dots dt_n.$$

Throughout the whole article, we always assume that  $\varphi$  is a nonnegative function in  $L^1_{\text{loc}}((0, \infty)^n)$  and denote by  $C$  a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol  $A \lesssim B$  means that  $A \leq CB$ . If  $A \lesssim B$  and  $B \lesssim A$ , then we write  $A \sim B$ .

### 2. Norm of $\mathcal{H}_\varphi$ on $L^p(\mathbb{R}^n)$

The main purpose of this section is to give the proof of Theorem 2. Let us first consider the operator  $\mathcal{H}_\varphi^*$  defined by

$$\mathcal{H}_\varphi^* f(x_1, \dots, x_n) = \int_0^\infty \dots \int_0^\infty f(t_1 x_1, \dots, t_n x_n) \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

Studying this operator on the spaces  $L^p(\mathbb{R}^n)$  is useful in proving the main theorem (Theorem 1) in the next section.

Remark that  $\mathcal{H}_\varphi^* = \mathcal{H}_{\overline{\varphi}}$  with  $\overline{\varphi}(\mathbf{t}) = \frac{\varphi(1/t_1, \dots, 1/t_n)}{t_1 \dots t_n}$  for all  $\mathbf{t} = (t_1, \dots, t_n) \in (0, \infty)^n$ . Hence, by Theorems 1 and 2, we obtain:

**THEOREM 3.**  *$\mathcal{H}_\varphi^*$  is bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  if and only if*

$$\int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n < \infty. \tag{3}$$

Moreover, in that case,

$$\|\mathcal{H}_\varphi^*\|_{H^1(\mathbb{R}\times\dots\times\mathbb{R})\rightarrow H^1(\mathbb{R}\times\dots\times\mathbb{R})} = \int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n.$$

**THEOREM 4.** *Let  $p \in [1, \infty]$ . Then  $\mathcal{H}_\varphi^*$  is bounded on  $L^p(\mathbb{R}^n)$  if and only if*

$$\int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1/p} \dots t_n^{1/p}} dt_1 \dots dt_n < \infty. \tag{4}$$

Moreover, in that case,

$$\|\mathcal{H}_\varphi^*\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} = \int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1/p} \dots t_n^{1/p}} dt_1 \dots dt_n.$$

By Theorems 2, 4 and the Fubini theorem,  $\mathcal{H}_\varphi^*$  can be viewed as the Banach space adjoint of  $\mathcal{H}_\varphi$  and vice versa. More precisely, we have:

**THEOREM 5.** *Let  $p \in [1, \infty]$  and  $1/p' + 1/p = 1$ .*

(i) *If (2) holds, then, for all  $f \in L^p(\mathbb{R}^n)$  and all  $g \in L^{p'}(\mathbb{R}^n)$ ,*

$$\int_{\mathbb{R}^n} \mathcal{H}_\varphi f(\mathbf{x})g(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x})\mathcal{H}_\varphi^* g(\mathbf{x})d\mathbf{x}.$$

(ii) *If (4) holds, then, for all  $f \in L^p(\mathbb{R}^n)$  and all  $g \in L^{p'}(\mathbb{R}^n)$ ,*

$$\int_{\mathbb{R}^n} \mathcal{H}_\varphi^* f(\mathbf{x})g(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x})\mathcal{H}_\varphi g(\mathbf{x})d\mathbf{x}.$$

As a consequence of the above theorem, we get the following.

**COROLLARY 1.** *Let  $p \in [1, 2]$ .*

(i) *If (2) holds, then, for all  $f \in L^p(\mathbb{R}^n)$ ,*

$$\widehat{\mathcal{H}_\varphi f} = \mathcal{H}_\varphi^* \hat{f}.$$

(ii) *If (4) holds, then, for all  $f \in L^p(\mathbb{R}^n)$ ,*

$$\widehat{\mathcal{H}_\varphi^* f} = \mathcal{H}_\varphi \hat{f}.$$

*Proof.* We prove only (i) since the proof of (ii) is similar. Moreover, from the Hausdorff-Young theorem and the fact that  $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n)$ , we

consider only the case  $p = 1$ . For all  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ , by Theorem 5(i) and the Fubini theorem, we get

$$\begin{aligned} \widehat{\mathcal{H}_\varphi f}(\mathbf{y}) &= \int_{\mathbb{R}^n} \mathcal{H}_\varphi f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} \int_0^\infty \dots \int_0^\infty e^{-2\pi i \sum_{j=1}^n t_j x_j y_j} \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_0^\infty \dots \int_0^\infty \hat{f}(t_1 y_1, \dots, t_n y_n) \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \mathcal{H}_\varphi^* \hat{f}(\mathbf{y}). \end{aligned}$$

This completes the proof of Corollary 1.  $\square$

*Proof of Theorem 2.* Since the case  $p = \infty$  is trivial, we consider only the case  $p \in [1, \infty)$ . Suppose that (2) holds. For any  $f \in L^p(\mathbb{R}^n)$ , by the Minkowski inequality, we obtain

$$\begin{aligned} \|\mathcal{H}_\varphi f\|_{L^p(\mathbb{R}^n)} &\leq \int_0^\infty \dots \int_0^\infty \left\| f\left(\frac{\cdot}{t_1}, \dots, \frac{\cdot}{t_n}\right) \right\|_{L^p(\mathbb{R}^n)} \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n \\ &= \|f\|_{L^p(\mathbb{R}^n)} \int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \dots dt_n. \end{aligned}$$

This proves that  $\mathcal{H}_\varphi$  is bounded on  $L^p(\mathbb{R}^n)$ , moreover,

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq \int_0^\infty \dots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \dots dt_n. \tag{5}$$

Conversely, suppose that  $\mathcal{H}_\varphi$  is bounded on  $L^p(\mathbb{R}^n)$ . For any  $\varepsilon > 0$ , take

$$f_\varepsilon(\mathbf{x}) = \prod_{j=1}^n |x_j|^{-1/p-\varepsilon} \chi_{\{y_j \in \mathbb{R}: |y_j| \geq 1\}}(x_j)$$

for all  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Then, it is easy to see that  $f_\varepsilon \in L^p(\mathbb{R}^n)$  and

$$\mathcal{H}_\varphi f_\varepsilon(\mathbf{x}) = \prod_{j=1}^n |x_j|^{-1/p-\varepsilon} \int_0^{|x_1|} dt_1 \dots \int_0^{|x_{n-1}|} dt_{n-1} \int_0^{|x_n|} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_n$$

for all  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Some simple computations give

$$\begin{aligned} \|\mathcal{H}_\varphi f_\varepsilon\|_{L^p(\mathbb{R}^n)} &\geq \int_0^{1/\varepsilon} \dots \int_0^{1/\varepsilon} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_1 \dots dt_n \times \\ &\quad \times \left( \prod_{j=1}^n \int_{\{x_j \in \mathbb{R}: |x_j| \geq 1\}} |x_j|^{-1-p\varepsilon} dx_j \right)^{1/p} \\ &= \int_0^{1/\varepsilon} \dots \int_0^{1/\varepsilon} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_1 \dots dt_n (\varepsilon^{n\varepsilon} \|f_\varepsilon\|_{L^p(\mathbb{R}^n)})^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} &\geq \frac{\|\mathcal{H}_\varphi f_\varepsilon\|_{L^p(\mathbb{R}^n)}}{\|f_\varepsilon\|_{L^p(\mathbb{R}^n)}} \\ &\geq \varepsilon^{n\varepsilon} \int_0^{1/\varepsilon} \cdots \int_0^{1/\varepsilon} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \cdots t_n^{1-1/p-\varepsilon}} dt_1 \cdots dt_n. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} \geq \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \cdots t_n^{1-1/p}} dt_1 \cdots dt_n.$$

This, together (5), implies that

$$\|\mathcal{H}_\varphi\|_{L^p(\mathbb{R}^n)\rightarrow L^p(\mathbb{R}^n)} = \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \cdots t_n^{1-1/p}} dt_1 \cdots dt_n,$$

and thus ends the proof of Theorem 2.  $\square$

### 3. Norm of $\mathcal{H}_\varphi$ on $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$

The main purpose of this section is to give the proof of Theorem 1 and to show that the norm of the Hausdorff operator  $\mathcal{H}_\varphi$  in Theorem 1 still holds when one replaces the norm  $\|\cdot\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R})}$  by the norm  $\|\cdot\|_*$  (see (12) and Theorem 8 below).

Let  $\mathbb{C}_+^n$  be the upper half-plan in  $\mathbb{C}^n$ , that is,

$$\mathbb{C}_+^n = \prod_{j=1}^n \{z_j = x_j + iy_j \in \mathbb{C} : y_j > 0\}.$$

Following Gundy-Stein [6] and Lacey [9], a function  $F : \mathbb{C}_+^n \rightarrow \mathbb{C}$  is said to be in the Hardy space  $\mathcal{H}_a^1(\mathbb{C}_+^n)$  if it is holomorphic in each variable separately and

$$\|F\|_{\mathcal{H}_a^1(\mathbb{C}_+^n)} := \sup_{(y_1, \dots, y_n) \in (0, \infty)^n} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty |F(x_1 + iy_1, \dots, x_n + iy_n)| dx_1 \cdots dx_n < \infty.$$

Let  $j \in \{1, \dots, n\}$ . For any  $f \in L^1(\mathbb{R}^n)$ , the Hilbert transform  $H_j f$  computed in the  $j^{\text{th}}$  variable is defined by

$$H_j f(\mathbf{x}) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^\infty \frac{f(x_1, \dots, x_j - y, \dots, x_n)}{y} dy.$$

For any  $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{E} := \{0, 1\}^n$ , denote

$$\mathbf{H}_\mathbf{e} = \prod_{j=1}^n H_j^{e_j}$$

with  $H_j^{e_j} = I$  for  $e_j = 0$  while  $H_j^{e_j} = H_j$  for  $e_j = 1$ .

The following two theorems are well-known, see for example [6, 9, 10, 18].

**THEOREM 6.** *A function  $f$  is in  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$  if and only if  $\mathbf{H}_e f$  is in  $L^1(\mathbb{R}^n)$  for all  $e \in \mathbb{E}$ . Moreover, in that case,*

$$\|f\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \sim \sum_{e \in \mathbb{E}} \|\mathbf{H}_e f\|_{L^1(\mathbb{R}^n)}.$$

**THEOREM 7.** *Let  $F \in \mathcal{H}_a^1(\mathbb{C}_+^n)$ . Then the boundary value function  $f$  of  $F$ , which is defined by*

$$f(x_1, \dots, x_n) = \lim_{(y_1, \dots, y_n) \rightarrow (0, \dots, 0)} F(x_1 + iy_1, \dots, x_n + iy_n),$$

*a. e.  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , is in  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$ . Moreover,*

$$\|f\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \sim \|f\|_{L^1(\mathbb{R}^n)} = \|F\|_{\mathcal{H}_a^1(\mathbb{C}_+^n)}$$

*and, for all  $\mathbf{x} + i\mathbf{y} = (x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}_+^n$ ,*

$$\begin{aligned} F(\mathbf{x} + i\mathbf{y}) &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1 - u_1, \dots, x_n - u_n) \prod_{j=1}^n \frac{1}{y_j} P\left(\frac{u_j}{y_j}\right) du_1 \dots du_n \\ &=: f * (\otimes_{j=1}^n P_{y_j})(\mathbf{x}), \end{aligned}$$

*where  $P(u) = \frac{1}{1+u^2}$ ,  $u \in \mathbb{R}$ , is the Poisson kernel on  $\mathbb{R}$ .*

In order to prove Theorem 1, we also need the following two lemmas.

**LEMMA 1.** *Let  $\varphi$  be such that  $\mathcal{H}_\varphi$  is bounded from  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$  into  $L^1(\mathbb{R}^n)$ . Then (1) holds.*

**LEMMA 2.** *Let  $\varphi$  be such that (1) holds. Then:*

(i)  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$ , moreover,

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \leq \int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

(ii) *If  $\text{supp } \varphi \subset [0, 1]^n$ , then*

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} = \int_0^1 \cdots \int_0^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

*Proof of Lemma 1.* Since the function

$$f(x) = \frac{x}{(1+x^2)^2}, \quad x \in \mathbb{R},$$

is in  $H^1(\mathbb{R})$  (see [7, Theorem 3.3]), Remark 1(ii) yields that

$$f \otimes \cdots \otimes f(\mathbf{x}) = \prod_{j=1}^n \frac{x_j}{(1+x_j^2)^2}, \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

is in  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ . Hence, the function

$$\mathcal{H}_\varphi(f \otimes \dots \otimes f)(\mathbf{x}) = \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{\frac{x_j}{t_j}}{\left[1 + \left(\frac{x_j}{t_j}\right)^2\right]^2} \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n,$$

$\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , is in  $L^1(\mathbb{R}^n)$  since  $\mathcal{H}_\varphi$  is bounded from  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$  into  $L^1(\mathbb{R}^n)$ . As a consequence,

$$\begin{aligned} & \left[ \int_0^\infty \frac{y}{(1+y^2)^2} dy \right]^n \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_{[0, \infty)^n} d\mathbf{x} \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{\frac{x_j}{t_j}}{\left[1 + \left(\frac{x_j}{t_j}\right)^2\right]^2} \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n \\ &\leq \| \mathcal{H}_\varphi(f \otimes \dots \otimes f) \|_{L^1(\mathbb{R}^n)} < \infty \end{aligned}$$

which proves (1), and thus ends the proof of Lemma 1.  $\square$

*Proof of Lemma 2.* (i) For any  $f \in H^1(\mathbb{R} \times \dots \times \mathbb{R})$ , by the Fubini theorem,

$$\begin{aligned} & M_\Phi(\mathcal{H}_\varphi f)(\mathbf{x}) \\ &= \sup_{(r_1, \dots, r_n) \in (0, \infty)^n} \left| \int_{\mathbb{R}^n} d\mathbf{y} \int_0^\infty \dots \int_0^\infty (\otimes_{j=1}^n \Phi_{r_j}^{(j)})(\mathbf{x}-\mathbf{y}) f\left(\frac{y_1}{t_1}, \dots, \frac{y_n}{t_n}\right) \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n \right| \\ &= \sup_{(r_1, \dots, r_n) \in (0, \infty)^n} \left| \int_0^\infty \dots \int_0^\infty \left(f * (\otimes_{j=1}^n \Phi_{r_j/t_j}^{(j)})\right)\left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}\right) \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n \right| \\ &\leq \mathcal{H}_\varphi(M_\Phi f)(\mathbf{x}) \end{aligned}$$

for all  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Hence, by Theorem 2,

$$\begin{aligned} \| \mathcal{H}_\varphi f \|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} &= \| M_\Phi(\mathcal{H}_\varphi f) \|_{L^1(\mathbb{R}^n)} \\ &\leq \| \mathcal{H}_\varphi(M_\Phi f) \|_{L^1(\mathbb{R}^n)} \\ &\leq \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \| M_\Phi f \|_{L^1(\mathbb{R}^n)} \\ &= \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \| f \|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}. \end{aligned}$$

This proves that  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ , moreover,

$$\| \mathcal{H}_\varphi \|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} \leq \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n. \tag{6}$$

(ii) Let  $\delta \in (0, 1)$  be arbitrary. Set  $\varphi_\delta(\mathbf{t}) := \varphi(\mathbf{t})\chi_{[\delta, 1]^n}(\mathbf{t})$  for all  $\mathbf{t} \in (0, \infty)^n$ . Then, by (6), we see that

$$\begin{aligned} \| \mathcal{H}_{\varphi_\delta} \|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} &\leq \int_0^\infty \dots \int_0^\infty \varphi_\delta(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_\delta^1 \dots \int_\delta^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n < \infty \end{aligned}$$

and

$$\begin{aligned} & \| \mathcal{H}_\varphi - \mathcal{H}_{\varphi_\delta} \|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} \\ & \leq \int_0^\infty \dots \int_0^\infty [\varphi(t_1, \dots, t_n) - \varphi_\delta(t_1, \dots, t_n)] dt_1 \dots dt_n \\ & = \int_{(0,1]^n \setminus [\delta,1]^n} \varphi(\mathbf{t}) dt < \infty. \end{aligned} \tag{7}$$

For any  $\varepsilon > 0$ , we define the function  $F_\varepsilon : \mathbb{C}_+^n \rightarrow \mathbb{C}$  by

$$F_\varepsilon(z_1, \dots, z_n) = \prod_{j=1}^n \frac{1}{(z_j + i)^{1+\varepsilon}}$$

where  $\zeta^{1+\varepsilon} = |\zeta|^{1+\varepsilon} e^{i(1+\varepsilon)\arg \zeta}$  for all  $\zeta \in \mathbb{C}$ . Denote by  $f_\varepsilon$  the boundary value function of  $F_\varepsilon$ , that is,  $f_\varepsilon(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow 0} F_\varepsilon(\mathbf{x} + i\mathbf{y})$ . Then, by Theorem 7,

$$\|f_\varepsilon\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} \sim \|F_\varepsilon\|_{\mathcal{H}_d^1(\mathbb{C}_+^n)} = \left[ \int_{-\infty}^\infty \frac{1}{\sqrt{x^2 + 1}^{1+\varepsilon}} dx \right]^n < \infty, \tag{8}$$

where the constants are independent of  $\varepsilon$ .

For all  $\mathbf{z} = \mathbf{x} + i\mathbf{y} = (x_1 + iy_1, \dots, x_n + iy_n) = (z_1, \dots, z_n) \in \mathbb{C}_+^n$ , by the Fubini theorem and Theorem 7, we get

$$\begin{aligned} & \left( \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t}) dt \right) * (\otimes_{j=1}^n P_{y_j})(\mathbf{x}) \\ & = \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{1}{(z_j + i)^{1+\varepsilon}} \frac{\varphi_\delta(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \dots dt_n \\ & \quad - \prod_{j=1}^n \frac{1}{(z_j + i)^{1+\varepsilon}} \int_0^\infty \dots \int_0^\infty \varphi_\delta(t_1, \dots, t_n) dt_1 \dots dt_n \\ & = \int_\delta^1 \dots \int_\delta^1 [\phi_{\varepsilon,\mathbf{z}}(t_1, \dots, t_n) - \phi_{\varepsilon,\mathbf{z}}(1, \dots, 1)] \varphi(t_1, \dots, t_n) dt_1 \dots dt_n, \end{aligned}$$

where  $\phi_{\varepsilon,\mathbf{z}}(t_1, \dots, t_n) := \prod_{j=1}^n \frac{t_j^\varepsilon}{(z_j + it_j)^{1+\varepsilon}}$ . For any  $\mathbf{t} = (t_1, \dots, t_n) \in [\delta, 1]^n$ , a simple calculus gives

$$\begin{aligned} & |\phi_{\varepsilon,\mathbf{z}}(t_1, \dots, t_n) - \phi_{\varepsilon,\mathbf{z}}(1, \dots, 1)| \\ & \leq \sup_{s \in [0,1]} \sum_{j=1}^n |t_j - 1| \left| \frac{\partial \phi_{\varepsilon,\mathbf{z}}}{\partial t_j}(t_j + s(1 - t_j)) \right| \\ & \leq \sum_{j=1}^n \left( \frac{\varepsilon \delta^{-2}}{\sqrt{x_j^2 + 1}^{1+\varepsilon}} + \frac{(1 + \varepsilon) \delta^{-2}}{\sqrt{x_j^2 + 1}^{2+\varepsilon}} \right) \prod_{\substack{k=1 \\ k \neq j}}^n \frac{\delta^{-1}}{\sqrt{x_k^2 + 1}^{1+\varepsilon}}. \end{aligned}$$

Therefore, by Theorem 7 again,

$$\begin{aligned} & \left\| \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t}) d\mathbf{t} \right\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})} \\ & \lesssim \left\| \sup_{(y_1, \dots, y_n) \in (0,\infty)^n} \left( \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t}) d\mathbf{t} \right) * (\otimes_{j=1}^n P_{y_j}) \right\|_{L^1(\mathbb{R}^n)} \\ & \leq \int_\delta^1 \dots \int_\delta^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \times \\ & \quad \times \sum_{j=1}^n \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \left( \frac{\varepsilon \delta^{-2}}{\sqrt{x_j^2 + 1}^{1+\varepsilon}} + \frac{(1 + \varepsilon) \delta^{-2}}{\sqrt{x_j^2 + 1}^{2+\varepsilon}} \right) \prod_{\substack{k=1 \\ k \neq j}}^n \frac{\delta^{-1}}{\sqrt{x_k^2 + 1}^{1+\varepsilon}} dx_1 \dots dx_n. \end{aligned}$$

This, together with (8), yields

$$\begin{aligned} & \frac{\left\| \mathcal{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t}) d\mathbf{t} \right\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}}{\|f_\varepsilon\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}} \tag{9} \\ & \lesssim \int_\delta^1 \dots \int_\delta^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \times \\ & \quad \times \sum_{j=1}^n \frac{\delta^{1-n} \left[ \varepsilon \delta^{-2} \int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2 + 1}^{1+\varepsilon}} dx_j + (1 + \varepsilon) \delta^{-2} \int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2 + 1}^{2+\varepsilon}} dx_j \right]}{\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2 + 1}^{1+\varepsilon}} dx_j} \\ & \lesssim \int_\delta^1 \dots \int_\delta^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \times \\ & \quad \times \sum_{j=1}^n \left[ \varepsilon \delta^{-1-n} + \frac{(1 + \varepsilon) \delta^{-1-n} \int_{-\infty}^\infty \frac{1}{x_j^2 + 1} dx_j}{\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2 + 1}^{1+\varepsilon}} dx_j} \right] \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . As a consequence,

$$\begin{aligned} \int_\delta^1 \dots \int_\delta^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n &= \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t}) d\mathbf{t} \\ &\leq \|\mathcal{H}_{\varphi_\delta}\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})}. \end{aligned}$$

This, together with (7), allows us to conclude that

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} \geq \int_0^1 \dots \int_0^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n$$

since  $\lim_{\delta \rightarrow 0} \int_{(0,1]^n \setminus [\delta,1]^n} \varphi(\mathbf{t}) d\mathbf{t} = 0$ . Hence, by (6),

$$\|\mathcal{H}_\varphi\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \dots \times \mathbb{R})} = \int_0^1 \dots \int_0^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

This completes the proof of Lemma 2.  $\square$

Now we are ready to give the proof of Theorem 1.

*Proof of Theorem 1.* By Lemma 2(i), it suffices to prove that

$$\int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \leq \| \mathcal{H}_\varphi \|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \tag{10}$$

provided  $\mathcal{H}_\varphi$  is bounded on  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$ . Indeed, by Lemma 1, we have

$$\int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n < \infty.$$

For any  $m > 0$ , set  $\varphi_m(\mathbf{t}) := \varphi(m\mathbf{t})\chi_{(0,1)^n}(\mathbf{t})$ . Then, by Lemma 2(i), we see that

$$\begin{aligned} & \left\| \mathcal{H}_\varphi - \mathcal{H}_{\varphi_m(\frac{\cdot}{m})} \right\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \tag{11} \\ &= \left\| \mathcal{H}_{\varphi - \varphi_m(\frac{\cdot}{m})} \right\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \\ &\leq \int_0^\infty \cdots \int_0^\infty \left[ \varphi(t_1, \dots, t_n) - \varphi_m\left(\frac{t_1}{m}, \dots, \frac{t_n}{m}\right) \right] dt_1 \dots dt_n \\ &= \int_{(0,\infty)^n \setminus (0,m)^n} \varphi(\mathbf{t}) d\mathbf{t}. \end{aligned}$$

Noting that

$$\left\| f\left(\frac{\cdot}{m}\right) \right\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R})} = m^n \|f(\cdot)\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \quad \text{and} \quad \mathcal{H}_{\varphi_m(\frac{\cdot}{m})} = \mathcal{H}_{\varphi_m} f\left(\frac{\cdot}{m}\right)$$

for all  $f \in H^1(\mathbb{R} \times \cdots \times \mathbb{R})$ , Lemma 2(ii) gives

$$\begin{aligned} \left\| \mathcal{H}_{\varphi_m(\frac{\cdot}{m})} \right\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} &= m^n \left\| \mathcal{H}_{\varphi_m} \right\|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \\ &= m^n \int_0^1 \cdots \int_0^1 \varphi_m(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_0^m \cdots \int_0^m \varphi(t_1, \dots, t_n) dt_1 \dots dt_n. \end{aligned}$$

Combining this with (11) allow us to conclude that

$$\| \mathcal{H}_\varphi \|_{H^1(\mathbb{R} \times \cdots \times \mathbb{R}) \rightarrow H^1(\mathbb{R} \times \cdots \times \mathbb{R})} \geq \int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n$$

since  $\lim_{m \rightarrow \infty} \int_{(0,\infty)^n \setminus (0,m)^n} \varphi(\mathbf{t}) d\mathbf{t} = 0$ . This proves (10), and thus ends the proof of Theorem 1.  $\square$

From Theorem 6, one can define  $H^1(\mathbb{R} \times \cdots \times \mathbb{R})$  as the space of functions  $f \in L^1(\mathbb{R}^n)$  such that

$$\|f\|_* := \sum_{\mathbf{e} \in \mathbb{E}} \| \mathbf{H}_{\mathbf{e}} f \|_{L^1(\mathbb{R}^n)} < \infty. \tag{12}$$

Our last result is the following:

**THEOREM 8.**  $\mathcal{H}_\varphi$  is bounded on  $(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)$  if and only if (1) holds. Moreover, in that case,

$$\|\mathcal{H}_\varphi\|_{(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*) \rightarrow (H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)} = \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n$$

and, for any  $\mathbf{e} \in \mathbb{E}$ ,  $\mathcal{H}_\varphi$  commutes with  $\mathbf{H}_\mathbf{e}$  on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ .

In order to prove Theorem 8, we need the following two lemmas.

**LEMMA 3.** Let  $\varphi$  be such that (1) holds. Then, for any  $\mathbf{e} \in \mathbb{E}$ ,  $\mathcal{H}_\varphi$  commutes with the Hilbert transform  $\mathbf{H}_\mathbf{e}$  on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ .

**LEMMA 4.** Let  $\varphi$  be such that (1) holds. Then:

(i)  $\mathcal{H}_\varphi$  is bounded on  $(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)$ , moreover,

$$\|\mathcal{H}_\varphi\|_{(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*) \rightarrow (H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)} \leq \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

(ii) If  $\text{supp } \varphi \subset [0, 1]^n$ , then

$$\|\mathcal{H}_\varphi\|_{(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*) \rightarrow (H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)} = \int_0^1 \dots \int_0^1 \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

*Proof of Lemma 3.* Since Theorem 1 and the fact that  $H_j$ 's are bounded on  $H^1(\mathbb{R} \times \dots \times \mathbb{R})$ , it suffices to prove

$$\mathcal{H}_\varphi H_j f = H_j \mathcal{H}_\varphi f \tag{13}$$

for all  $j \in \{1, \dots, n\}$  and all  $f \in H^1(\mathbb{R} \times \dots \times \mathbb{R})$ . Indeed, thanks to the ideas from [1, 15, 16] and Lemma 1(i), for almost every  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ ,

$$\begin{aligned} \widehat{\mathcal{H}_\varphi H_j f}(\mathbf{y}) &= \int_0^\infty \dots \int_0^\infty \widehat{H_j f}(t_1 y_1, \dots, t_n y_n) \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_0^\infty \dots \int_0^\infty (-i \text{sign}(t_j y_j)) \widehat{f}(t_1 y_1, \dots, t_n y_n) \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= (-i \text{sign } y_j) \widehat{\mathcal{H}_\varphi f}(\mathbf{y}) = \widehat{H_j \mathcal{H}_\varphi f}(\mathbf{y}). \end{aligned}$$

This proves (13), and thus ends proof of Lemma 3, since the uniqueness of the Fourier transform.  $\square$

*Proof of Lemma 4.* (i) For all  $f \in H^1(\mathbb{R} \times \dots \times \mathbb{R})$  and all  $\mathbf{e} \in \mathbb{E}$ , by Lemma 3 and Theorem 2, we get

$$\begin{aligned} \|\mathbf{H}_\mathbf{e} \mathcal{H}_\varphi f\|_{L^1(\mathbb{R}^n)} &= \|\mathcal{H}_\varphi \mathbf{H}_\mathbf{e} f\|_{L^1(\mathbb{R}^n)} \\ &\leq \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \|\mathbf{H}_\mathbf{e} f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

This proves that

$$\|\mathcal{H}_\varphi\|_{(H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*) \rightarrow (H^1(\mathbb{R} \times \dots \times \mathbb{R}), \|\cdot\|_*)} \leq \int_0^\infty \dots \int_0^\infty \varphi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

(ii) The proof is similar to that of Lemma 2(ii) and will be omitted. The key point is the estimate (9) and the fact that  $\|\cdot\|_* \sim \|\cdot\|_{H^1(\mathbb{R} \times \dots \times \mathbb{R})}$ .  $\square$

*Proof of Theorem 8.* The proof is similar to that of Theorem 1 by Lemma 4. We leave the details to the interested readers.  $\square$

*Acknowledgements.* The authors would like to thank the referees for their carefully reading and helpful suggestions.

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(Received July 26, 2017)

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