

CHARACTERIZATION OF OPERATOR CONVEX FUNCTIONS BY CERTAIN OPERATOR INEQUALITIES

HIROYUKI OSAKA, YUKIHIRO TSURUMI AND SHUHEI WADA

(Communicated by I. Perić)

Abstract. For $\lambda \in (0, 1)$, let ψ be a non-constant, non-negative, continuous function on $(0, \infty)$ and let $\Gamma_\lambda(\psi)$ be the set of all non-trivial operator means σ such that an inequality

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$$

holds for all $A, B \in B(H)^{++}$. Then we have:

1. ψ is a decreasing operator convex function if and only if

$$\Gamma_\lambda(\psi) = \{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\}.$$

2. ψ is an operator convex function which is not a decreasing function if and only if

$$\Gamma_\lambda(\psi) = \{\nabla_\lambda\}.$$

The first result is a weighted version of Ando and Hiai's characterization of an operator monotone decreasing function and these two results imply each other.

1. Introduction

A bounded operator A , acting on a Hilbert space H is said to be positive if $(Ax, x) \geq 0$ for all $x \in H$. We denote this by $A \geq 0$. Let $B(H)^+$ be the set of all positive operators on H , and let $B(H)^{++}$ be the set of all positive invertible operators on H .

A real-valued function f on $(0, \infty)$ is called operator monotone if $0 < A \leq B$ implies $f(A) \leq f(B)$. The two functions $f(t) = t^s$ ($s \in [0, 1]$) and $f(t) = \log t$ are well known examples of operator monotone functions.

In [8], Kubo and Ando developed an axiomatic theory concerning operator connections and means for pairs of positive operators. That is, a binary operation σ acting on the class of positive operators, $(A, B) \mapsto A\sigma B$, is called an operator connection if the following requirements are fulfilled:

(I) If $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$.

(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.

Mathematics subject classification (2010): 47A64, 47A63.

Keywords and phrases: Operator means, operator monotone functions, operator convex functions.

The first author was supported in this research by the JSPS grant for Scientific Research No. 26400125.

(III) If $A_n \searrow A$ and $B_n \searrow B$, then $A_n \sigma B_n \searrow A \sigma B$.

An operator mean is a connection satisfying the normalization condition:

(IV) $1 \sigma 1 = 1$.

Kubo and Ando showed that an affine order-isomorphism exists from the class of operator connections onto the class of positive operator monotone functions, by the correspondence $\sigma \mapsto f_\sigma(t) = 1 \sigma(t)1$.

It is well known that if $f : (0, \infty) \rightarrow (0, \infty)$ is operator monotone, then the transpose $f'(t) = t f(\frac{1}{t})$, the adjoint $f^*(t) = \frac{1}{f(\frac{1}{t})}$, and the dual $f^\perp = \frac{t}{f(t)}$ are also operator monotone ([8]). Furthermore, we call f symmetric if $f = f'$ and self-adjoint if $f = f^*$. It was shown in [8] that if f is symmetric with $f(1) = 1$, then the corresponding operator mean exists between the harmonic mean $!$ and the arithmetic mean ∇ . That is, $! \leq \sigma_f \leq \nabla$.

Let f be a non-negative continuous function f on $(0, \infty)$. It is said that f is operator convex if $f(A \nabla B) \leq f(A) \nabla f(B)$ holds for all $A, B \in B(H)^{++}$. It is also said that f is operator monotone decreasing if $A, B \in B(H)^{++}$ satisfy $A \leq B$, then $f(A) \geq f(B)$ holds. It is known [1] that f is operator monotone decreasing if and only if it is operator convex and numerically non-increasing. It is also well known that f is operator monotone if and only if it is operator concave (i.e., $-f$ is operator convex).

In [1], Ando and Hiai gave a characterization of an operator monotone decreasing function by means of certain operator inequalities. In this paper, we show a weighted version of this result. To do this, for a non-negative continuous function ψ on $(0, \infty)$ and $\lambda \in (0, 1)$, we consider the set $\Gamma_\lambda(\psi)$ of operator means σ such that the inequality

$$\psi(A \nabla_\lambda B) \leq \psi(A) \sigma \psi(B)$$

holds for all $A, B \in B(H)^{++}$. Our main results (Theorem 3.2) are the following:

(1) ψ is a decreasing operator convex function if and only if

$$\Gamma_\lambda(\psi) = \{ \sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda \}.$$

(2) ψ is an operator convex function which is not a decreasing function if and only if

$$\Gamma_\lambda(\psi) = \{ \nabla_\lambda \}.$$

The first result is a weighted version of Ando and Hiai's characterization of an operator monotone decreasing function and these two results imply each other.

2. λ -weighted means and operator convexity

From the theory of operator means, an operator mean σ is identified with an operator monotone function $t \mapsto 1 \sigma t$ on $(0, \infty)$. Specifically, a non-negative value $\frac{d(1 \sigma t)}{dt} \Big|_{t=1}$ often indicates some properties of σ (see [2]). We call this value the weight of σ . Since $1 \leq 1 \sigma t \leq t$ for all $t \geq 1$, we have

$$\frac{d(1 \sigma t)}{dt} \Big|_{t=1} \leq \lim_{t \rightarrow 1^+} \frac{t-1}{t-1} = 1.$$

DEFINITION 2.1. Let $\lambda \in [0, 1]$. An operator mean σ is called λ -weighted if

$$\left. \frac{d(1\sigma t)}{dt} \right|_{t=1} = \lambda$$

and is called non-trivial if the weight of σ is in $(0, 1)$.

Note that σ is the left trivial mean ($A\sigma B = A$) if $\lambda = 0$ and the right trivial mean ($A\sigma B = B$) if $\lambda = 1$.

In the rest of the paper, we consider a continuous function ψ satisfying

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B) \tag{2.1}$$

for all $A, B \in B(H)^{++}$ and for a certain operator mean σ . From the following result, it is natural to assume that ψ is operator convex.

PROPOSITION 2.2. Let ψ be a non-negative continuous function on $(0, \infty)$. Then the following are equivalent:

- (1) ψ is operator convex;
- (2) $\psi(A\nabla_\lambda B) \leq \psi(A)\nabla_\lambda\psi(B)$ for all $A, B \in B(H)^{++}$ and for all $\lambda \in (0, 1)$;
- (3) $\psi(A\nabla_\lambda B) \leq \psi(A)\nabla_\lambda\psi(B)$ for all $A, B \in B(H)^{++}$ and for some $\lambda \in (0, 1)$;
- (4) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all $A, B \in B(H)^{++}$ and for some $\lambda \in (0, 1)$ and for some non-trivial operator mean σ .

Proof. It is sufficient to show (4) \rightarrow (1). For every $A, B \in B(H)^{++}$, we define sequences by

$$\begin{aligned} A_0 &:= A, & B_0 &:= B, \\ A_n &:= (A_{n-1}\nabla_{1-\lambda}B_{n-1})\nabla_\lambda(A_{n-1}\nabla_\lambda B_{n-1}), \\ B_n &:= A + B - A_n \end{aligned}$$

for $n \geq 1$. Since

$$\begin{aligned} \begin{bmatrix} A_n \\ B_n \end{bmatrix} &= \begin{bmatrix} 2\lambda(1-\lambda) & \lambda^2 + (1-\lambda)^2 \\ \lambda^2 + (1-\lambda)^2 & 2\lambda(1-\lambda) \end{bmatrix} \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 2\lambda(1-\lambda) & \lambda^2 + (1-\lambda)^2 \\ \lambda^2 + (1-\lambda)^2 & 2\lambda(1-\lambda) \end{bmatrix}^n \begin{bmatrix} A \\ B \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -(2\lambda - 1)^{2n} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}, \end{aligned}$$

the sequences $\{A_n\}$ and $\{B_n\}$ have the same limit $A\nabla B$ in the operator norm topology.

Put $\gamma = \left. \frac{d(1\sigma t)}{dt} \right|_{t=1}$. We define sequences $\{A^{(n)}\}$ and $\{B^{(n)}\}$ by

$$A^{(0)} := \psi(A), \quad B^{(0)} := \psi(B),$$

$$A^{(n)} := (A^{(n-1)} \nabla_{1-\gamma} B^{(n-1)}) \nabla_{\gamma} (A^{(n-1)} \nabla_{\gamma} B^{(n-1)}),$$

$$B^{(n)} := \psi(A) + \psi(B) - A^{(n)}.$$

These sequences tend to $\psi(A) \nabla \psi(B)$ using the same argument as in the preceding sequence.

It follows from the assumption that

$$\begin{aligned} \psi(A_n) &= \psi((A_{n-1} \nabla_{1-\lambda} B_{n-1}) \nabla_{\lambda} (A_{n-1} \nabla_{\lambda} B_{n-1})) \\ &\leq \psi(A_{n-1} \nabla_{1-\lambda} B_{n-1}) \sigma \psi(A_{n-1} \nabla_{\lambda} B_{n-1}) \\ &\leq \psi(A_{n-1} \nabla_{1-\lambda} B_{n-1}) \nabla_{\gamma} \psi(A_{n-1} \nabla_{\lambda} B_{n-1}) \\ &= \psi(B_{n-1} \nabla_{\lambda} A_{n-1}) \nabla_{\gamma} \psi(A_{n-1} \nabla_{\lambda} B_{n-1}) \\ &\leq (\psi(B_{n-1}) \sigma \psi(A_{n-1})) \nabla_{\gamma} (\psi(A_{n-1}) \sigma \psi(B_{n-1})) \\ &\leq (\psi(B_{n-1}) \nabla_{\gamma} \psi(A_{n-1})) \nabla_{\gamma} (\psi(A_{n-1}) \nabla_{\gamma} \psi(B_{n-1})) \\ &\leq (\psi(A_{n-1}) \nabla_{1-\gamma} \psi(B_{n-1})) \nabla_{\gamma} (\psi(A_{n-1}) \nabla_{\gamma} \psi(B_{n-1})) \leq A^{(n)}, \end{aligned}$$

which implies that

$$\psi(A \nabla B) = \lim_{n \rightarrow \infty} \psi(A_n) \leq \lim_{n \rightarrow \infty} A^{(n)} = \psi(A) \nabla \psi(B),$$

where $\lim_{n \rightarrow \infty}$ is the limit in the operator norm topology. \square

PROPOSITION 2.3. For $\lambda \in (0, 1)$, let ψ be a non-negative, non-constant, continuous function on $(0, \infty)$ and let σ be a non-trivial operator mean. Suppose that

$$\psi(A \nabla_{\lambda} B) \leq \psi(A) \sigma \psi(B)$$

for all $A, B \in B(H)^{++}$. Then, σ is λ -weighted.

LEMMA 2.4. For $\lambda \in [0, 1]$, let ψ be a non-negative continuous function on $(0, \infty)$ with a non-zero derivative at 1 and let σ be a non-trivial operator mean. Suppose that

$$\psi(A \nabla_{\lambda} B) \leq \psi(A) \sigma \psi(B)$$

for all $A, B \in B(H)^{++}$. Then, σ is λ -weighted.

Proof. Put $\gamma = \frac{d(1\sigma t)}{dt} \Big|_{t=1}$. It follows from the fact $\sigma \leq \nabla_{\gamma}$ that the inequality

$$\psi(A \nabla_{\lambda} B) \leq \psi(A) \nabla_{\gamma} \psi(B)$$

holds for all $A, B \in B(H)^{++}$.

Thus, it is sufficient to show the case $\sigma = \nabla_{\gamma}$. Moreover, since $\psi'(1) \neq 0$ and ψ is operator convex by Proposition 2.2, we may assume that $\psi(1) = 1$ and hence $\psi(t) > 0$ for all $t > 0$.

By assumption, the inequality

$$\frac{1\sigma\psi(t) - 1\sigma\psi(1)}{t - 1} \geq \frac{\psi((1 - \lambda) + t\lambda) - \psi(1)}{t - 1}$$

holds for all $t > 1$, which implies that

$$\lim_{t \downarrow 1} \frac{1\sigma\psi(t) - 1\sigma\psi(1)}{t - 1} = \frac{d}{dt}(1\sigma t) \Big|_{t=1} \frac{d\psi}{dt} \Big|_{t=1} \geq \lambda \frac{d\psi}{dt} \Big|_{t=1}.$$

We also obtain

$$\lim_{t \uparrow 1} \frac{1\sigma\psi(t) - 1\sigma\psi(1)}{t - 1} = \frac{d}{dt}(1\sigma t) \Big|_{t=1} \frac{d\psi}{dt} \Big|_{t=1} \leq \lambda \frac{d\psi}{dt} \Big|_{t=1}.$$

Therefore, $\frac{d}{dt}(1\sigma t) \Big|_{t=1} \frac{d\psi}{dt} \Big|_{t=1} = \lambda \frac{d\psi}{dt} \Big|_{t=1}$, which implies the desired result. \square

Proof of Proposition 2.3. By Proposition 2.2, it is clear that ψ is operator convex and is differentiable at 1. The case when ψ has a non-zero derivative at 1 is discussed in Lemma 2.4. Therefore, we only consider the case when ψ has a zero derivative at 1. Considering the scalar multiple, we may assume that $\psi(1) = 1$.

Put $\varphi(t) = \psi(t + 1) - 1$ and $\gamma = \frac{d(1\sigma t)}{dt} \Big|_{t=1}$. We show that φ and ∇_γ satisfy the assumption of Lemma 2.4.

From the facts that ψ is a non-negative operator convex function and $\frac{d\psi}{dt} \Big|_{t=1} = 0$, φ is a non-negative operator convex function with $\varphi(0) = 0$. Thus, it follows from [3] that φ can be written as $\varphi(t) = tf(t)$ by using a non-negative operator monotone function f on $(0, \infty)$. If $f = 0$, then $\psi = 1$ on $[1, \infty)$, which implies $\psi = 1$ on $(0, \infty)$. This contradicts the assumption. Therefore, $f \neq 0$ and

$$\frac{d\varphi}{dt} \Big|_{t=1} = f(1) + \frac{df}{dt} \Big|_{t=1} > 0.$$

Furthermore,

$$\begin{aligned} \varphi(A\nabla_\lambda B) &= \psi(A\nabla_\lambda B + 1) - 1 \\ &= \psi((A + 1)\nabla_\lambda(B + 1)) - 1 \\ &\leq \psi(A + 1)\sigma\psi(B + 1) - 1 \\ &\leq \psi(A + 1)\nabla_\gamma\psi(B + 1) - 1 \\ &= \varphi(A)\nabla_\gamma\varphi(B) \end{aligned}$$

for $A, B \geq 0$. Now, it is obtained that φ and ∇_γ satisfy the assumption of Lemma 2.4. Hence, ∇_γ is λ -weighted, namely $\gamma = \lambda$. \square

Now, we can characterize a non-negative operator convex function on $(0, \infty)$.

COROLLARY 2.5. For $\lambda \in (0, 1)$, let ψ be a non-constant, non-negative, continuous function on $(0, \infty)$ and let $\Gamma_\lambda(\psi)$ be the set of all non-trivial operator means σ such that inequality (2.1) holds for all $A, B \in B(H)^{++}$. Then, ψ is an operator convex function if and only if

$$\{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\} \supseteq \Gamma_\lambda(\psi) \supseteq \{\nabla_\lambda\}.$$

COROLLARY 2.6. For $\lambda \in (0, 1)$, let ϕ be a positive operator concave function on $(0, \infty)$ with non-zero derivative at 1 and $\phi(1) = 1$ and let σ be a non-trivial operator mean. Then, the following are equivalent:

- (1) σ is λ -weighted;
- (2) $\phi(A)\sigma\phi(B) \leq \phi(A\nabla_\lambda B)$ for all $A, B \in B(H)^{++}$;
- (3) $\phi^*(A!_\lambda B) \leq \phi^*(A)\sigma^*\phi^*(B)$ for all $A, B \in B(H)^{++}$, where $\phi^*(x) = (\phi(x^{-1}))^{-1}$.

Proof. (1) \rightarrow (2): Because σ is λ -weighted, we have $\sigma \leq \nabla_\lambda$. This means that

$$\phi(A)\sigma\phi(B) \leq \phi(A)\nabla_\lambda\phi(B) \leq \phi(A\nabla_\lambda B).$$

The last inequality follows from the operator concavity of ϕ .

(2) \rightarrow (1): Note that, because ϕ is non-constant positive operator concave on $(0, \infty)$, $\frac{1}{\phi(t)}$ is non-constant operator convex with a non-zero derivative at 1. From the assumptions,

$$\phi(A)^{-1}\sigma^*\phi(B)^{-1} \geq \phi(A\nabla_\lambda B)^{-1}$$

holds for all $A, B > 0$, where σ^* is the adjoint of σ , so that $A\sigma^*B = (A^{-1}\sigma B^{-1})^{-1}$. Hence, σ^* is λ -weighted by Proposition 2.3. Because

$$\frac{d}{dt}(1\sigma t)|_{t=1} = \frac{d}{dt}(1\sigma^*t)|_{t=1} = \lambda,$$

σ is λ -weighted.

(2) \leftrightarrow (3):

We have

$$\begin{aligned} \phi(A\nabla_\lambda B) &\geq \phi(A)\sigma\phi(B) \text{ for all } A, B \in B(H)^{++} \\ \Leftrightarrow \phi^*(A!_\lambda B) &\leq \phi^*(A)\sigma^*\phi^*(B) \text{ for all } A, B \in B(H)^{++}. \quad \square \end{aligned}$$

Because ϕ is operator concave, equivalently operator monotone, ϕ^* is operator monotone and so operator concave, with $\phi^*(1) = 1$.

3. Characterization of operator convex functions

The following is a weighted version of [1, Theorem 2.1].

PROPOSITION 3.1. For $\lambda \in (0, 1)$, let ψ be a non-negative continuous function on $(0, \infty)$. Then, the following conditions are equivalent:

- (1) ψ is operator monotone decreasing;
- (2) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all $A, B \in B(H)^{++}$ and for all λ -weighted operator means σ ;
- (3) $\psi(A\nabla_\lambda B) \leq \psi(A)\#_\lambda\psi(B)$ for all $A, B \in B(H)^{++}$;
- (4) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all $A, B \in B(H)^{++}$ and for some λ -weighted operator mean $\sigma \neq \nabla_\lambda$,

where $A\#_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$.

Proof. We first demonstrate (1) \rightarrow (2). It is sufficient to prove the case $\psi > 0$. Since a mapping $t \mapsto \frac{1}{\psi(t)}$ is an operator concave function on $(0, \infty)$, we have

$$\frac{1}{\psi(A\nabla_\lambda B)} \geq \frac{1}{\psi(A)}\nabla_\lambda \frac{1}{\psi(B)}$$

for $A, B \in B(H)^{++}$. This implies $\psi(A\nabla_\lambda B) \leq \psi(A)\!_\lambda\psi(B) \leq \psi(A)\sigma\psi(B)$.

The implications of (2) \rightarrow (3) \rightarrow (4) are trivial. Lastly, we demonstrate (4) \rightarrow (1). By Proposition 2.2, the operator convexity of ψ is obtained. Therefore, we have

$$\begin{aligned} \psi(A\nabla B) &= \psi\left(\frac{A\nabla_\lambda B + A\nabla_{1-\lambda} B}{2}\right) \\ &\leq \frac{1}{2}\psi(A\nabla_\lambda B) + \frac{1}{2}\psi(A\nabla_{1-\lambda} B) \\ &\leq \psi(A)\tau\psi(B) \end{aligned}$$

for all $A, B \in B(H)^{++}$, where τ is a symmetric operator mean such that $1\tau t = \frac{1\sigma t + t\sigma 1}{2}$. From the assumption $\sigma \leq \nabla_\lambda$, there exists $t_0 > 0$ such that

$$1\tau t_0 = \frac{1\sigma t_0 + t_0\sigma 1}{2} < \frac{1\nabla_\lambda t_0 + t_0\nabla_\lambda 1}{2} = \frac{1+t_0}{2},$$

which signifies that $\tau \leq \nabla$. It follows from [1, Theorem 2.1] that ψ is operator monotone decreasing. \square

Combining the above results, our main theorem is obtained:

THEOREM 3.2. For $\lambda \in (0, 1)$, let ψ be a non-constant, non-negative, continuous function on $(0, \infty)$ and let $\Gamma_\lambda(\psi)$ be the set of all non-trivial operator means σ such that the inequality

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$$

holds for all $A, B \in B(H)^{++}$. Then, the following holds:

(1) ψ is a decreasing operator convex function if and only if

$$\Gamma_\lambda(\psi) = \{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\}.$$

(2) ψ is an operator convex function which is not a decreasing function if and only if

$$\Gamma_\lambda(\psi) = \{\nabla_\lambda\}.$$

Proof. From (2) in Proposition 3.1 and Proposition 2.3, the first statement is true. Next, we present the second one. Assume ψ is operator convex and is not decreasing. Then a relation $\Gamma_\lambda(\psi) \supseteq \{\nabla_\lambda\}$ holds by Proposition 2.2. If $\Gamma_\lambda(\psi) \setminus \{\nabla_\lambda\} \neq \emptyset$, then ψ is decreasing by (4) in Proposition 3.1, which contradicts the assumption. Hence, $\Gamma_\lambda(\psi) \setminus \{\nabla_\lambda\} = \emptyset$.

Conversely, if $\Gamma_\lambda(\psi) = \{\nabla_\lambda\}$, then ψ is operator convex by Proposition 2.2. From the first statement in this theorem, the operator convex function ψ with $\Gamma_\lambda(\psi) \neq \{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\}$ is not a decreasing function. \square

It is known that a non-negative operator convex function ψ on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(1) = 1$ is strictly increasing. Therefore, the following is a direct result of the preceding theorem.

COROLLARY 3.3. Let $\lambda \in (0, 1)$, and let σ be a non-trivial operator mean. Suppose that ψ is a non-negative operator convex function on $[0, \infty)$, with $\psi(0) = 0$ and $\psi(1) = 1$. Then, the following are equivalent:

1. $\sigma = \nabla_\lambda$;
2. $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all $A, B \in B(H)^{++}$.

REMARK 3.4. In Theorem 3.2, the first statement implies the second one and can be proven using Corollary 3.3 and the arguments from the proof of [1, Theorem 2.1]. Thus, these three statements (two statements in Theorem 3.2 and Corollary 3.3) are equivalent.

4. Matrix 2-convex functions

If ψ is a non-negative 2-convex function on $[0, \infty)$ with $\psi(0) = 0$, then ψ is a C^2 -function on $(0, \infty)$, by [7] (Cf. [4, Theorem 2.4.2]). Recall that ψ is said to be 2-convex if for all $A, B \in M_2(\mathbf{C})^+$ and $\lambda \in [0, 1]$ $\psi(\lambda A + (1 - \lambda)B) \leq \lambda \psi(A) + (1 - \lambda)\psi(B)$. Moreover, if ψ is non-constant, then it is strictly monotone increasing on $(0, \infty)$. Indeed, by [11, Theorem 2.2] there exists a monotone function f on $(0, \infty)$,

such that $\psi(t) = tf(t)$ for all $t > 0$. Let us show that $f(t) > 0$ for all $t > 0$. Assume on the contrary that $f(t_0) = 0$ for some $t_0 > 0$. Then, since f is monotone, we have $f(t) = 0$ for all $t \in (0, t_0]$. By [4, Theorem 2.4.2] (or [6, Theorem 6.6.52 (2)]), ψ is linear on $(0, \infty)$ so that ψ is constant zero, a contradiction. Then, for any $0 < x_1 < x_2$, we have

$$\begin{aligned} \psi(x_1) &= x_1 f(x_1) \leq x_1 f(x_2) \\ &< x_2 f(x_2) = \psi(x_2). \end{aligned}$$

Using this, we present an extension of Corollary 3.3.

PROPOSITION 4.1. *Let $\lambda \in (0, 1)$, and let σ be a non-trivial operator mean. Suppose that ψ is a non-negative operator 2-convex function on $[0, \infty)$, with $\psi(0) = 0$ and $\psi(1) = 1$. Then, the following are equivalent:*

- (1) $\sigma = \nabla_\lambda$;
- (2) $\psi(A \nabla_\lambda B) \leq \psi(A) \sigma \psi(B)$ for all positive definite 2×2 matrices A, B .

Proof. It is sufficient to demonstrate (2) \rightarrow (1). From the argument in Proposition 2.2, it follows that ψ is a 2-convex function. Let P, Q be orthogonal projections in $M_2(\mathbb{C})$ with $P \wedge Q = 0$. Applying the inequality in the assumption to $A_\varepsilon := P + \varepsilon I_2$ and $B_\varepsilon := Q + \varepsilon I_2$ for an arbitrary $\varepsilon > 0$, we obtain

$$\psi(A_\varepsilon \nabla_\lambda B_\varepsilon) \leq \psi(A_\varepsilon) \sigma \psi(B_\varepsilon).$$

Because $A_\varepsilon \nabla_\lambda B_\varepsilon = P \nabla_\lambda Q + \varepsilon I_H \rightarrow P \nabla_\lambda Q$, $\psi(A_\varepsilon \nabla_\lambda B_\varepsilon) \rightarrow \psi(P \nabla_\lambda Q)$ as $\varepsilon \rightarrow 0$ in the operator norm topology. Furthermore, because $\psi(A_\varepsilon) \searrow \psi(P) = P$, $\psi(B_\varepsilon) \searrow \psi(Q) = Q$ as $\varepsilon \rightarrow 0$ in the strong operator topology and the operator mean is continuous in it under the downward convergence, we have

$$\psi(P \nabla_\lambda Q) \leq P \sigma Q. \tag{4.1}$$

Furthermore, $P \sigma Q = aP + bQ$ by [8, Theorem 3.7], where $a = \inf_x f_\sigma(x)$, $b = \lim_{x \rightarrow \infty} \frac{f_\sigma(x)}{x}$, with f_σ denoting the representing function on $(0, \infty)$ corresponding to σ . Choose two orthogonal projections as

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \quad (0 < \theta < \frac{\pi}{2})$$

in the realization of the 2×2 matrix algebra in $B(H)$. Then, $P \wedge Q = 0$ and

$$\psi(P \nabla_\lambda Q) = \psi \left(\begin{bmatrix} (1-\lambda) + \lambda \cos^2 \theta & \lambda \cos \theta \sin \theta \\ \lambda \cos \theta \sin \theta & \lambda \sin^2 \theta \end{bmatrix} \right).$$

Because ψ is continuous, letting $\theta \rightarrow 0$ gives (4.1) as

$$\lim_{\theta \rightarrow 0} \psi(P \nabla_\lambda Q) = \psi \left(\begin{bmatrix} (1-\lambda) + \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \leq \lim_{\theta \rightarrow 0} P \sigma Q = \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing the (1,1)-entries of both sides of the above inequality, we find that

$$1 \leq a + b. \quad (4.2)$$

Furthermore, because f_σ is an operator monotone function, there exists a positive Radon measure μ on $[0, \infty]$ such that

$$f_\sigma(x) = 1\sigma x = a + bx + \int_{(0, \infty)} \frac{(t+1)x}{t+x} d\mu(t),$$

where $a = \lim_{x \rightarrow 0^+} f_\sigma(x)$ and $b = \lim_{x \rightarrow \infty} \frac{f_\sigma(x)}{x}$. Therefore,

$$f_\sigma(1) = a + b + \int_{(0, \infty)} d\mu(t) = 1,$$

and hence $\mu = 0$, by (4.2). Then, we have

$$f_\sigma(x) = a + bx, \quad 1 = a + b.$$

It follows from Proposition 2.3 that $\lambda = b$. \square

Similarly, we have the following characterization of the λ -weighted harmonic mean.

PROPOSITION 4.2. *Let ψ be a non-negative continuous function on $[0, \infty)$ with $\psi(1) = 1$ and $\lim_{x \rightarrow \infty} \psi(x) = +\infty$, and assume that $\lambda \in (0, 1)$. If a non-trivial operator mean σ satisfies*

$$\psi(A!_\lambda B) \geq \psi(A)\sigma\psi(B)$$

for all positive definite 2×2 matrices A, B , then $\sigma = !_\lambda$.

Proof. We have $\psi(A!_\lambda B) \geq \psi(A)\sigma\psi(B)$ for positive definite 2×2 matrices A, B $\Leftrightarrow \psi^*(A\nabla_\lambda B) \leq \psi^*(A)\sigma^*\psi^*(B)$ for positive definite 2×2 matrices A, B , where σ^* is the adjoint of σ , so that $A\sigma^*B = (A^{-1}\sigma B^{-1})^{-1}$ and $\psi^*(x) = (\psi(x^{-1}))^{-1}$. Thus, ψ^* is 2-convex by Proposition 2.2. Because $\psi^*(0) = \lim_{x \rightarrow 0} \psi^*(x) = 0$ and $\psi^*(1) = 1$, we have $\sigma^* = \nabla_\lambda$ by Proposition 4.1. Therefore, $\sigma = !_\lambda$. \square

REFERENCES

- [1] T. ANDO AND F. HIAI, *Operator log-convex functions and operator means*, Math. Ann. **350** (2011), 611–630.
- [2] J. I. FUJII, *Operator means and Range inclusion*, Linear Algebra Appl. **170** (1992), 137–146.
- [3] F. HANSEN AND G. K. PEDERSEN, *Jensen's inequality for operator and Löwner's theorem*, Math. Ann. **258** (1982) 229–241.
- [4] F. HIAI, *Matrix analysis: matrix monotone functions, matrix means, and Majorization*, Interdiscip. Inform. Sci. vol 16 (2010), no. 2, 139–248.
- [5] F. HIAI AND D. PETZ, *Introduction to matrix analysis and applications*, Universitext, Springer, New Delhi, 2014.
- [6] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, 1990.

- [7] F. KRAUS, *Über Konvexe Mathtrixfunctiounen*, Math. Z. **41** (1936) 18–42.
- [8] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann. **246** (1980) 205–224.
- [9] K. LÖWNER, *Über monotone matrixfunktionen*, Math. Z. **38** (1934) 177–216.
- [10] C. P. NICULESCU AND L.-E. PERSSON, *Convex functions and their applications. A contemporary approach*, CMS Books in Mathematics vol. 23, Springer, New York, 2006.
- [11] H. OSAKA AND J. TOMIYAMA, *Double piling structure of matrix monotone functions and of matrix convex functions*, Linear Algebra Appl. **431** (2009) 1825–1832.

(Received February 8, 2018)

Hiroyuki Osaka
Department of Mathematical Sciences
Ritsumeikan University
Kusatsu, Shiga 525-8577, Japan
e-mail: osaka@se.ritsumei.ac.jp

Yukihiro Tsurumi
Department of Mathematical Sciences
Ritsumeikan University
Kusatsu, Shiga 525-8577, Japan
e-mail: tsurumi8269@gmail.com

Shuhei Wada
Department of Information and Computer Engineering
National Institute of Technology, Kisarazu College,
Kisarazu Chiba 292-0041, Japan
e-mail: wada@j.kisarazu.ac.jp