

WEIGHTED MORREY ESTIMATES FOR HAUSDORFF OPERATOR AND ITS COMMUTATOR ON THE HEISENBERG GROUP

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Abstract. In this paper, we study the high-dimensional Hausdorff operators, defined via a general linear mapping A , and their commutators on the weighted Morrey spaces in the setting of the Heisenberg group. Particularly, under some assumption on the mapping A , we establish their sharp boundedness on the power weighted Morrey spaces.

1. Introduction

Let \mathbf{R}^n be the Euclidean space of dimension n . Lerner and Liflyand in [18] studied the Hausdorff operator $H_{\Phi,A}$ defined by

$$H_{\Phi,A}(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f(xA(y)) dy,$$

where $A(y)$ is an $n \times n$ matrix satisfying $\det A(y) \neq 0$ (almost everywhere in the support of a fixed integrable function Φ). By choosing

$$A(y) = \text{diag}[1/|y|, 1/|y|, \dots, 1/|y|],$$

one then defines $H_{\Phi,A}$ in this special case by

$$H_{\Phi}(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy.$$

In the definition of $H_{\Phi,A}(f)$, for simplicity, one may always assume that functions f initially lie in the Schwartz space S . After we establish the boundedness of $H_{\Phi,A}(f)$ for $f \in S$ on a normed (or quasi-normed) space X , we can use a standard dense argument together with the Hahn-Banach theorem to easily extend the boundedness of $H_{\Phi,A}$ to the whole space X . For the Lebesgue space L^p ($p \geq 1$) and the Hardy space H^1 , the boundedness of H_{Φ} (even $H_{\Phi,A}$) are well established (see [4, 7, 19, 20, 23, 24, 27, 30, 35]). Besides spaces L^p and H^1 , the boundedness of H_{Φ} and its special cases

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on other function spaces was recently also studied by many authors (see, for instance, [5, 15, 22, 26, 31, 32, 36, 37, 38] and the references therein). Here, we recommend two recent survey papers [6] and [21] for understanding further the background and historical development of this research topic. Particularly, it is notable that many well known operators in analysis can be derived from the Hausdorff operator if one chooses suitable generating functions Φ [6].

This paper is aimed to study the Hausdorff operator on Morrey spaces. The classical Morrey spaces introduced by Morrey [28] are a useful work frame in the study of the existence and regularity of partial differential equations. It has been obtained that many properties of solutions to partial differential equations are concerned with the boundedness of some operators on Morrey type spaces. Therefore, in recent years there has been an explosion of interest on the boundedness of operators in Morrey type spaces. For this information, one can refer to [1, 3, 9] and references therein. On the other hand, Chiarenza and Frasca [8] established the boundedness of the Hardy-Littlewood maximal operator, of the fractional integral operator, and of the singular integral operator on Morrey spaces. Subsequently, Komori and Shirai [17] extended the results of [8] to the weighted Morrey spaces. Alvarez, Lakey and Guzmán-Partida [2] studied the central Morrey spaces.

Inspired by above mentioned research, the purpose of this paper is to study the boundedness of Hausdorff operator, as well as its commutator, on the weighted central Morrey spaces in the setting of the Heisenberg group $\dot{L}^{p,\lambda}(\mathbb{H}^n; w)$ (see next section for the definition). We remark that the Hausdorff operator is a linear operator, while we can view its commutator as a bilinear operator (see Definition 1.1).

The Heisenberg group \mathbb{H}^n is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbf{R}^{2n} \times \mathbf{R}$ and the group law

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2 \sum_{j=1}^n (y_j x_{n+j} - x_j y_{n+j}) \right),$$

where $x = (x_1, x_2, \dots, x_{2n+1})$, $y = (y_1, y_2, \dots, y_{2n+1})$. The geometric motions on the Heisenberg group \mathbb{H}^n are quite different from those on \mathbf{R}^n due to the loss of interchangeability. On the other hand we find that \mathbb{H}^n inherits some basic structures of \mathbf{R}^n . These inheritances are good enough for us to study the Hausdorff operator on \mathbb{H}^n . Also, since the Heisenberg group plays significant roles in many math branches such as representation theory, several complex analysis, harmonic analysis, partial differential equations and quantum mechanics (see [12, 34] for more details), an extension of Hausdorff operator to the Heisenberg group seems interesting and encouraging.

By the definition, the identity element on \mathbb{H}^n is $0 \in \mathbf{R}^{2n+1}$, while the inverse element of x is $-x$. The corresponding Lie algebra is generated by the left-invariant

vector fields:

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2x_{n+j} \frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{n+j} &= \frac{\partial}{\partial x_{n+j}} - 2x_j \frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n, \\ X_{2n+1} &= \frac{\partial}{\partial x_{2n+1}}. \end{aligned}$$

The only non-trivial commutator relations are

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

\mathbb{H}^n is a homogeneous group in the sense of Folland and Stein [10] with dilations

$$\delta_r(x_1, x_2, \dots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \dots, rx_{2n}, r^2x_{2n+1}), \quad r > 0.$$

The Haar measure on \mathbb{H}^n coincides with the usual Lebesgue measure on $\mathbf{R}^{2n} \times \mathbf{R}$. We denote the measure of any measurable set $E \subset \mathbb{H}^n$ by $|E|$. It is easy to check that

$$|\delta_r(E)| = r^Q |E|, \quad d(\delta_r x) = r^Q dx.$$

In the above, $Q = 2n + 2$ is the homogeneous dimension of \mathbb{H}^n .

The Heisenberg distance

$$d(p, q) = d(q^{-1}p, 0) = |q^{-1}p|_h$$

is derived from the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2 \right) + x_{2n+1}^2 \right]^{\frac{1}{4}},$$

where $x = (x_1, x_2, \dots, x_{2n}, x_{2n+1})$.

This distance d is left-invariant in the sense that $d(p, q)$ remains unchanged when p and q are both left-translated by some fixed vector on \mathbb{H}^n . Furthermore, d satisfies the triangular inequality (p. 320 in [16])

$$d(p, q) \leq d(p, x) + d(x, q), \quad p, x, q \in \mathbb{H}^n.$$

For $r > 0$ and $x \in \mathbb{H}^n$, the ball and sphere with center x and radius r on \mathbb{H}^n are given by

$$B(x, r) = \{y \in \mathbb{H}^n : d(x, y) < r\},$$

and

$$S(x, r) = \{y \in \mathbb{H}^n : d(x, y) = r\},$$

respectively. We know that

$$|B(x, r)| = |B(0, r)| = \Omega_Q r^Q,$$

where

$$\Omega_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})}, \tag{1}$$

is the volume of the unit ball $B(0, 1)$ on \mathbb{H}^n . The area of $S(0, 1)$ on \mathbb{H}^n is $\omega_Q = Q\Omega_Q$. For more details about the Heisenberg group one can refer to [10].

Now we provide the definition of Hausdorff operators and their commutators on the Heisenberg group in the following.

DEFINITION 1. Let Φ be a locally integrable function on \mathbb{H}^n . The Hausdorff operators on \mathbb{H}^n are defined by

$$\begin{aligned} \mathcal{H}_\Phi f(x) &= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(\delta_{|y|_h^{-1}}x) dy, \\ \mathcal{H}_{\Phi,A} f(x) &= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) dy, \end{aligned}$$

where $A(y)$ is a matrix-valued function and $\det A(y) \neq 0$ almost everywhere in the support of Φ .

If $b \in L_{loc}(\mathbb{H}^n)$, The commutator of Hausdorff operator is defined by

$$\mathcal{H}_{\Phi,A}^b f = b\mathcal{H}_{\Phi,A} f - \mathcal{H}_{\Phi,A}(bf).$$

In the above definition, we note that $\mathcal{H}_{\Phi,A} = \mathcal{H}_\Phi$ if we choose a special matrix A . For a matrix M , we will use the norm $\|M\| = \sup_{x \in \mathbb{H}^n, x \neq 0} |Mx|_h / |x|_h$. By Lemma 3.1 in [33],

$$\|M\|^{-Q} \leq |\det M^{-1}| \leq \|M^{-1}\|^Q, \tag{2}$$

where M is any invertible $(2n+1) \times (2n+1)$ matrix. Define

$$\begin{aligned} (\mathcal{H}_{\Phi,A}^{b,1} f)(x) &= \int_{\|A(y)\| \leq 1} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) [b(x) - b(A(y)x)] dy, \\ (\mathcal{H}_{\Phi,A}^{b,2} f)(x) &= \int_{\|A(y)\| > 1} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) [b(x) - b(A(y)x)] dy. \end{aligned}$$

It is not difficult to see that the commutator can be rewritten by

$$\mathcal{H}_{\Phi,A}^b f = \mathcal{H}_{\Phi,A}^{b,1} f + \mathcal{H}_{\Phi,A}^{b,2} f.$$

Here and throughout this paper, we use the notation $A \preceq B$ to denote that there is a constant $C > 0$ independent of all essential values and variables such that $A \leq CB$. We use the notation $A \simeq B$, if there exists a positive constant C independent of all essential values and variables, such that $C^{-1}B \leq A \leq CB$. Also, the class A_p denotes the set of all A_p weights whose definition can be found in the next section.

Now we are in a position to state our results.

THEOREM 1. *Let $1 \leq p_1, p_2, q < \infty$ and $-1/p_1 \leq \lambda < 0$. Suppose that $w \in A_q$ with the critical index r_w for the reverse Hölder condition. If $p_1 > p_2 q r_w / (r_w - 1)$, then we have that, for any $1 < \delta < r_w$,*

$$\|\mathcal{H}_{\Phi, A} f\|_{L^{p_2, \lambda}(\mathbb{H}^n; w)} \leq C_1 \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)},$$

where

$$C_1 = \int_{\|A(y)\| > 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} dy \\ + \int_{\|A(y)\| \leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda q} dy.$$

THEOREM 2. *Let $1 \leq p, p_1, p_2, q < \infty$ and $-1/p_1 \leq \lambda < 0$. Suppose that $w \in A_q$ with the critical index r_w for the reverse Hölder condition. If $1/p > (1/p_1 + 1/p_2) q r_w / (r_w - 1)$ and $q \leq p_2$, then we have that, for any $1 < \delta < r_w$,*

$$\|\mathcal{H}_{\Phi, A}^b f\|_{L^{p, \lambda}(\mathbb{H}^n; w)} \leq C_2 \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \|f\|_{CMO^{p_2}(\mathbb{H}^n; w)},$$

where

$$C_2 = \int_{\|A(y)\| > 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} \max \left\{ \frac{\|A(y)\|_Q^Q}{|\det A(y)|}, \log_2 \|A(y)\| \right\} dy \\ + \int_{\|A(y)\| \leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda q} \max \left\{ \frac{\|A(y)\|_Q^Q}{|\det A(y)|}, \log_2 \frac{1}{\|A(y)\|} \right\} dy.$$

When the weight is reduced to the power function, we have the following enhanced results.

THEOREM 3. *Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$ and $-Q < \alpha < \infty$. We have that*

$$\|\mathcal{H}_{\Phi, A} f\|_{L^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \leq C_3(\alpha) \|f\|_{L^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)},$$

where

$$C_3(\alpha) = \begin{cases} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p)}}{|\det A(y)|^{1/p}} \|A^{-1}(y)\|^{\alpha/p} dy, & 0 < \alpha < \infty, \\ \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p)}}{|\det A(y)|^{1/p}} \|A(y)\|^{-\alpha/p} dy, & -Q < \alpha \leq 0. \end{cases}$$

THEOREM 4. *Let $1 \leq p, p_1, p_2 < \infty$, $-1/p_1 \leq \lambda < 0$ and $1/p = 1/p_1 + 1/p_2$. (i) If $-Q < \alpha \leq 0$, then we have that*

$$\|\mathcal{H}_{\Phi, A}^b f\|_{L^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \leq C_4 \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \|b\|_{CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)},$$

where

$$C_4 = \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|y|_h^Q |\det A(y)|^{1/p_1}} \|A(y)\|^{-\alpha/p_1} \max \left\{ \frac{\|A(y)\|^Q}{|\det A(y)|}, |\log_2 \|A(y)\|| \right\} dy.$$

(ii) If $0 < \alpha < \infty$ and $p_2 > (Q + \alpha)/Q$, then we have that

$$\|\mathcal{H}_{\Phi,A}^b f\|_{\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \preceq C_5 \|f\|_{\dot{L}^{p_1,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \|b\|_{CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)},$$

where

$$C_5 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p)}}{|\det A(y)|^{1/p}} \|A^{-1}(y)\|^{\alpha/p} + \frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|\det A(y)|^{1/p_1}} \|A^{-1}(y)\|^{\alpha/p_1} \max \left\{ \frac{\|A(y)\|^Q}{|\det A(y)|}, |\log_2 \|A(y)\|| \right\} \right) dy.$$

Especially, if $\|A^{-1}(y)\|$ and $\|A(y)\|^{-1}$ are comparable, the following sharp results hold

THEOREM 5. *Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$, $-Q < \alpha < \infty$ and Φ be a non-negative function. Suppose that there is a constant C_0 independent of y such that $\|A^{-1}(y)\| \leq C_0 \|A(y)\|^{-1}$ for all $y \in \text{supp}(\Phi)$. Then $\mathcal{H}_{\Phi,A}$ is bounded on $\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ if and only if*

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} dy < \infty. \tag{3}$$

THEOREM 6. *Let $1 \leq p$, $p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $-1/p_1 < \lambda < 0$, $-Q < \alpha < \infty$ and $p_2 > (Q + \alpha)/Q$ if $0 < \alpha < \infty$ or $p_2 \geq 1$ if $-Q < \alpha \leq 0$. Suppose that Φ is a nonnegative function and there is a constant C_0 independent of y such that $\|A^{-1}(y)\| \leq C_0 \|A(y)\|^{-1}$ for all $y \in \text{supp}(\Phi)$. If $b \in CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)$ and (3) holds, then we have the following conclusions.*

(i) $\mathcal{H}_{\Phi,A}^{b,1}$ is bounded from $\dot{L}^{p_1,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ to $\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ if and only if

$$\int_{\|A(y)\| \leq 1} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} |\log_2 \|A(y)\|| dy < \infty.$$

(ii) $\mathcal{H}_{\Phi,A}^{b,2}$ is bounded from $\dot{L}^{p_1,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ to $\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ if and only if

$$\int_{\|A(y)\| > 1} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \log_2 \|A(y)\| dy < \infty.$$

Finally in this section, we want to make a few remarks about our main theorems.

REMARK 1. Suppose $A(y) = \text{diag}[1/\lambda_1(y), \dots, 1/\lambda_{2n}(y), 1/\lambda_{2n+1}(y)]$ with $\lambda_i(y) \neq 0$, for $i = 1, \dots, 2n + 1$. Denote

$$M(y) = \max\{|\lambda_1(y)|, \dots, |\lambda_{2n}(y)|, |\lambda_{2n+1}(y)|^{1/2}\},$$

$$m(y) = \min\{|\lambda_1(y)|, \dots, |\lambda_{2n}(y)|, |\lambda_{2n+1}(y)|^{1/2}\}.$$

If there is a constant $C \geq 1$ independent of y such that $M(y) \leq Cm(y)$, then it is easy to check that $A(y)$ satisfies the assumptions of Theorem 5 and Theorem 6.

REMARK 2. By checking the proof of necessity of Theorem 6, we find that the necessary condition in (ii) with $C_0 = 1$ and in (i) are also true without the assumption (3). Therefore, comparing with Theorem 4 and Theorem 6, we raise the following two questions.

- (i) Do the statements also hold for $\lambda = -1/p_1$ in Theorem 6?
- (ii) Is $\mathcal{H}_{\Phi, A}^b$ bounded from $\dot{L}^{p_1, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ to $\dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ if and only if

$$\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} (1 + |\log_2 \|A(y)\||) dy < \infty?$$

In the second section, we will introduce some necessary notation and definitions, as well as some known results to be used later in the paper. We will prove the main theorems in Section 3.

2. Notation and definitions

We first recall some standard definitions and notation. The theory of A_p weight was first introduced by Muckenhoupt in the Euclidean spaces for studying the weighted L^p boundedness of Hardy-Littlewood maximal functions in [29]. For A_p weights on the Heisenberg group one can refer to [11, 13]. A weight is a nonnegative, locally integrable function on \mathbb{H}^n .

DEFINITION 2. Let $1 < p < \infty$. We say that a weight $w \in A_p(\mathbb{H}^n)$ if there exists a constant C such that for all balls B ,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C.$$

We say that a weight $w \in A_1(\mathbb{H}^n)$ if there is a constant C such that for all balls B ,

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess\,inf}_{x \in B} w(x).$$

We define

$$A_\infty(\mathbb{H}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{H}^n).$$

Following proofs of Propositions 1.4.1, 1.4.2 in [25] together with the reverse Hölder inequality on the Heisenberg group [13], we have the following results.

PROPOSITION 1. (i) $A_p(\mathbb{H}^n) \subsetneq A_q(\mathbb{H}^n)$, for $1 \leq p < q < \infty$.

(ii) If $w \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, then there is an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $w \in A_{p-\varepsilon}(\mathbb{H}^n)$.

A close relation to $A_\infty(\mathbb{H}^n)$ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant C such that

$$\left(\frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq \frac{C}{|B|} \int_B w(x) dx$$

for all balls $B \subset \mathbb{H}^n$, we then say that w satisfies the *reverse Hölder condition of order r* and write $w \in RH_r(\mathbb{H}^n)$. According to Theorem 19 and Corollary 21 in [14], $w \in A_\infty(\mathbb{H}^n)$ if and only if there exists some $r > 1$ such that $w \in RH_r(\mathbb{H}^n)$. Moreover, if $w \in RH_r(\mathbb{H}^n)$, $r > 1$, then $w \in RH_{r+\varepsilon}(\mathbb{H}^n)$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in RH_r(\mathbb{H}^n)\}$ to denote the *critical index of w for the reverse Hölder condition*.

An important example of $A_p(\mathbb{H}^n)$ weight is the power function $|x|_h^\alpha$. By the similar proofs of Propositions 1.4.3 and 1.4.4 in [25], we obtain the following properties of power weights.

PROPOSITION 2. Let $x \in \mathbb{H}^n$. Then

- (i) $|x|_h^\alpha \in A_1(\mathbb{H}^n)$ if and only if $-Q < \alpha \leq 0$;
- (ii) $|x|_h^\alpha \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, if and only if $-Q < \alpha < Q(p - 1)$.

We will denote by q_w the *critical index* for w , that is, the infimum of all the q such that w satisfies the condition A_q . From Proposition 1, we see that unless $q_w = 1$, w is never an A_{q_w} weight. Also by Proposition 1 and Proposition 2 we see that if $0 < \alpha < \infty$, then

$$|x|_h^\alpha \in \bigcap_{\frac{Q+\alpha}{Q} < p < \infty} A_p, \tag{4}$$

where $(Q + \alpha)/Q$ is the critical index of $|x|_h^\alpha$.

For any $w \in A_\infty(\mathbb{H}^n)$ and any Lebesgue measurable set E , write $w(E) = \int_E w(x) dx$. We have the following standard characterization of A_p weights (see [33]).

PROPOSITION 3. Let $w \in A_p(\mathbb{H}^n) \cap RH_r(\mathbb{H}^n)$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset E of a ball B . Especially, for any $\lambda > 1$,

$$w(B(x_0, \lambda R)) \leq C \lambda^{np} w(B(x_0, R)).$$

PROPOSITION 4. If $w \in A_p(\mathbb{H}^n)$, $1 \leq p < \infty$, then for any $f \in L^1_{loc}(\mathbb{H}^n)$ and any ball $B \subset \mathbb{H}^n$,

$$\frac{1}{|B|} \int_B |f(x)| dx \leq C \left(\frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right)^{1/p}.$$

Given a weight function w on \mathbb{H}^n , for any measurable set $E \subset \mathbb{H}^n$, as usual we denote by $L^p(E; w)$ the weighted Lebesgue space of all functions satisfying

$$\|f\|_{L^p(E; w)} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

We denote $L^\infty(\mathbb{H}^n; w) = L^\infty(\mathbb{H}^n)$ and $\|f\|_{L^\infty(\mathbb{H}^n; w)} = \|f\|_{L^\infty(\mathbb{H}^n)}$ for $p = \infty$.

DEFINITION 3. Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$ and w be a weight on \mathbb{H}^n . A function $f \in L^p_{loc}(\mathbb{H}^n; w)$ is said to belong to the weighted central Morrey spaces $\dot{L}^{p, \lambda}(\mathbb{H}^n; w)$ if

$$\|f\|_{\dot{L}^{p, \lambda}(\mathbb{H}^n; w)} = \sup_{r>0} \left(\frac{1}{w(B(0, r))^{1+p\lambda}} \int_{B(0, r)} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

When $\lambda = -1/p$, then $\dot{L}^{p, \lambda}(\mathbb{H}^n; w) = L^p(\mathbb{H}^n; w)$. If $w \equiv 1$, one can easily check that $\dot{L}^{p, \lambda}(\mathbb{H}^n)$ reduces to 0 when $\lambda < -1/p$.

DEFINITION 4. Let $1 \leq p < \infty$ and w be a weight on \mathbb{H}^n . A function $f \in L^p_{loc}(\mathbb{H}^n; w)$ is said to be in the weighted central BMO spaces $CMO^p(\mathbb{H}^n; w)$ if

$$\|f\|_{CMO^p(\mathbb{H}^n; w)} = \sup_{r>0} \left(\frac{1}{w(B(0, r))} \int_{B(0, r)} |f(x) - f_B|^p w(x) dx \right)^{1/p} < \infty,$$

where $f_B = \int_{B(0, r)} f(x) dx / |B(0, r)|$.

The spaces $CMO^p(\mathbb{H}^n; w)$ are quasi-Banach spaces. When $1 < p < \infty$, then $CMO^p(\mathbb{H}^n; w)$ are Banach spaces after identifying the functions that differ by a constant almost everywhere. Hölder's inequality shows that $CMO^{p_1}(\mathbb{H}^n; w) \subset CMO^{p_2}(\mathbb{H}^n; w)$ if $1 \leq p_2 < p_1 < \infty$. If $w \equiv 1$, we denote the central BMO spaces by $CMO^p(\mathbb{H}^n)$ and we can see that $BMO(\mathbb{H}^n) \subset CMO^p(\mathbb{H}^n)$, $1 \leq p < \infty$.

3. Proof of the theorems

In this section, we use $tB(0, r)$ to denote $B(0, tr)$ for any central ball $B(0, r)$ in \mathbb{H}^n and any $t > 0$. For a $(2n+1) \times (2n+1)$ matrix M , we denote $MB(0, r)$ the set $\{z \in \mathbb{H}^n \mid z = Mx, x \in B(0, r)\}$.

3.1. Proof of Theorem 1

By the definition and the Minkowski inequality,

$$\begin{aligned} & \| \mathcal{H}_{\Phi, A} f \|_{L^{p_2, \lambda}(\mathbb{H}^n; w)} \\ &= \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p_2}} \left\| \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)\cdot) dy \right\|_{L^{p_2}(B(0, r); w)} \\ &\leq \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p_2}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot)\|_{L^{p_2}(B(0, r); w)} dy. \end{aligned} \tag{5}$$

Since $p_1 > p_2 q r_w / (r_w - 1) = p_2 q r'_w$, there is $1 < \gamma < r_w$ such that $p_1/q = p_2 \gamma' = p_2 \gamma / (\gamma - 1)$. In view of the Hölder inequality and the reverse Hölder condition, we obtain that

$$\begin{aligned} & \|f(A(y)\cdot)\|_{L^{p_2}(B(0, r); w)} \\ &\preceq \left(\int_{B(0, r)} |f(A(y)x|^{p_1/q} dx \right)^{q/p_1} \left(\int_{B(0, r)} w(x)^\gamma dx \right)^{1/(\gamma p_2)} \\ &\preceq |\det A^{-1}(y)|^{q/p_1} \left(\int_{A(y)B(0, r)} |f(x)|^{p_1/q} dx \right)^{q/p_1} \left(\int_{B(0, r)} w(x)^\gamma dx \right)^{1/(\gamma p_2)} \\ &\preceq |\det A^{-1}(y)|^{q/p_1} |B(0, r)|^{-q/p_1} w(B(0, r))^{1/p_2} \left(\int_{A(y)B(0, r)} |f(x)|^{p_1/q} dx \right)^{q/p_1}. \end{aligned}$$

Proposition 4 and (2) show that

$$\begin{aligned} & \left(\int_{A(y)B(0, r)} |f(x)|^{p_1/q} dx \right)^{q/p_1} \\ &\leq \left(\int_{B(0, \|A(y)\|_r)} |f(x)|^{p_1/q} dx \right)^{q/p_1} \\ &\preceq \|A(y)\|^{2q/p_1} |B(0, r)|^{q/p_1} \left(\frac{1}{w(B(0, \|A(y)\|_r))} \int_{B(0, \|A(y)\|_r)} |f(x)|^{p_1} w(x) dx \right)^{1/p_1}, \end{aligned}$$

which implies that

$$\begin{aligned} & \|f(A(y)\cdot)\|_{L^{p_2}((B(0, r); w)} \\ &\preceq |\det A^{-1}(y)|^{q/p_1} \|A(y)\|^{2q/p_1} \frac{w(B(0, r))^{1/p_2}}{w(B(0, \|A(y)\|_r))^{1/p_1}} \|f\|_{L^{p_1}(B(0, \|A(y)\|_r); w)}. \end{aligned} \tag{6}$$

Therefore, we infer from (5) and (6) that

$$\begin{aligned} & \| \mathcal{H}_{\Phi, A} f \|_{L^{p_2, \lambda}(\mathbb{H}^n; w)} \\ &\preceq \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \sup_{r>0} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_h^Q}{|\det A(y)|} \right)^{q/p_1} \left(\frac{w(B(0, \|A(y)\|_r)}{w(B(0, r))} \right)^\lambda dy. \end{aligned} \tag{7}$$

If $\|A(y)\| > 1$, Proposition 3 shows that, for any $1 < \delta < r_w$,

$$\frac{w(B(0, \|A(y)\|r))}{w(B(0, r))} \geq \left(\frac{|B(0, \|A(y)\|r)|}{|B(0, r)|} \right)^{(\delta-1)/\delta} = \|A(y)\|^{Q(\delta-1)/\delta}. \quad (8)$$

If $\|A(y)\| \leq 1$, by Proposition 3 again, we have

$$\frac{w(B(0, \|A(y)\|r))}{w(B(0, r))} \geq \left(\frac{|B(0, \|A(y)\|r)|}{|B(0, r)|} \right)^q = \|A(y)\|^{Qq}. \quad (9)$$

Thus we complete the proof of Theorem 1 by (7)–(9). \square

3.2. Proof of Theorem 2

By the definition,

$$\begin{aligned} & \|\mathcal{H}_{\Phi, A}^b f\|_{L^{p, \lambda}(\mathbb{H}^n; w)} \\ & \leq \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p}} \left(\int_{B(0, r)} \left| \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) [b(x) - b_{B(0, r)}] dy \right|^p w(x) dx \right)^{1/p} \\ & \quad + \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p}} \\ & \quad \times \left(\int_{B(0, r)} \left| \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) [b_{B(0, r)} - b_{A(y)B(0, r)}] dy \right|^p w(x) dx \right)^{1/p} \\ & \quad + \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p}} \\ & \quad \times \left(\int_{B(0, r)} \left| \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) [b_{A(y)B(0, r)} - b(A(y)x)] dy \right|^p w(x) dx \right)^{1/p} \\ & := I + II + III. \end{aligned} \quad (10)$$

Hölder's inequality and Theorem 1 show that

$$\begin{aligned} I & = \sup_{r>0} \frac{1}{w(B(0, r))^{\lambda+1/p}} \left(\int_{B(0, r)} |\mathcal{H}_{\Phi, A}^b f(x)|^p |b(x) - b_{B(0, r)}|^p w(x) dx \right)^{1/p} \\ & \leq \|\mathcal{H}_{\Phi, A}^b f\|_{L^{p_3, \lambda}(\mathbb{H}^n; w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)} \\ & \leq \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)} \left(\int_{\|A(y)\|>1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_h^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} dy \right. \\ & \quad \left. + \int_{\|A(y)\|\leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_h^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda q} dy \right), \end{aligned} \quad (11)$$

where $1/p = 1/p_2 + 1/p_3$.

According to the Minkowski inequality,

$$\begin{aligned}
 II &\preceq \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |b_{B(0,r)} - b_{A(y)B(0,r)}| \\
 &\quad \times \left(\int_{B(0,r)} |f(A(y)x)|^p w(x) dx \right)^{1/p} dy.
 \end{aligned}$$

By the similar argument as Theorem 1, we have, for any $1 < \delta < r_w$,

$$\begin{aligned}
 &\sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \|f(A(y)\cdot)\|_{L^p(B(0,r);w)} \\
 &\preceq \|f\|_{\dot{L}^{p_1,\lambda}(B(0,r);w)} \begin{cases} \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta}, & \|A(y)\| > 1, \\ \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda q}, & \|A(y)\| \leq 1. \end{cases} \tag{12}
 \end{aligned}$$

Next, we estimate $\sup_{r>0} |b_{B(0,r)} - b_{A(y)B(0,r)}|$. If $\|A(y)\| > 1$, there exists a non-negative integer $k_0 \geq 0$ satisfying

$$2^{k_0} < \|A(y)\| \leq 2^{k_0+1}.$$

Therefore

$$\begin{aligned}
 &|b_{B(0,r)} - b_{A(y)B(0,r)}| \\
 &\preceq |b_{B(0,r)} - b_{2B(0,r)}| + |b_{2B(0,r)} - b_{4B(0,r)}| + \dots + |b_{2^{k_0}B(0,r)} - b_{2^{k_0+1}B(0,r)}| \\
 &\quad + |b_{2^{k_0+1}B(0,r)} - b_{A(y)B(0,r)}|. \tag{13}
 \end{aligned}$$

Since for any $k \in \mathbf{Z}$, Proposition 4 implies that

$$\begin{aligned}
 &|b_{2^k B(0,r)} - b_{2^{k+1} B(0,r)}| \\
 &\leq \frac{2^Q}{|2^{k+1} B(0,r)|} \int_{2^{k+1} B(0,r)} |b(x) - b_{2^{k+1} B(0,r)}| dx \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 &\preceq \left(\frac{1}{w(2^{k+1} B(0,r))} \int_{2^{k+1} B(0,r)} |b(x) - b_{2^{k+1} B(0,r)}|^q w(x) dx \right)^{1/q} \\
 &\preceq \left(\frac{1}{w(2^{k+1} B(0,r))} \int_{2^{k+1} B(0,r)} |b(x) - b_{2^{k+1} B(0,r)}|^{p_2} w(x) dx \right)^{1/p_2} \\
 &\preceq \|b\|_{CMOP^2(\mathbb{H}^n;w)}, \tag{15}
 \end{aligned}$$

where the third inequality is achieved by Hölder’s inequality and $q \leq p_2$.

On the other hand,

$$\begin{aligned}
 & \left| b_{2^{k_0+1}B(0,r)} - b_{A(y)B(0,r)} \right| \\
 \leq & \frac{1}{|A(y)B(0,r)|} \int_{A(y)B(0,r)} \left| b(x) - b_{2^{k_0+1}B(0,r)} \right| dx \\
 \leq & \frac{2^{Q(k_0+1)}}{|\det A(y)|} \frac{1}{|2^{k_0+1}B(0,r)|} \int_{2^{k_0+1}B(0,r)} \left| b(x) - b_{2^{k_0+1}B(0,r)} \right| dx \\
 \leq & \frac{\|A(y)\|^Q}{|\det A(y)|} \left(\frac{1}{w(2^{k_0+1}B(0,r))} \int_{2^{k_0+1}B(0,r)} \left| b(x) - b_{2^{k_0+1}B(0,r)} \right|^q w(x) dx \right)^{1/q} \\
 \leq & \frac{\|A(y)\|^Q}{|\det A(y)|} \|b\|_{CMOP^2(\mathbb{H}^n;w)}. \tag{16}
 \end{aligned}$$

Then, it follows from (13)–(16) that, for $\|A(y)\| > 1$,

$$\begin{aligned}
 |b_{B(0,r)} - b_{A(y)B(0,r)}| & \leq \left(k_0 + 1 + \frac{\|A(y)\|^Q}{|\det A(y)|} \right) \|b\|_{CMOP^2(\mathbb{H}^n;w)} \\
 & \leq \max \left\{ \log_2 \|A(y)\|, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} \|b\|_{CMOP^2(\mathbb{H}^n;w)}. \tag{17}
 \end{aligned}$$

Similar to the proceeding argument, for $\|A(y)\| \leq 1$,

$$|b_{B(0,r)} - b_{A(y)B(0,r)}| \leq \max \left\{ \log_2 \frac{1}{\|A(y)\|}, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} \|b\|_{CMOP^2(\mathbb{H}^n;w)}. \tag{18}$$

Therefore (12), (17) and (18) yield that

$$\begin{aligned}
 II & \leq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n;w)} \|b\|_{CMOP^2(\mathbb{H}^n;w)} \tag{19} \\
 & \times \left(\int_{\|A(y)\|>1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} \right. \\
 & \times \max \left\{ \log_2 \|A(y)\|, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} dy \\
 & \left. + \int_{\|A(y)\|\leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^Q}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{Q\lambda q} \max \left\{ \log_2 \frac{1}{\|A(y)\|}, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} dy \right).
 \end{aligned}$$

Now we turn to estimate the term III. Using the Minkowski inequality again,

$$III \leq \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot) [b_{A(y)B(0,r)} - b(A(y)\cdot)]\|_{L^p(B(0,r);w)} dy.$$

On the other hand, since $1/p > (1/p_1 + 1/p_2)qr'_w$, we can choose p_4, p_5 satisfying $1/p_4 > qr'_w/p_1, 1/p_5 > qr'_w/p_2$ and $1/p = 1/p_4 + 1/p_5$. Then (12) implies that,

$$\begin{aligned}
 & \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \|f(A(y)\cdot) [b_{A(y)B(0,r)} - b(A(y)\cdot)]\|_{L^p(B(0,r);w)} \\
 \leq & \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \|f(A(y)\cdot)\|_{L^{p_4}(B(0,r);w)} \|b_{A(y)B(0,r)} - b(A(y)\cdot)\|_{L^{p_5}(B(0,r);w)}
 \end{aligned}$$

$$\begin{aligned}
 &= \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p_4}} \|f(A(y)\cdot)\|_{L^{p_4}(B(0,r);w)} \\
 &\quad \times \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x)|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\leq \sup_{r>0} \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x)|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\quad \times \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n;w)} \begin{cases} \left(\frac{\|A(y)\|_{\mathcal{Q}}}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{\mathcal{Q}\lambda(\delta-1)/\delta}, & \|A(y)\| > 1, \\ \left(\frac{\|A(y)\|_{\mathcal{Q}}}{|\det A(y)|} \right)^{q/p_1} \|A(y)\|^{\mathcal{Q}\lambda q}, & \|A(y)\| \leq 1. \end{cases} \tag{20}
 \end{aligned}$$

It follows from Proposition 4 and $q \leq p_2$ that

$$\begin{aligned}
 &\left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x)|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\leq \frac{1}{w(B(0,r))^{1/p_5}} \left(\int_{B(0,r)} |b(A(y)x) - b_{B(0,\|A(y)\|r)}|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\quad + |b_{B(0,\|A(y)\|r)} - b_{A(y)B(0,r)}| \\
 &\leq \frac{1}{w(B(0,r))^{1/p_5}} \left(\int_{B(0,r)} |b(A(y)x) - b_{B(0,\|A(y)\|r)}|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\quad + \frac{\|A(y)\|_{\mathcal{Q}}}{|\det A(y)|} \left(\frac{1}{w(B(0,\|A(y)\|r))} \int_{B(0,\|A(y)\|r)} |b(x) - b_{B(0,\|A(y)\|r)}|^{p_2} w(x) dx \right)^{1/p_2} \\
 &\leq \frac{1}{w(B(0,r))^{1/p_5}} \left(\int_{B(0,r)} |b(A(y)x) - b_{B(0,\|A(y)\|r)}|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\quad + \frac{\|A(y)\|_{\mathcal{Q}}}{|\det A(y)|} \|b\|_{CMOp_2(\mathbb{H}^n;w)}.
 \end{aligned}$$

Since $1/p_5 > qr'_w/p_2$, there exists $1 < \beta < r_w$ such that $1/p_5 = q\beta'/p_2 = q\beta/(p_2(\beta-1))$. Due to the Hölder inequality and the reverse Hölder condition, we have

$$\begin{aligned}
 &\frac{1}{w(B(0,r))^{1/p_5}} \left(\int_{B(0,r)} |b(A(y)x) - b_{B(0,\|A(y)\|r)}|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\leq \frac{1}{w(B(0,r))^{1/p_5}} \left(\int_{B(0,r)} |b(A(y)x) - b_{B(0,\|A(y)\|r)}|^{p_2/q} dx \right)^{q/p_2} \left(\int_{B(0,r)} w(x)^\beta dx \right)^{1/(\beta p_5)} \\
 &\leq \frac{1}{w(B(0,r))^{1/p_5} |\det A(y)|^{q/p_2}} \left(\int_{B(0,\|A(y)\|r)} |b(x) - b_{B(0,\|A(y)\|r)}|^{p_2/q} dx \right)^{q/p_2} \\
 &\quad \times \left(\int_{B(0,r)} w(x)^\beta dx \right)^{1/(\beta p_5)}
 \end{aligned}$$

$$\begin{aligned}
 &\preceq \frac{1}{w(B(0,r))^{1/p_5}} |B(0,r)|^{(1-\beta)/(\beta p_5)} \left(\int_{B(0,r)} w(x) dx \right)^{1/p_5} \left(\frac{|B(0, \|A(y)\|_r)|}{|\det A(y)|} \right)^{q/p_2} \\
 &\quad \times \left(\frac{1}{w(B(0, \|A(y)\|_r))} \int_{B(0, \|A(y)\|_r)} |b(x) - b_{B(0, \|A(y)\|_r)}|^{p_2} w(x) dx \right)^{1/p_2} \\
 &\preceq \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{q/p_2} \left(\frac{1}{w(B(0, \|A(y)\|_r))} \int_{B(0, \|A(y)\|_r)} |b(x) - b_{B(0, \|A(y)\|_r)}|^{p_2} w(x) dx \right)^{1/p_2} \\
 &\preceq \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{q/p_2} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)} \\
 &\preceq \frac{\|A(y)\|_Q}{|\det A(y)|} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)},
 \end{aligned}$$

which yields that

$$\begin{aligned}
 &\sup_{r>0} \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x)|^{p_5} w(x) dx \right)^{1/p_5} \\
 &\preceq \frac{\|A(y)\|_Q}{|\det A(y)|} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)}. \tag{21}
 \end{aligned}$$

Then, we infer from (20) and (21) that

$$\begin{aligned}
 III &\preceq \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n; w)} \\
 &\quad \times \left(\int_{\|A(y)\|>1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{1+q/p_1} \|A(y)\|^{Q\lambda(\delta-1)/\delta} dy \right. \\
 &\quad \left. + \int_{\|A(y)\|\leq 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|_Q}{|\det A(y)|} \right)^{1+q/p_1} \|A(y)\|^{Q\lambda q} dy \right). \tag{22}
 \end{aligned}$$

By combining (10), (11), (19) and (22), we finish the proof of the theorem. \square

Here and in after, we some time use $w(\cdot)$ for $|\cdot|_h^\alpha$ for the sake of convenience.

3.3. Proof of Theorem 3

Similar to the proof of the Theorem 1, we have

$$\| \mathcal{H}_{\Phi, A} f \|_{L^{p, \lambda}(\mathbb{H}^n; w)} \preceq \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot)\|_{L^p(B(0,r); w)} dy. \tag{23}$$

A changing of variables $A(y)x = z$ yields that

$$\begin{aligned}
 &\|f(A(y)x)\|_{L^p(B(0,r); |x|^\alpha)} \tag{24} \\
 &\preceq \begin{cases} |\det A^{-1}(y)|^{1/p} \|A^{-1}(y)\|^{\alpha/p} \left(\int_{B(0, \|A(y)\|_r)} |f(z)|^p |z|_h^\alpha dz \right)^{1/p}, & 0 < \alpha < \infty, \\ |\det A^{-1}(y)|^{1/p} \|A(y)\|^{-\alpha/p} \left(\int_{B(0, \|A(y)\|_r)} |f(z)|^p |z|_h^\alpha dz \right)^{1/p}, & -Q < \alpha \leq 0. \end{cases}
 \end{aligned}$$

For simplicity, we just consider the case $0 < \alpha < \infty$, since the proof of the case $-Q < \alpha \leq 0$ is essentially similar. Combining (23),(24) and the simple fact

$$w(B(0, r)) = \int_{|x|_h < r} |x|_h^\alpha dx = \frac{\omega_Q}{Q + \alpha} r^{Q+\alpha}, \quad -Q < \alpha < \infty, \tag{25}$$

where ω_Q is the surface area of the unit sphere in \mathbb{H}^n , we have

$$\begin{aligned} & \| \mathcal{H}_{\Phi, A} f \|_{L^{p, \lambda}(\mathbb{H}^n; w)} \\ & \preceq \| f \|_{L^{p, \lambda}(\mathbb{H}^n; w)} \sup_{r > 0} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/p} \|A^{-1}(y)\|^{\alpha/p} \left(\frac{w(B(0, \|A(y)\|r))}{w(B(0, r))} \right)^{\lambda+1/p} dy \\ & \preceq \| f \|_{L^{p, \lambda}(\mathbb{H}^n; w)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/p} \|A^{-1}(y)\|^{\alpha/p} \|A(y)\|^{(\lambda+1/p)(Q+\alpha)} dy. \end{aligned}$$

This proves the theorem. \square

3.4. Proof of Theorem 4

According to the same argument as Theorem 2, we have

$$\| \mathcal{H}_{\Phi, A}^b f \|_{L^{p, \lambda}(\mathbb{H}^n; w)} \preceq I + II + III,$$

here I, II, III are the same as in (10).

We first consider the case that $0 < \alpha < \infty$. The Hölder inequality and Theorem 3 show that

$$\begin{aligned} I & \preceq \| \mathcal{H}_{\Phi, A} f \|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \| b \|_{CMOP^2(\mathbb{H}^n; w)} \\ & \preceq \| f \|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \| b \|_{CMOP^2(\mathbb{H}^n; w)} \\ & \quad \times \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|y|_h^Q |\det A(y)|^{1/p_1}} \|A^{-1}(y)\|^{\alpha/p_1} dy. \end{aligned} \tag{26}$$

By the Minkowski inequality and Hölder’s inequality again,

$$\begin{aligned} II & \preceq \sup_{r > 0} \frac{1}{w(B(0, r))^{\lambda+1/p}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |b_{B(0, r)} - b_{A(y)B(0, r)}| \\ & \quad \times \left(\int_{B(0, r)} |f(A(y)x)|^p |x|_h^\alpha dx \right)^{1/p} dy \\ & \preceq \sup_{r > 0} \frac{1}{w(B(0, r))^{\lambda+1/p_1}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |b_{B(0, r)} - b_{A(y)B(0, r)}| \\ & \quad \times \left(\int_{B(0, r)} |f(A(y)x)|^{p_1} |x|_h^\alpha dx \right)^{1/p_1} dy \end{aligned}$$

$$\begin{aligned} & \lesssim \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \\ & \times \sup_{r>0} \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|y|_h^Q |\det A(y)|^{1/p_1}} \|A^{-1}(y)\|^{\alpha/p_1} |b_{B(0,r)} - b_{A(y)B(0,r)}| dy, \end{aligned} \tag{27}$$

where the last inequality is achieved by a similar argument as in the proof of Theorem

3. A similar discussion as in (13)–(18) shows that

$$\sup_{r>0} |b_{B(0,r)} - b_{A(y)B(0,r)}| \lesssim \max \left\{ \|\log_2 \|A(y)\|\|, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} \|b\|_{CMOp_2(\mathbb{H}^n; w)}, \tag{28}$$

where $p_2 > (Q + \alpha)/Q$. Then we infer from (27) and (28) that

$$\begin{aligned} II & \lesssim \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \|b\|_{CMOp_2(\mathbb{H}^n; w)} \\ & \times \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|y|_h^Q |\det A(y)|^{1/p_1}} \|A^{-1}(y)\|^{\alpha/p_1} \max \left\{ \|\log_2 \|A(y)\|\|, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} dy. \end{aligned} \tag{29}$$

It is not difficult to check that

$$III \lesssim \sup_{r>0} \frac{1}{w(B(0,r))^{\lambda+1/p}} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot)[b_{A(y)B(0,r)} - b(A(y)\cdot)]\|_{L^p(B(0,r); w)} dy,$$

and for any $r > 0$,

$$\begin{aligned} & \frac{1}{w(B(0,r))^{\lambda+1/p}} \|f(A(y)\cdot)[b_{A(y)B(0,r)} - b(A(y)\cdot)]\|_{L^p(B(0,r); w)} \\ & \lesssim \frac{1}{w(B(0,r))^{\lambda+1/p_1}} \|f(A(y)\cdot)\|_{L^{p_1}(B(0,r); w)} \\ & \quad \times \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x|^{p_2} |x|_h^\alpha dx \right)^{1/p_2} \\ & \lesssim \|f\|_{L^{p_1, \lambda}(\mathbb{H}^n; w)} \frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|\det A(y)|^{1/p_1}} \|A^{-1}(y)\|^{\alpha/p_1} \\ & \quad \times \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x|^{p_2} |x|_h^\alpha dx \right)^{1/p_2}. \end{aligned} \tag{30}$$

On the other hand, the Minkowski inequality and Proposition 4 imply that

$$\begin{aligned} & \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b_{A(y)B(0,r)} - b(A(y)x|^{p_2} |x|_h^\alpha dx \right)^{1/p_2} \\ & \leq \left(\frac{1}{w(B(0,r))} \int_{B(0,r)} |b(A(y)x) - b_{B(0, \|A(y)\|r)}|^{p_2} |x|_h^\alpha dx \right)^{1/p_2} \\ & \quad + |b_{A(y)B(0,r)} - b_{B(0, \|A(y)\|r)}| \end{aligned}$$

$$\begin{aligned}
 &\preceq \|A^{-1}(y)\|^{\alpha/p_2} |\det A^{-1}(y)|^{1/p_2} \\
 &\quad \times \left(\frac{1}{w(B(0,r))} \int_{A(y)B(0,r)} |b(z) - b_{B(0,\|A(y)\|r)}|^{p_2} |z|^\alpha dz \right)^{1/p_2} \\
 &\quad + \frac{1}{|A(y)B(0,r)|} \int_{A(y)B(0,r)} |b(z) - b_{B(0,\|A(y)\|r)}| dz \\
 &\preceq \|A^{-1}(y)\|^{\alpha/p_2} |\det A^{-1}(y)|^{1/p_2} \left(\frac{w(B(0,\|A(y)\|r))}{w(B(0,r))} \right)^{1/p_2} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \\
 &\quad + \frac{\|A(y)\|^Q}{|\det A(y)|} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \\
 &\preceq \left(\|A^{-1}(y)\|^{\alpha/p_2} |\det A^{-1}(y)|^{1/p_2} \|A(y)\|^{(Q+\alpha)/p_2} + \frac{\|A(y)\|^Q}{|\det A(y)|} \right) \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)}. \tag{31}
 \end{aligned}$$

We infer from (30), (31) and (2) that

$$\begin{aligned}
 III &\preceq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A^{-1}(y)\|^{\alpha/p} \|A(y)\|^{(Q+\alpha)(\lambda+1/p)}}{|\det A(y)|^{1/p}} \right. \\
 &\quad \left. + \frac{\|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)} \|A^{-1}(y)\|^{\alpha/p_1} \|A(y)\|^Q}{|\det A(y)|^{1/p_1} |\det A(y)|} \right) dy. \tag{32}
 \end{aligned}$$

Thus we complete the proof of the case $0 < \alpha < \infty$ by (26), (29) and (32).

Next we consider the case that $-Q < \alpha \leq 0$. By the previously used argument, we have

$$\begin{aligned}
 I &\preceq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \\
 &\quad \times \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)(\lambda+1/p_1)}}{|y|_h^Q |\det A(y)|^{1/p_1}} \|A(y)\|^{-\alpha/p_1} dy \\
 &= \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)\lambda+Q/p_1}}{|y|_h^Q |\det A(y)|^{1/p_1}} dy. \tag{33}
 \end{aligned}$$

$$\begin{aligned}
 II &\preceq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \\
 &\quad \times \int_{\mathbb{H}^n} \frac{|\Phi(y)| \|A(y)\|^{(Q+\alpha)\lambda+Q/p_1}}{|y|_h^Q |\det A(y)|^{1/p_1}} \max \left\{ |\log_2 \|A(y)\||, \frac{\|A(y)\|^Q}{|\det A(y)|} \right\} dy. \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 III &\preceq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(\frac{\|A(y)\|^{(Q+\alpha)\lambda+Q/p}}{|\det A(y)|^{1/p}} \right. \\
 &\quad \left. + \frac{\|A(y)\|^{(Q+\alpha)\lambda+Q/p_1} \|A(y)\|^Q}{|\det A(y)|^{1/p_1} |\det A(y)|} \right) dy \\
 &\preceq \|f\|_{L^{p_1,\lambda}(\mathbb{H}^n,w)} \|b\|_{CMO^{p_2}(\mathbb{H}^n,w)} \int_{\mathbb{H}^n} \frac{\|A(y)\|^{(Q+\alpha)\lambda+Q/p_1} \|A(y)\|^Q}{|\det A(y)|^{1/p_1} |\det A(y)|} dy, \tag{35}
 \end{aligned}$$

where the last inequality is obtained by (2). Thus we finish the proof of the theorem. \square

3.5. Proof of Theorem 5

If $\|A^{-1}(y)\| \preceq \|A(y)\|^{-1}$, then (2) gives that

$$\|A^{-1}(y)\|^Q \simeq \|A(y)\|^{-Q} \simeq |\det A^{-1}(y)|. \tag{36}$$

The “if” part of Theorem 5 is easily obtained from Theorem 3. Next we will show the “only if” part.

We just consider the case that $-1/p < \lambda < 0$, since the theorem is exactly the Corollary 1.6 in [33] if $\lambda = -1/p$. Let $f^*(x) = |x|_h^{(Q+\alpha)\lambda}$. By the fact of (25), it is not difficult to check that

$$\|f^*\|_{L^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} = \frac{(Q + \alpha)^\lambda}{\omega_Q^\lambda (1 + p\lambda)^{1/p}}.$$

Simple calculation shows that, for $\Phi \geq 0$,

$$\begin{aligned} (\mathcal{H}_{\Phi,A} f^*)(x) &= \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|^{(Q+\alpha)\lambda} dy \\ &\geq f^*(x) \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} dy, \end{aligned}$$

which completes the proof of Theorem 5. \square

3.6. Proof of Theorem 6

The sufficient part is easily obtained by (36) and Theorem 4. It now remains to prove the necessary part.

(i) Let $b^*(x) = \ln|x|_h$. A simple calculation tells us that

$$b_{B(0,r)}^* = \frac{1}{|B(0,r)|} \int_{B(0,r)} \ln|x|_h dx = \ln r - 1/Q,$$

which implies that

$$\begin{aligned} \int_{B(0,r)} |b^*(x) - b_{B(0,r)}^*|^{p_2} |x|_h^\alpha dx &= \omega_Q \int_0^r |1/Q - \ln(r/\rho)|^{p_2} \rho^{\alpha+Q-1} d\rho \\ &= \omega_Q r^{Q+\alpha} \int_1^\infty |1/Q - \ln t|^{p_2} t^{-(Q+\alpha+1)} dt. \end{aligned}$$

Therefore,

$$\|b^*\|_{CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)} = c(Q + \alpha)^{1/p_2},$$

where

$$c = \left(\int_1^\infty |1/Q - \ln t|^{p_2} t^{-(Q+\alpha+1)} dt \right)^{1/p_2} < \infty.$$

Let $f^*(x) = |x|_h^{(Q+\alpha)\lambda}$. Then $f^* \in \dot{L}^{p_1, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha) \cap \dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ and

$$\|f^*\|_{\dot{L}^{p_1, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} = \frac{(Q+\alpha)^\lambda}{\omega_Q^\lambda (1+p_1\lambda)^{1/p_1}}, \quad \|f^*\|_{\dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} = \frac{(Q+\alpha)^\lambda}{\omega_Q^\lambda (1+p\lambda)^{1/p}}. \quad (37)$$

By definition of the commutator of the Hausdorff operator,

$$\left(\mathcal{H}_{\Phi, A}^{b^*, 1} f^* \right) (x) = \int_{\|A(y)\| \leq 1} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|_h^{(Q+\alpha)\lambda} \ln \left(\frac{|x|_h}{|A(y)x|_h} \right) dy.$$

Since $\|A(y)\| \leq 1$,

$$\ln \left(\frac{|x|_h}{|A(y)x|_h} \right) \geq \ln \left(\frac{1}{\|A(y)\|} \right) \geq 0.$$

This inequality implies that

$$\|\mathcal{H}_{\Phi, A}^{b^*, 1} f^*\|_{\dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \geq \|f^*\|_{\dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \int_{\|A(y)\| \leq 1} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \ln \left(\frac{1}{\|A(y)\|} \right) dy.$$

Therefore, the boundedness of $\mathcal{H}_{\Phi, A}^{b^*, 1}$ from $\dot{L}^{p_1, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ to $\dot{L}^{p, \lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ and (37) show that

$$\int_{\|A(y)\| \leq 1} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \ln \left(\frac{1}{\|A(y)\|} \right) dy < \infty.$$

(ii) Let $\tilde{b}(x) = \ln(1/|x|_h)$. Then $\tilde{b} \in CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)$ and

$$\|\tilde{b}\|_{CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)} = \|b^*\|_{CMO^{p_2}(\mathbb{H}^n; |\cdot|_h^\alpha)},$$

where b^* is as in (i). Without loss of generality, we assume that the constant $C_0 > 1$, since the case of $C_0 = 1$ is easier to deal with. Taking f^* be as in (i), we have

$$\left(\mathcal{H}_{\Phi, A}^{\tilde{b}, 2} f^* \right) (x) = \int_{\|A(y)\| > 1} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|_h^{(Q+\alpha)\lambda} \ln \left(\frac{|A(y)x|_h}{|x|_h} \right) dy = I_1 + I_2, \quad (38)$$

where

$$I_1 = \int_{\|A(y)\| > C_0} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|_h^{(Q+\alpha)\lambda} \ln \left(\frac{|A(y)x|_h}{|x|_h} \right) dy,$$

$$I_2 = \int_{1 < \|A(y)\| \leq C_0} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|_h^{(Q+\alpha)\lambda} \ln \left(\frac{|A(y)x|_h}{|x|_h} \right) dy.$$

Since $\|A(y)\|^{-1} \leq \|A^{-1}(y)\| \leq C_0 \|A(y)\|^{-1}$, if $\|A(y)\| > C_0$,

$$\ln \left(\frac{|A(y)x|_h}{|x|_h} \right) \geq \ln \left(\frac{1}{\|A^{-1}(y)\|} \right) \geq \ln \left(\frac{\|A(y)\|}{C_0} \right) \geq 0, \quad (39)$$

and if $1 < \|A(y)\| \leq C_0$,

$$\left| \ln \left(\frac{|A(y)x|_h}{|x|_h} \right) \right| \leq \ln C_0. \quad (40)$$

It follows from (39) and (40) that

$$\begin{aligned} I_1 &\geq |x|_h^{(Q+\alpha)\lambda} \int_{\|A(y)\| \geq C_0} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \ln \left(\frac{\|A(y)\|}{C_0} \right) dy \geq 0, \\ |I_2| &\leq |x|_h^{(Q+\alpha)\lambda} \ln C_0 \int_{1 < \|A(y)\| < C_0} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} dy. \end{aligned}$$

These two inequalities and (38) tell us that

$$\begin{aligned} &\|f^*\|_{\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \int_{\|A(y)\| > C_0} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \ln \left(\frac{\|A(y)\|}{C_0} \right) dy \\ &\leq \|\mathcal{H}_{\Phi,A}^{\tilde{b},2} f^*\|_{\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} + \|f^*\|_{\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)} \ln C_0 \int_{1 < \|A(y)\| \leq C_0} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} dy. \end{aligned}$$

Therefore, the boundedness of $H_{\Phi,A}^{\tilde{b},2}$ from $\dot{L}^{p_1,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ to $\dot{L}^{p,\lambda}(\mathbb{H}^n; |\cdot|_h^\alpha)$ and (3), (37) show that

$$\int_{\|A(y)\| > C_0} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{(Q+\alpha)\lambda} \ln \left(\frac{\|A(y)\|}{C_0} \right) dy < \infty.$$

Using (3) again, we finish the proof of Theorem 6. \square

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