

## COMMUTATORS OF RIESZ POTENTIAL IN THE VANISHING GENERALIZED WEIGHTED MORREY SPACES WITH VARIABLE EXPONENT

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*Abstract.* Let  $\Omega \subset \mathbb{R}^n$  be an unbounded open set. We consider the generalized weighted Morrey spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  and the vanishing generalized weighted Morrey spaces  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  with variable exponent  $p(x)$  and a general function  $\varphi(x,r)$  defining the Morrey-type norm. The main result of this paper are the boundedness of Riesz potential and its commutators on the spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  and  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ . This result generalizes several existing results for Riesz potential and its commutators on Morrey type spaces. Especially, it gives a unified result for generalized Morrey spaces and variable Morrey spaces which currently gained a lot of attentions from researchers in theory of function spaces.

### 1. Introduction

The variable exponent generalized weighted Morrey spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  over an open set  $\Omega \subset \mathbb{R}^n$  was introduced and the boundedness of the Hardy-Littlewood maximal operator, the singular integral operators and their commutators on these spaces was proven in [28]. The main focus of this article is to prove that the Riesz potential and its commutators are bounded on generalized weighted Morrey spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  and vanishing generalized weighted Morrey spaces  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  with variable exponents. Also some Sobolev-type inequalities for Riesz potentials on spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  and  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  are proved.

The classical Morrey spaces were introduced by Morrey [38] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey-type spaces are defined in the process of study. Mizuhara [39] and Nakai [42] introduced generalized Morrey spaces  $M^{p,\varphi}(\mathbb{R}^n)$  (see, also [21]); Komori and Shirai [33] defined weighted Morrey spaces  $L^{p,\kappa}(w)$ ; Guliyev [22] gave a concept of the generalized weighted Morrey spaces  $M_w^{p,\varphi}(\mathbb{R}^n)$  which could be viewed as extension of both  $M^{p,\varphi}(\mathbb{R}^n)$  and  $L^{p,\kappa}(w)$ .

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Vanishing Morrey spaces  $VM^{p,\varphi}(\mathbb{R}^n)$  are subspaces of functions in Morrey spaces which were introduced by Vitanza [52] satisfying the condition

$$\lim_{r \rightarrow 0} \sup_{\substack{x \in \mathbb{R}^n \\ 0 < t < r}} r^{-\frac{\lambda}{p}} \|f \chi_{B(x,t)}\|_{L^{p(\cdot)}(B(x,t))} = 0$$

and applied there to obtain a regularity result for elliptic partial differential equations. Also Ragusa [44] proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces  $VM^{p,\lambda}(\mathbb{R}^n)$ . The vanishing generalized Morrey spaces  $VM^{p,\varphi}(\mathbb{R}^n)$  were introduced and studied by Samko in [46], see also [3, 17, 37].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [16, 32, 47], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and Riesz potential on Lebesgue spaces with variable exponent we refer to [9, 10, 13, 14, 15, 31, 35, 45].

Variable exponent Morrey spaces  $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ , were introduced and studied in [4] and [40] in the Euclidean setting. The boundedness of Riesz potential in variable exponent Morrey spaces  $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$  under the log-condition on  $p(\cdot), \lambda(\cdot)$  and a Sobolev type  $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \rightarrow \mathcal{L}^{q(\cdot),\lambda(\cdot)}$ -theorem for potential operators of variable order  $\alpha(x)$  was proved in [31]. In the case of constant  $\alpha$ , there was also proved a boundedness theorem in the limiting case  $p(x) = \frac{n-\lambda(x)}{\alpha}$ , when the potential operator  $I^\alpha$  acts from  $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$  into  $BMO$  was proved in [4]. In [40] the maximal operator and potential operators were considered in a somewhat more general space, but under more restrictive conditions on  $p(x)$ .

Generalized Morrey spaces of such a kind in the case of constant  $p$  were studied in [18, 21, 39, 42, 50, 51]. In the case of bounded sets the boundedness of the maximal operator, singular integral operator and the potential operators in generalized variable exponent Morrey type spaces was proved in [24, 25, 26] and in the case of unbounded sets in [27]. Also, in the case of bounded sets the boundedness of these operators in generalized variable exponent weighted Morrey spaces for the power weights was proved in [30].

In the case of constant  $p$  and  $\lambda$ , the results on the boundedness of potential operators go back to [1] and [43], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [12]; for further results in the case of constant  $p$  and  $\lambda$  (see, for instance, [5, 19]).

In the spaces  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  over open sets  $\Omega \subset \mathbb{R}^n$  we consider the following operators:

1) the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\bar{B}(x,r)} |f(y)| dy,$$

2) Riesz potential operator

$$I^\alpha f(x) = \int_{\Omega} |x - y|^{\alpha-n} f(y) dy, \quad 0 < \alpha < n,$$

3) the fractional maximal operator

$$M^\alpha f(x) = \sup_{r>0} |B(x, r)|^{\frac{\alpha}{n}-1} \int_{\tilde{B}(x,r)} |f(y)| dy, \quad 0 \leq \alpha < n.$$

We find the condition on the function  $\varphi(x, r)$  for the boundedness of the Riesz potential  $I^\alpha$  and its commutators in the generalized weighted Morrey space  $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$  and the vanishing generalized weighted Morrey spaces  $V\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$  with variable  $p(x)$  under the log-condition on  $p(\cdot)$ .

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue, generalized weighted Morrey spaces and vanishing generalized weighted Morrey spaces. In Section 3 we prove the boundedness of Riesz potential and its commutators on the variable exponent generalized weighted Morrey spaces. In Section 4 we prove the boundedness of Riesz potential and its commutators on the variable exponent vanishing generalized weighted Morrey spaces.

The main results are given in Theorems 4, 5, 8, 9, 10, 1 and 2. We emphasize that the results we obtain for generalized Morrey spaces are new even in the case when  $p(x)$  is constant, because we do not impose any monotonicity type condition on  $\varphi(r)$ .

We use the following notation:  $\mathbb{R}^n$  is the  $n$ -dimensional Euclidean space,  $\Omega \subset \mathbb{R}^n$  is an open set,  $\chi_E(x)$  is the characteristic function of a set  $E \subseteq \mathbb{R}^n$ ,  $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ ,  $\tilde{B}(x, r) = B(x, r) \cap \Omega$ , by  $c, C, c_1, c_2$  etc, we denote various absolute positive constants, which may have different values even in the same line.

## 2. Preliminaries on variable exponent weighted Lebesgue, generalized weighted Morrey spaces and vanishing generalized weighted Morrey spaces

We refer to the book [14] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let  $p(\cdot)$  be a measurable function on  $\Omega$  with values in  $(1, \infty)$ . An open set  $\Omega$  which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{2.1}$$

where  $p_- := \text{ess inf}_{x \in \Omega} p(x)$ ,  $p_+ := \text{ess sup}_{x \in \Omega} p(x)$ . By  $L^{p(\cdot)}(\Omega)$  we denote the space of all measurable functions  $f(x)$  on  $\Omega$  such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : I_{p(\cdot)} \left( \frac{f}{\eta} \right) \leq 1 \right\},$$

this is a Banach function space. By  $p'(\cdot) = \frac{p(x)}{p(x)-1}$ ,  $x \in \Omega$ , we denote the conjugate exponent.

The space  $L^{p(\cdot)}(\Omega)$  coincides with the space

$$\left\{ f(x) : \left| \int_{\Omega} f(y)g(y)dy \right| < \infty \text{ for all } g \in L^{p'(\cdot)}(\Omega) \right\} \tag{2.2}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \leq 1} \left| \int_{\Omega} f(y)g(y)dy \right|, \tag{2.3}$$

see [36, Proposition 2.2], see also [34, Theorem 2.3], or [48, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [49], [34].

$\mathcal{P}(\Omega)$  is the set of bounded measurable functions  $p : \Omega \rightarrow [1, \infty)$ ;

$\mathcal{P}^{log}(\Omega)$  is the set of exponents  $p \in \mathcal{P}(\Omega)$  satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega, \tag{2.4}$$

where  $A = A(p) > 0$  does not depend on  $x, y$ ;

$\mathcal{A}^{log}(\Omega)$  is the set of bounded exponents  $p : \Omega \rightarrow \mathbb{R}^n$  satisfying the condition (2.4);

$\mathbb{P}^{log}(\Omega)$  is the set of exponents  $p \in \mathcal{P}^{log}(\Omega)$  with  $1 < p_- \leq p_+ < \infty$ ;

for  $\Omega$  which may be unbounded, by  $\mathcal{P}_{\infty}(\Omega)$ ,  $\mathcal{P}_{\infty}^{log}(\Omega)$ ,  $\mathbb{P}_{\infty}^{log}(\Omega)$ ,  $\mathcal{A}_{\infty}^{log}(\Omega)$  we denote the subsets of the above sets of exponents satisfying the decay condition (when  $\Omega$  is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_{\infty}}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n, \tag{2.5}$$

where  $p_{\infty} = \lim_{x \rightarrow \infty} p(x) > 1$ .

We will also make use of the estimate provided by the following lemma (see [14], Corollary 4.5.9).

$$\|\chi_{\tilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\theta_p(x,r)}, \quad x \in \Omega, \quad p \in \mathbb{P}_{\infty}^{log}(\Omega), \tag{2.6}$$

where  $\theta_p(x, r) = \begin{cases} \frac{n}{p(x)}, & r \leq 1, \\ \frac{n}{p(\infty)}, & r > 1 \end{cases}$ .

By  $\omega$  we always denote a weight, i.e. a positive, locally integrable function with domain  $\Omega$ . The weighted Lebesgue space  $L_{\omega}^{p(\cdot)}(\Omega)$  is defined as the set of all measurable functions for which

$$\|f\|_{L_{\omega}^{p(\cdot)}(\Omega)} = \|f\omega\|_{L^{p(\cdot)}(\Omega)}.$$

Let us define the class  $A_{p(\cdot)}(\Omega)$  (see [16], [35]) to consist of those weights  $\omega$  for which

$$[\omega]_{A_{p(\cdot)}} \equiv \sup_B |B|^{-1} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,r))} < \infty.$$

A weight function  $\omega$  belongs to the class  $A_{p(\cdot),q(\cdot)}(\Omega)$  if

$$[\omega]_{A_{p(\cdot),q(\cdot)}} \equiv \sup_B |B|^{\frac{1}{p(x)} - \frac{1}{q(x)} - 1} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,r))} < \infty.$$

LEMMA 1. Let  $p, q$  satisfy condition (2.1) and  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$ , then  $\omega^{-1} \in A_{q'(\cdot),p'(\cdot)}(\Omega)$ , with  $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ .

Proof. Let  $p, q$  satisfy condition (2.1) and  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$ . Then  $\varphi = \omega^{-1} \in A_{q'(\cdot),p'(\cdot)}(\Omega)$ . Indeed,

$$\begin{aligned} &|B|^{\frac{1}{q'(x)} - \frac{1}{p'(x)} - 1} \|\varphi\|_{L^{p'(\cdot)}(\tilde{B}(x,r))} \|\varphi^{-1}\|_{L^{q'(\cdot)}(\tilde{B}(x,r))} \\ &= |B|^{\frac{1}{p(x)} - \frac{1}{q(x)} - 1} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,r))}. \quad \square \end{aligned}$$

THEOREM 1. [29, Thorem 1.1] Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set and  $p \in \mathbb{P}_\infty^{log}(\Omega)$ . Then  $M : L_\omega^{p(\cdot)}(\Omega) \rightarrow L_\omega^{p(\cdot)}(\Omega)$  if and only if  $\omega \in A_{p(\cdot)}(\Omega)$ .

For unbounded sets, say  $\Omega = \mathbb{R}^n$ , and constant orders a the corresponding Sobolev theorem proved in [8, 9] runs as follows.

THEOREM 2. Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$  and  $p \in \mathbb{P}_\infty^{log}(\Omega)$ . Let also  $p_+ < \frac{n}{\alpha}$ . Then the operators  $M^\alpha$  and  $I^\alpha$  are bounded from  $L^{p(\cdot)}(\Omega)$  to  $L^{q(\cdot)}(\Omega)$  with  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ .

Let  $\lambda(x)$  be a measurable function on  $\Omega$  with values in  $[0, n]$ . The variable Morrey space  $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$  and variable weighted Morrey space  $\mathcal{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)$  is defined as the set of integrable functions  $f$  on  $\Omega$  with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, t > 0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\tilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathcal{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, t > 0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

Let  $\omega$  be a nonnegative measurable function on  $\mathbb{R}^n$  such that  $\omega^p$  is locally integrable on  $\mathbb{R}^n$ . Then a Radon measure  $\mu$  is canonically associated with the weight  $\omega(\cdot)^{p(\cdot)}$ , that is,

$$\mu(E) = \int_E \omega(y)^{p(y)} dy.$$

We denote by  $\mathcal{L}^{p(\cdot),\lambda}(\mathbb{R}^n, d\mu)$  the set of all measurable functions  $f$  with finite norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda}(\mathbb{R}^n, d\mu)} = \inf \left\{ \eta > 0 : \sup_{x \in \Omega, t > 0} \frac{t^\lambda}{\mu(B(x,t))} \int_{B(x,t)} \left( \frac{|f(y)|}{\eta} \right)^{p(y)} d\mu(y) \leq 1 \right\}.$$

**THEOREM 3.** [41] *Let  $0 < \alpha < n$ ,  $0 \leq \lambda < n$ ,  $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ ,  $p_+ < \frac{\lambda}{\alpha}$ ,  $\frac{1}{p(x)} = \frac{1}{p(x)} - \frac{\alpha}{\lambda}$ ,  $\omega \in A_{p(\cdot)}(\mathbb{R}^n)$ . Then the operator  $I^\alpha$  is bounded from  $\mathcal{L}^{p(\cdot),\lambda}(\mathbb{R}^n, d\mu)$  to  $\mathcal{L}^{q(\cdot),\lambda}(\mathbb{R}^n, d\mu)$ .*

In view of the well known pointwise estimate  $M^\alpha f(x) \leq C(I^\alpha|f|)(x)$ , it suffices to treat only the case of the operator  $I^\alpha$ .

**COROLLARY 1.** ([11], [41]) *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{\log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$ . Then the operators  $M^\alpha$  and  $I^\alpha$  are bounded from  $L_\omega^{p(\cdot)}(\Omega)$  to  $L_\omega^{q(\cdot)}(\Omega)$ .*

Let  $M^\sharp$  be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy,$$

where  $f_{\tilde{B}(x,r)}(x) = |\tilde{B}(x,r)|^{-1} \int_{\tilde{B}(x,r)} f(y) dy$ .

**DEFINITION 1.** We define the  $BMO(\Omega)$  space as the set of all locally integrable functions  $f$  with finite norm

$$\|f\|_{BMO} = \sup_{x \in \Omega} M^\sharp f(x) = \sup_{x \in \Omega, r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy.$$

**DEFINITION 2.** We define the  $BMO_{p(\cdot),\omega}(\Omega)$  space as the set of all locally integrable functions  $f$  with finite norm

$$\|f\|_{BMO_{p(\cdot),\omega}} = \sup_{x \in \Omega, r>0} \frac{\|(f(\cdot) - f_{\tilde{B}(x,r)}) \chi_{\tilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}{\|\chi_{\tilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}.$$

**THEOREM 4.** [36] *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $p \in \mathbb{P}_\infty^{\log}(\Omega)$  and  $\omega$  be a Lebesgue measurable function. If  $\omega \in A_{p(\cdot)}(\Omega)$ , then the norms  $\|\cdot\|_{BMO_{p(\cdot),\omega}}$  and  $\|\cdot\|_{BMO}$  are mutually equivalent.*

Everywhere in the sequel the functions  $\varphi(x,r)$ ,  $\varphi_1(x,r)$  and  $\varphi_2(x,r)$  used in the body of the paper, are non-negative measurable function on  $\Omega \times (0, \infty)$ . We find it convenient to define the variable exponent generalized weighted Morrey spaces in the form as follows.

**DEFINITION 3.** Let  $1 \leq p(x) < \infty$ ,  $x \in \Omega$ . The variable exponent generalized Morrey space  $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$  and variable exponent generalized weighted Morrey space  $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  are defined by the norms

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}} = \sup_{x \in \Omega, r>0} \frac{1}{\varphi(x,r)r^{\theta_{p(x,r)}}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))},$$

$$\|f\|_{\mathcal{M}_\omega^{p(\cdot),\varphi}} = \sup_{x \in \Omega, r > 0} \frac{1}{\varphi(x,r)\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))}.$$

According to this definition, we recover the space  $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$  under the choice  $\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}$ :

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\varphi(\cdot)}(\Omega) \Big|_{\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}}.$$

**DEFINITION 4.** (Vanishing generalized weighted Morrey space) The vanishing generalized weighted Morrey space  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  is defined as the space of functions  $f \in \mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  such that

$$\limsup_{r \rightarrow 0} \sup_{x \in \Omega} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r \rightarrow 0} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \inf_{x \in \Omega} \varphi(x,t)} = 0 \tag{2.7}$$

and

$$\sup_{0 < r < \infty} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \inf_{x \in \Omega} \varphi(x,t)} = 0, \tag{2.8}$$

which makes the spaces  $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$  non-trivial, because bounded functions with compact support belong then to this space.

**THEOREM 5.** [28] *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $p \in \mathbb{P}_\infty^{\text{log}}(\Omega)$ ,  $\omega \in A_{p(\cdot)}(\Omega)$  and the function  $\varphi_1(x,r)$  and  $\varphi_2(x,r)$  satisfy the condition*

$$\sup_{t > r} \frac{\text{ess inf}_{t < s < \infty} \varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \leq C\varphi_2(x,r). \tag{2.9}$$

*Then the maximal operator  $M$  is bounded from the space  $\mathcal{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$  the space  $\mathcal{M}_\omega^{p(\cdot),\varphi_2}(\Omega)$ .*

### 3. Riesz potential and its commutators in the spaces $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. Let  $b \in BMO(\mathbb{R}^n)$ . A well known result of Chanillo [7] states that the commutator operator  $[b, I^\alpha]f = I^\alpha(bf) - bI^\alpha f$  is bounded from  $L_p(\mathbb{R}^n)$  to  $L_q(\mathbb{R}^n)$  with  $1/q = 1/p - \alpha/n$ ,  $1 < p < n/\alpha$ .

Let  $L^\infty(\mathbb{R}_+, v)$  be the weighted  $L^\infty$ -space with the norm

$$\|g\|_{L^\infty(\mathbb{R}_+, v)} = \operatorname{ess\,sup}_{t>0} v(t)g(t).$$

In the sequel  $\mathfrak{M}(\mathbb{R}_+)$ ,  $\mathfrak{M}^+(\mathbb{R}_+)$  and  $\mathfrak{M}^+(\mathbb{R}_+; \uparrow)$  stand for the set of Lebesgue-measurable functions on  $\mathbb{R}_+$ , and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(\mathbb{R}_+; \uparrow) : \lim_{t \rightarrow 0^+} \varphi(t) = 0 \right\}.$$

Let  $u$  be a continuous and non-negative function on  $\mathbb{R}_+$ . We define the supremal operator  $\overline{S}_u$  by

$$(\overline{S}_u g)(t) := \|u g\|_{L_t(0,t)}, \quad t \in (0, \infty).$$

In the following theorem proved in [6], we use the notation

$$\tilde{v}_1(t) = \sup_{0 < \xi < t} v_1(\xi).$$

**THEOREM 1.** *Suppose that  $v_1$  and  $v_2$  are nonnegative measurable functions such that  $0 < \|v_1\|_{L_\infty(0,t)} < \infty$  for every  $t > 0$ . Let  $u$  be a continuous nonnegative function on  $\mathbb{R}$ . Then the operator  $\overline{S}_u$  is bounded from  $L_\infty(\mathbb{R}_+, v_1)$  to  $L_\infty(\mathbb{R}_+, v_2)$  on the cone  $\mathbb{A}$  if and only if*

$$\left\| v_2 \overline{S}_u \left( \|v_1\|_{L_\infty(0,\cdot)}^{-1} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^1 g(s)w(s)ds, \quad H_w^* g(t) := \int_t^1 \left(1 + \ln \frac{s}{t}\right) g(s)w(s)ds, \quad 0 < t < \infty,$$

where  $w$  is a weight.

The following theorem was proved in [23].

**THEOREM 2.** [23] *Let  $v_1, v_2$  and  $w$  be weights on  $(0, \infty)$  and  $v_1(t)$  be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)H_w g(t) \leq C \sup_{t>0} v_1(t)g(t)$$

holds for some  $C > 0$  for all non-negative and non-decreasing  $g$  on  $(0, \infty)$  if and only if

$$B := \sup_{t>0} v_2(t) \int_t^1 \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

**THEOREM 3.** [22] *Let  $v_1, v_2$  and  $w$  be weights on  $(0, \infty)$  and  $v_1(t)$  be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \leq C \sup_{t>0} v_1(t) g(t) \tag{3.1}$$

*holds for some  $C > 0$  for all non-negative and non-decreasing  $g$  on  $(0, \infty)$  if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^1 \left(1 + \ln \frac{s}{t}\right) \frac{w(s) ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

*Moreover, the value  $C = B$  is the best constant for (3.1).*

The following weighted local estimates are valid.

**THEOREM 4.** *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{\log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$ . Then*

$$\|I^\alpha f\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}, \tag{3.2}$$

*where  $C$  does not depend on  $f, x$  and  $t$ .*

*Proof.* We represent  $f$  as

$$f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{\tilde{B}(x,2t)}(y), \quad f_2(y) = f(y) \chi_{\Omega \setminus \tilde{B}(x,2t)}(y), \quad t > 0, \tag{3.3}$$

and have

$$I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x).$$

By Corollary 1 we obtain

$$\|I^\alpha f_1\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \leq \|I^\alpha f_1\|_{L_\omega^{q(\cdot)}(\Omega)} \leq C \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} = C \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}.$$

Then

$$\|I^\alpha f_1\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \leq C \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))},$$

where the constant  $C$  is independent of  $f$ .

On the other hand,

$$\begin{aligned} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))} &\approx |B|^{1-\frac{\alpha}{n}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))} \int_{2t}^1 \frac{ds}{s^{n+1-\alpha}} \\ &\leq |B|^{1-\frac{\alpha}{n}} \int_{2t}^1 \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \frac{ds}{s^{n+1-\alpha}} \\ &\lesssim \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \|w^{-1}\|_{L_\omega^{p'(\cdot)}(\tilde{B}(x,t))} \int_t^1 \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \frac{ds}{s^{n+1-\alpha}} \\ &\lesssim \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \int_t^1 \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega^{-1}\|_{L_\omega^{p'(\cdot)}(\tilde{B}(x,s))} \frac{ds}{s^{n+1-\alpha}} \\ &\lesssim [\omega]_{A_{p(\cdot), q(\cdot)}} \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \int_t^1 \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \end{aligned} \tag{3.4}$$

Taking into account that

$$\|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,t))} \leq C\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

we get

$$\|I^\alpha f_1\|_{L^{q(\cdot)}_{\omega}(\tilde{B}(x,t))} \leq C\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{3.5}$$

When  $|x-z| \leq t$ ,  $|z-y| \geq 2t$ , we have  $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$ , and therefore

$$\begin{aligned} |I^\alpha f_2(x)| &\leq \int_{\Omega \setminus \tilde{B}(x,2t)} |z-y|^{\alpha-n} |f(y)| dy \\ &\leq 2^{n-\alpha} \int_{\Omega \setminus \tilde{B}(x,2t)} |x-y|^{\alpha-n} |f(y)| dy. \end{aligned}$$

We obtain

$$\begin{aligned} \int_{\Omega \setminus \tilde{B}(x,2t)} |f(y)| dy &= \int_{\Omega \setminus \tilde{B}(x,2t)} |f(y)| \left( \int_{|x-y|}^\infty s^{\alpha-n-1} ds \right) dy \\ &\lesssim \int_{2t}^\infty s^{\alpha-n-1} \left( \int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds \\ &\lesssim \int_t^\infty s^{\alpha-n-1} \|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} ds \\ &\lesssim \int_t^\infty \|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \end{aligned}$$

Hence

$$\|I^\alpha f_2\|_{L^{q(\cdot)}_{\omega}(\tilde{B}(x,t))} \lesssim \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L^{p(\cdot)}_{\omega}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

which together with (3.5) yields (3.2).  $\square$

**THEOREM 5.** Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$  and the functions  $\varphi_1(x,t)$  and  $\varphi_2(x,t)$  fulfill the condition

$$\int_t^\infty \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x,r) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))} ds}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \lesssim \varphi_2(x,t). \tag{3.6}$$

Then the operator  $I^\alpha$  is bounded from  $\mathcal{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$  to  $\mathcal{M}_\omega^{q(\cdot),\varphi_2}(\Omega)$ .

*Proof.* Let  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$ , by condition (3.6) and Theorems 4, 2 with  $v_2(r) = \varphi_2(x,r)^{-1}$ ,  $v_1(r) = \varphi_1(x,r)^{-1} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1}$ ,  $g(r) = \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}$  and

$w(r) = \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1} r^{-1}$  we obtain

$$\begin{aligned} & \|I^\alpha f\|_{\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)} \\ & \lesssim \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_2(x, t)} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s} \\ & \lesssim \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} = \|f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)}. \quad \square \end{aligned}$$

Now we consider the commutators Riesz potential defined by

$$[b, I^\alpha]f(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]f(y)|x - y|^{\alpha-n} dy.$$

The commutator generated by  $M$  and a suitable function  $b$  is formally defined by

$$[M, b]f = M(bf) - bM(f).$$

Given a measurable function  $b$  the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |b(x) - b(y)| |f(y)| dy$$

for all  $x \in \mathbb{R}^n$ .

LEMMA 2. [20] Let  $b \in BMO(\mathbb{R}^n)$ ,  $1 < s < \infty$ . Then

$$M^\sharp([b, I^\alpha]f(x)) \leq C \|b\|_{BMO} \left[ (M|I^\alpha f(x)|^s)^{\frac{1}{s}} + (M^{s\alpha}|f(x)|^s)^{\frac{1}{s}} \right],$$

where  $C > 0$  is independent of  $f$  and  $x$ .

LEMMA 3. [11] Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $p \in \mathbb{P}_\infty^{log}(\Omega)$  and  $\omega \in A_{p(\cdot)}(\Omega)$ . Then

$$\|f\omega\|_{L^{p(\cdot)}} \leq C \|\omega M^\sharp f\|_{L^{p(\cdot)}}$$

with a constant  $C > 0$  not depending on  $f$ .

THEOREM 6. [2, Theorem 1.13] Let  $b \in BMO(\mathbb{R}^n)$ . Suppose that  $X$  is a Banach space of measurable functions defined on  $\mathbb{R}^n$ . Assume that  $M$  is bounded on  $X$ . Then the operator  $M_b$  is bounded on  $X$ , and the inequality

$$\|M_b f\|_X \leq C \|b\|_* \|f\|_X$$

holds with constant  $C$  independent of  $f$ .

COROLLARY 2. Let  $b \in BMO(\Omega)$ ,  $p \in \mathbb{P}_\infty^{log}(\Omega)$  and  $\omega \in A_{p(\cdot)}(\Omega)$ , then the operator  $M_b$  is bounded on  $L_\omega^{p(\cdot)}(\mathbb{R}^n)$ .

**THEOREM 7.** [28] *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $p \in \mathbb{P}_\infty^{\log}(\Omega)$ ,  $\omega \in A_{p(\cdot)}(\Omega)$ ,  $b \in BMO(\Omega)$  and the function  $\varphi_1(x, r)$  and  $\varphi_2(x, r)$  satisfy the condition*

$$\sup_{t>r} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x, s) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \leq C \varphi_2(x, r), \tag{3.7}$$

where  $C$  does not depend on  $x \in \Omega$  and  $t$ . Then the operator  $M_b$  is bounded from the space  $\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$  to the space  $\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)$ .

**THEOREM 8.** *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$  and  $p \in \mathbb{P}_\infty^{\log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$ . The following assertions are equivalent:*

- (i) *The operator  $[b, I^\alpha]$  is bounded from  $L_\omega^{p(\cdot)}(\Omega)$  to  $L_\omega^{q(\cdot)}(\Omega)$ .*
- (ii)  *$b \in BMO(\Omega)$ .*

*Proof.* (ii)  $\Rightarrow$  (i) Let  $f \in L_\omega^{p(\cdot)}(\Omega)$  and  $b \in BMO(\Omega)$ . By the Lemma 3, we have

$$\|[b, I^\alpha]f\|_{L_\omega^{q(\cdot)}(\Omega)} \lesssim \|M^\sharp([b, I^\alpha]f)\|_{L_\omega^{q(\cdot)}(\Omega)}.$$

From Lemma 2, we have

$$\begin{aligned} \|M^\sharp([b, I^\alpha]f)\|_{L_\omega^{q(\cdot)}(\Omega)} &\lesssim \|b\|_* \left\| (M|I^\alpha f|^s)^{\frac{1}{s}} + (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_\omega^{q(\cdot)}(\Omega)} \\ &\lesssim \|b\|_* \left[ \left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_\omega^{q(\cdot)}(\Omega)} + \left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_\omega^{q(\cdot)}(\Omega)} \right]. \end{aligned}$$

By Theorem 1 and Corollary 1, we have

$$\left\| (M|I^\alpha f|^s)^{\frac{1}{s}} \right\|_{L_\omega^{q(\cdot)}(\Omega)} \lesssim \| |I^\alpha f|^s \|_{L_{\omega^{s q(\cdot)}}^{\frac{q(\cdot)}{s}}(\Omega)}^{\frac{1}{s}} = \|I^\alpha f\|_{L_\omega^{p(\cdot)}(\Omega)} \lesssim \|f\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

By Corollary 1, we have

$$\left\| (M^{\alpha s}|f|^s)^{\frac{1}{s}} \right\|_{L_\omega^{q(\cdot)}(\Omega)} \lesssim \|f\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

Therefore

$$\|[b, I^\alpha]f\|_{L_\omega^{q(\cdot)}(\Omega)} \lesssim \|b\|_* \|f\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

(i)  $\Rightarrow$  (ii) Now, let us prove the “only if” part. Let  $[b, I^\alpha]$  be bounded from

$L_\omega^{p(\cdot)}(\Omega)$  to  $L_\omega^{q(\cdot)}(\Omega)$ ,  $1 < p_+ < \frac{n}{\alpha}$ . Then

$$\begin{aligned} & |B(x,t)| \int_{\tilde{B}(x,t)} |b(z) - b_{B(x,t)}| dz \\ &= \frac{1}{|B(x,t)|} \int_{\tilde{B}(x,t)} \left| b(z) - \frac{1}{|B(x,t)|} \int_{\tilde{B}(x,t)} b(y) dy \right| dz \\ &\leq \frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}} \int_{\tilde{B}(x,t)} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \left| \int_{\tilde{B}(x,t)} (b(z) - b(y)) dy \right| dz \\ &\leq \frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}} \int_{\tilde{B}(x,t)} \left| \int_{\tilde{B}(x,t)} (b(z) - b(y)) |x-y|^{\alpha-n} dy \right| dz \\ &\leq \frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}} \int_{\tilde{B}(x,t)} |[b, I_\alpha] \chi_{B(x,t)}(z)| dz \\ &\leq Ct^{-n-\alpha} \|[b, I_\alpha] \chi_{B(x,t)}\|_{L_\omega^{q(\cdot)}} \|\chi_{B(x,t)}\|_{L_{\omega^{-1}}^{q'(\cdot)}} \\ &\leq Ct^{-n-\alpha} \|\omega\|_{L^{p(\cdot)}(B(x,t))} \|\omega^{-1}\|_{L^{q'(\cdot)}(B(x,t))} \leq C. \end{aligned}$$

Hence we get

$$|B(x,t)|^{-1} \int_{\tilde{B}(x,t)} |b(y) - b_{B(x,t)}| dy \leq C.$$

This shows that  $b \in BMO(\Omega)$ .

The theorem has been proved.  $\square$

**THEOREM 9.** Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$ ,  $b \in BMO(\Omega)$ . Then

$$\begin{aligned} \|[b, I^\alpha] f\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} &\leq C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \\ &\quad \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}, \end{aligned} \quad (3.8)$$

where  $C$  does not depend on  $f$ ,  $x$  and  $t$ .

*Proof.* We represent  $f$  as

$$f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{\tilde{B}(x,2t)}(y), \quad f_2(y) = f(y) \chi_{\Omega \setminus \tilde{B}(x,2t)}(y), \quad t > 0, \quad (3.9)$$

and have

$$[b, I^\alpha] f(x) = [b, I^\alpha] f_1(x) + [b, I^\alpha] f_2(x).$$

By Theorem 8 we obtain

$$\begin{aligned} \|[b, I^\alpha] f_1\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} &\leq \|[b, I^\alpha] f_1\|_{L_\omega^{q(\cdot)}(\Omega)} \lesssim \|b\|_* \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} \\ &= \|b\|_* \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}. \end{aligned}$$

Then

$$\| [b, I^\alpha] f_1 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \| b \|_* \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))},$$

where the constant  $C$  is independent of  $f$ .

Taking into account that from the inequality (3.4) we have

$$\| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \| b \|_* \| \omega \|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \| \omega \|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

and then

$$\| [b, I^\alpha] f_1 \|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \leq C \| b \|_* \| \omega \|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \| \omega \|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \tag{3.10}$$

When  $|x - z| \leq t$ ,  $|z - y| \geq 2t$ , we have  $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$ , and therefore

$$\begin{aligned} \| [b, I^\alpha] f_2(x) \| &\leq \int_{\Omega \setminus \tilde{B}(x,2t)} |b(y) - b(z)| |z - y|^{\alpha-n} |f(y)| dy \\ &\leq C \int_{\Omega \setminus \tilde{B}(x,2t)} |b(y) - b(z)| |x - y|^{\alpha-n} |f(y)| dy. \end{aligned}$$

We obtain

$$\begin{aligned} &\int_{\Omega \setminus \tilde{B}(x,2t)} |b(y) - b(z)| |x - y|^{\alpha-n} |f(y)| dy \\ &= \int_{\Omega \setminus \tilde{B}(x,2t)} |b(y) - b(z)| |f(y)| \left( \int_{|x-y|}^\infty s^{\alpha-n-1} ds \right) dy \\ &\leq C \int_{2t}^\infty s^{\alpha-n-1} \left( \int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |b(y) - b_{\tilde{B}(x,t)}| |f(y)| dy \right) ds \\ &\quad + C |b(z) - b_{\tilde{B}(x,t)}| \int_{2t}^\infty s^{\alpha-n-1} \left( \int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds = V_1 + V_2. \end{aligned}$$

To estimate  $V_1$ :

$$\begin{aligned} V_1 &= C \int_{2t}^\infty s^{\alpha-n-1} \left( \int_{\{y \in \Omega : 2t \leq |x-y| \leq s\}} |b(y) - b_{\tilde{B}(x,t)}| |f(y)| dy \right) ds \\ &\leq C \int_t^\infty s^{\alpha-n-1} \| b(\cdot) - b_{\tilde{B}(x,s)} \|_{L_\omega^{p'(\cdot)}(\tilde{B}(x,s))} \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} ds \\ &\quad + C \int_t^\infty s^{\alpha-n-1} |b_{\tilde{B}(x,t)} - b_{\tilde{B}(x,s)}| \left( \int_{\tilde{B}(x,s)} |f(y)| dy \right) ds \\ &\leq C \| b \|_* \int_t^\infty s^{\alpha-n-1} \| \omega^{-1} \|_{L^{p'(\cdot)}(\tilde{B}(x,s))} \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} ds \\ &\quad + C \| b \|_* \int_t^\infty s^{\alpha-n-1} \ln \frac{s}{t} \| \omega^{-1} \|_{L^{p'(\cdot)}(\tilde{B}(x,s))} \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} ds \\ &\leq C \| b \|_* \int_t^\infty \left( 1 + \ln \frac{s}{t} \right) \| \omega \|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \| f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \frac{ds}{s}. \end{aligned} \tag{3.11}$$

To estimate  $V_2$ :

$$\begin{aligned}
 V_2 &= C|b(z) - b_{\tilde{B}(x,t)}| \int_{2t}^\infty s^{\alpha-n-1} \left( \int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |f(y)| dy \right) ds \\
 &\leq C|B(x,t)|^{-1} \int_{\tilde{B}(x,t)} |b(z) - b(y)| dy \int_{2t}^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\
 &\leq CM_b \chi_{B(x,t)}(z) \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}, \tag{3.12}
 \end{aligned}$$

where  $C$  does not depend on  $x, t$ . Then by Corollary 2 and (3.11), (3.12) we have

$$\begin{aligned}
 &\| [b, I^\alpha] f_2 \|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \\
 &\leq \|V_1\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} + \|V_2\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \\
 &\leq C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\
 &\quad + C \|M_b \chi_{B(x,t)}\|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\
 &\leq C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\
 &\quad + C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\
 &\leq C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\| [b, I^\alpha] f_2 \|_{L_\omega^{q(\cdot)}(\tilde{B}(x,t))} \\
 &\leq C \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},
 \end{aligned}$$

which together with (3.10) yields (3.8).  $\square$

**THEOREM 10.** Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{\log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$ ,  $b \in BMO(\Omega)$  and the functions  $\varphi_1(x, t)$  and  $\varphi_2(x, t)$  fulfill the condition

$$\int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\operatorname{ess\,inf}_{s < r < 1} \varphi_1(x, r) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} \leq C \varphi_2(x, t). \tag{3.13}$$

Then the operators  $[b, I^\alpha]$  is bounded from  $\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$  to  $\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)$ .

*Proof.* Let  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$ , by condition (3.13) and Theorems 9, 3 with  $v_2(r) = \varphi_2(x, r)^{-1}$ ,  $v_1(r) = \varphi_1(x, r)^{-1} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1}$ ,  $g(r) = \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))}$  and

$w(r) = \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1} r^{-1}$  we obtain

$$\begin{aligned} & \| [b, I^\alpha] f \|_{\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)} \\ & \lesssim \|b\|_* \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_2(x, t)} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\ & \lesssim \|b\|_* \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} = \|b\|_* \|f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)}. \quad \square \end{aligned}$$

**4. Riesz potential and its commutators in the spaces  $V\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$**

**THEOREM 1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}_\infty^{log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot), q(\cdot)}(\Omega)$  and the functions  $\varphi_1(x, t)$  and  $\varphi_2(x, t)$  fulfill the conditions*

$$C_{\gamma_0} := \int_{\gamma_0}^1 \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x, r) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))}}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} < \infty \tag{4.1}$$

for every  $\gamma_0 > 0$ , and

$$\int_t^\infty \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x, r) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))}}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} \leq C \varphi_2(x, t). \tag{4.2}$$

Then the operators  $I^\alpha$  is bounded from  $V\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$  to  $V\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)$ .

*Proof.* The norm inequalities follow from Theorem 5, so we only have to prove that if

$$\limsup_{r \rightarrow 0, x \in \mathbb{R}^n} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} = 0,$$

then

$$\limsup_{r \rightarrow 0, x \in \mathbb{R}^n} \frac{1}{\varphi_2(x, t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|I^\alpha f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{q(\cdot)}(\Omega)} = 0 \tag{4.3}$$

otherwise.

To show that  $\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x, t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|I^\alpha f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{q(\cdot)}(\Omega)} < \varepsilon$  for small  $r$ , we split the right-hand side of (3.2):

$$\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x, t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|I^\alpha f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C_0 (I_{1, \gamma_0}(x, t) + I_{2, \gamma_0}(x, t)), \tag{4.4}$$

where  $\gamma_0 > 0$  will be chosen as shown below (we may take  $\gamma_0 < 1$ ),

$$I_{1, \gamma_0}(x, t) := \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^{\gamma_0} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

$$I_{2,\gamma_0}(x,t) := \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_{\gamma_0}^{\infty} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

and it is supposed that  $t < \gamma_0$ . Now we choose any fixed  $\gamma_0 > 0$  such that

$$\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_1(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f\chi_{\tilde{B}(x,t)}\|_{L_{\omega}^{p(\cdot)}(\Omega)} < \frac{\varepsilon}{2CC_0}, \text{ for all } 0 < t < \gamma_0,$$

where  $C$  and  $C_0$  are constants from (4.2) and (4.4), which is possible since  $f \in V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$ . Then

$$\sup_{x \in \mathbb{R}^n} CI_{1,\gamma_0}(x,t) < \frac{\varepsilon}{2}, \quad 0 < t < \gamma_0,$$

by (4.2).

The estimation of the second term now may be made already by the choice of  $t$  sufficiently small thanks to the condition (4.1). We have

$$I_{2,\gamma_0}(x,t) \leq C\gamma_0 \frac{\varphi_2(x,t)}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f\|_{V\mathcal{M}_{\omega}^{q(\cdot),\varphi_2}(\Omega)},$$

where  $C\gamma_0$  is the constant from (4.1). Then, by (4.1) it suffices to choose  $r$  small enough such that

$$\frac{\varphi_2(x,t)}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} < \frac{\varepsilon}{2CC_0\|f\|_{V\mathcal{M}_{\omega}^{q(\cdot),\varphi_2}(\Omega)}},$$

which completes the proof of (4.3).  $\square$

**THEOREM 2.** *Let  $\Omega \subset \mathbb{R}^n$  be an open unbounded set,  $0 < \alpha < n$ ,  $p \in \mathbb{P}^{log}(\Omega)$ ,  $p_+ < \frac{n}{\alpha}$ ,  $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ ,  $\omega \in A_{p(\cdot),q(\cdot)}(\Omega)$  and the functions  $\varphi_1(x,t)$  and  $\varphi_2(x,t)$  fulfill the conditions*

$$C_{\gamma} := \int_{\gamma}^{\infty} \left(1 + \ln \frac{s}{t}\right) \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x,r) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))}}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} < \infty \tag{4.5}$$

for every  $\gamma$ , and

$$\int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \frac{\text{ess inf}_{s < r < \infty} \varphi_1(x,r) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,r))}}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}} \frac{ds}{s} \leq C\varphi_2(x,t). \tag{4.6}$$

Then the operators  $[b, I^{\alpha}]$  is bounded from  $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$  to  $V\mathcal{M}_{\omega}^{q(\cdot),\varphi_2}(\Omega)$ .

*Proof.* The norm inequalities follow from Theorem 10, so we only have to prove that

$$\begin{aligned} \limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_1(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f\chi_{\tilde{B}(x,t)}\|_{L_{\omega}^{p(\cdot)}(\Omega)} &= 0 \Rightarrow \\ \limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|[b, I^{\alpha}]f\chi_{\tilde{B}(x,t)}\|_{L_{\omega}^{q(\cdot)}(\Omega)} &= 0 \end{aligned} \tag{4.7}$$

otherwise.

To show that  $\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \| [b, I^\alpha] f \chi_{\tilde{B}(x,t)} \|_{L^{q(\cdot)}(\Omega)} < \varepsilon$  for small  $r$ , we split the right-hand side of (3.8):

$$\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \| [b, I^\alpha] f \chi_{\tilde{B}(x,t)} \|_{L^{q(\cdot)}(\Omega)} \leq C_0 (I_{1,\gamma}(x,r) + I_{2,\gamma}(x,r)), \quad (4.8)$$

where  $\gamma > 0$  will be chosen as shown below (we may take  $\gamma < 1$ ),

$$I_{1,\gamma}(x,t) := \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_t^\gamma \left(1 + \ln \frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

$$I_{2,\gamma}(x,t) := \|b\|_* \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))} \int_\gamma^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}$$

and it is supposed that  $t < \gamma$ . Now we choose any fixed  $\gamma > 0$  such that

$$\sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_1(x,t) \|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f \chi_{\tilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)} < \frac{\varepsilon}{2CC_0 \|b\|_*}, \quad \text{for all } 0 < t < \gamma,$$

where  $C$  and  $C_0$  are constants from (4.6) and (4.8), which is possible since  $f \in V\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$ . Then

$$\sup_{x \in \mathbb{R}^n} CI_{1,\gamma}(x,t) < \frac{\varepsilon}{2}, \quad 0 < t < \gamma,$$

by (4.6).

The estimation of the second term now may be made already by the choice of  $r$  sufficiently small thanks to the condition (4.5). We have

$$I_{2,\gamma}(x,t) \leq C_\gamma \|b\|_* \frac{\varphi_2(x,t)}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} \|f\|_{V\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)},$$

where  $C_\gamma$  is the constant from (4.5). Then, by (4.5) it suffices to choose  $r$  small enough such that

$$\frac{\varphi_2(x,t)}{\|\omega\|_{L^{q(\cdot)}(\tilde{B}(x,t))}} < \frac{\varepsilon}{2CC_\gamma \|b\|_* \|f\|_{V\mathcal{M}_\omega^{q(\cdot), \varphi_2}(\Omega)}},$$

which completes the proof of (4.6).  $\square$

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