

SCHUR-CONVEXITY OF THE WEIGHTED QUADRATURE FORMULA

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(Communicated by J. Jakšetić)

Abstract. We explore the error of the weighted quadrature formulae and give the sufficient and necessary conditions for this type of quadrature formula to have Schur-convexity property. Some special cases of the weighted quadrature formulae are considered.

1. Introduction

Let us begin by recalling some definitions of convex, n -convex and Schur-convex functions.

DEFINITION 1. A function f is *convex* on an interval I if for any two points $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1)$$

If the inequality (1) is reversed, then f is said to be *concave*.

DEFINITION 2. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be n -convex, $n \geq 0$ on $[a, b]$ if for all choices of $(n + 1)$ distinct points in $[a, b]$,

$$[x_0, \dots, x_n]f \geq 0, \quad (2)$$

where $[x_0, \dots, x_n]f$ denotes the n -th order divided difference of f . If the inequality (2) is reversed, then f is said to be n -concave.

REMARK 1. For $n = 0, 1, 2$ respectively, 0 -convex functions are nonnegative functions, 1 -convex functions are increasing functions and 2 -convex functions are convex functions.

THEOREM 1. If $f^{(n)}$ exists, then f is n -convex if $f^{(n)} \geq 0$.

For this definitions and results see [5].

Mathematics subject classification (2010): 26D15, 65D30, 65D32.

Keywords and phrases: Weighted function, Schur-convexity, two-point quadrature formula, w -harmonic sequences of functions.

DEFINITION 3. Function $F : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *Schur-convex* on A if

$$F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n) \tag{3}$$

for every $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in A$ such that $\mathbf{x} \prec \mathbf{y}$, i.e. such that

$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]} \text{ and } \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \text{ for } k = 1, \dots, n-1, \tag{4}$$

where $x_{[i]}$ denotes the i -th-largest component in \mathbf{x} . Function F is said to be *Schur-concave* on A if $-F$ is Schur-convex.

REMARK 2. Every convex and symmetric function is Schur-convex.

Schur-convexity has been investigated by many researchers. The following result was proved in [3] for arithmetic integral mean.

THEOREM 2. Let f be a continuous function on an interval I with a non-empty interior. Then

$$F(x, y) = \begin{cases} \frac{1}{y-x} \int_x^y f(t) dt, & x, y \in I, x \neq y, \\ f(x), & x = y \in I \end{cases} \tag{5}$$

is *Schur-convex* (*Schur-concave*) on I^2 if and only if f is convex (*concave*) on I .

The next result for the Schur-convexity of the weighted arithmetic integral mean was proved several years later [6].

THEOREM 3. Let f be a continuous function on $I \subset \mathbb{R}$ and let w be a positive continuous weight on I . Then the function

$$F_w(x, y) = \begin{cases} \frac{1}{\int_x^y w(t) dt} \int_x^y w(t) f(t) dt, & x, y \in I, x \neq y, \\ f(x), & x = y \in I \end{cases} \tag{6}$$

is *Schur-convex* (*Schur-concave*) on I^2 if and only if the inequality

$$\frac{\int_x^y w(t) f(t) dt}{\int_x^y w(t) dt} \leq \frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \tag{7}$$

holds (reverses) for all $x, y \in I$.

Authors have left the open problem: Under what conditions do the following inequalities

$$f\left(\frac{xw(x) + yw(y)}{w(x) + w(y)}\right) \leq \frac{\int_x^y w(t) f(t) dt}{\int_x^y w(t) dt} \leq \frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \tag{8}$$

hold for all $x, y \in I$?

The following result ([1]) was the motivation for our paper:

THEOREM 4. *Suppose $f : I \rightarrow \mathbb{R}$ is a continuous function. Function*

$$M(x,y) = \begin{cases} \frac{1}{y-x} \int_x^y f(t)dt - f\left(\frac{x+y}{2}\right), & x,y \in I, x \neq y, \\ 0, & x = y \in I \end{cases} \tag{9}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I . Furthermore, function

$$T(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x} \int_x^y f(t)dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I \end{cases} \tag{10}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I .

In [2], the new proof for the Schur-convexity of functions $M(x,y)$ and $T(x,y)$ is given.

The objective of this work is to give the necessary and sufficient condition for the function $M_w : I^2 \rightarrow \mathbb{R}$ defined as

$$M_w(x,y) = \begin{cases} \frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt - f\left(\frac{x+y}{2}\right), & x,y \in I, x \neq y, \\ 0, & x = y \in I \end{cases} \tag{11}$$

and the function $T_w : I^2 \rightarrow \mathbb{R}$ defined as

$$T_w(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I \end{cases} \tag{12}$$

is Schur-convex (Schur-concave) on I^2 .

Let us recall the weighted onepoint quadrature formula ([4]). If $f : [x,y] \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is a piecewise continuous function, then we have

$$\int_x^y w(t)f(t)dt = \sum_{j=1}^n A_j(z)f^{(j-1)}(z) + (-1)^n \int_x^y W_{n,w}(t,z)f^{(n)}(t)dt, \tag{13}$$

where for $j = 1, \dots, n$

$$A_j(z) = \frac{(-1)^{j-1}}{(j-1)!} \int_x^y (z-s)^{j-1} w(s)ds \tag{14}$$

and

$$W_{n,w}(t,z) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int_x^t (t-s)^{n-1} w(s)ds, & t \in [x,z], \\ w_{2n}(t) = \frac{1}{(n-1)!} \int_y^t (t-s)^{n-1} w(s)ds, & t \in (z,y]. \end{cases} \tag{15}$$

Weighted trapezoidal quadrature formula: If $f : [x,y] \rightarrow \mathbb{R}$ is such that f'' is continuous on $[x,y]$, then there exists $\eta \in (x,y)$ such that

$$\begin{aligned} \int_x^y w(t)f(t)dt &= \frac{f(x)+f(y)}{2} \int_x^y w(t)dt \\ &+ \frac{1}{2} \left(\int_x^y [(y-s)^2 - (y-x)(y-s)] w(s)ds \right) f''(\eta). \end{aligned} \tag{16}$$

Weighted midpoint quadrature formula: If $f : [x, y] \rightarrow \mathbb{R}$ is such that f'' is continuous on $[x, y]$, then there exists $\eta \in (x, y)$ such that

$$\int_x^y w(t)f(t)dt = f\left(\frac{x+y}{2}\right) \int_x^y w(t)dt + \frac{1}{2} \int_x^y \left(\frac{x+y}{2} - s\right)^2 w(s)ds \cdot f''(\eta). \tag{17}$$

In order to prove our result, we shall use the following characterization of Schur-convexity ([5]):

LEMMA 1. *Let $f : I^n \rightarrow \mathbb{R}$ be a continuous symmetric function. If f is differentiable on I^n , then f is Schur-convex on I^n if and only if*

$$(x_i - x_j) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \geq 0 \tag{18}$$

for all $x_i, x_j \in I, x_i \neq x_j, i, j = 1, 2, \dots, n$. Function f is Schur-concave if and only if the reversed inequality sign holds.

2. Main result

THEOREM 5. *If $f \in C^2(I)$, then the function $M_w(x, y)$ defined by (11) is Schur-convex on I^2 if and only if inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \leq \frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \tag{19}$$

holds for all $x, y \in I$.

Proof. Obviously, M_w is symmetric, continuous and differentiable function, so we shall prove this by using Lemma 1. Let $x, y \in I, x < y$. We have

$$(y-x) \left(\frac{\partial M_w}{\partial y} - \frac{\partial M_w}{\partial x} \right) = \frac{(y-x)(w(x) + w(y))}{\left(\int_x^y w(t)dt\right)^2} \cdot \left[\frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \int_x^y w(t)dt - \int_x^y w(t)f(t)dt \right] \geq 0$$

if and only if

$$\frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \geq \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}. \quad \square$$

REMARK 3. For $w(t) = \frac{1}{y-x}, t \in [x, y]$, the condition (19) becomes the right-hand side of the Hermite-Hadamard inequality:

$$\frac{1}{y-x} \int_x^y f(t)dt \leq \frac{f(x) + f(y)}{2}.$$

In [7] it is proved that f is convex iff at least one of the inequalities in Hermite-Hadamard is valid. Therefore, Theorem 5 for weighted version of the function $M_w(x, y)$ is the generalization of the result in [1].

COROLLARY 1. If $M_w(x, y)$ is Schur-convex function on I^2 , then for all $x, y \in I$

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt.$$

Proof. Since $M_w(x, y)$ is Schur-convex, then for $x, y \in I$ we have $(\frac{x+y}{2}, \frac{x+y}{2}) \prec (x, y)$, so by definition of Schur-convexity we have

$$\begin{aligned} M_w\left(\frac{x+y}{2}, \frac{x+y}{2}\right) &\leq M_w(x, y) \\ 0 &\leq \frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt - f\left(\frac{x+y}{2}\right) \\ f\left(\frac{x+y}{2}\right) &\leq \frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt, \end{aligned}$$

which is the left-hand side of the Fejer-Hermite-Hadamard inequality. \square

THEOREM 6. If $f \in C^2(I)$, then the function $T_w(x, y)$ defined by (12) is Schur-convex on I^2 if f is convex and

$$\frac{\int_x^y tw(t)dt}{\int_x^y w(t)dt} = \frac{xw(x) + yw(y)}{w(x) + w(y)} \tag{20}$$

and

$$2\frac{w(x)w(y)(y-x)}{w(x) + w(y)} \leq \int_x^y w(t)dt \tag{21}$$

hold for all $x, y \in I$.

Proof. Obviously, T_w is symmetric, continuous and differentiable function. Let $x, y \in I, x < y$. Suppose f is convex function. We have

$$\begin{aligned} (y-x) \left(\frac{\partial T_w}{\partial y} - \frac{\partial T_w}{\partial x} \right) &= \frac{(y-x)(w(x) + w(y))}{\int_x^y w(t)dt} \\ &\cdot \left[\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \right] \\ &+ \frac{\int_x^y w(t)dt}{w(x) + w(y)} \cdot \left[\frac{f'(y) - f'(x)}{2} \right]. \end{aligned} \tag{22}$$

Since f is convex and according to the inequality (21), we have

$$\left[\frac{\int_x^y w(t)dt}{2} - \frac{w(y) \int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} \right] \cdot [f'(y) - f'(x)] \geq 0. \tag{23}$$

Applying condition (20) to the last inequality we get:

$$\left[\frac{\int_x^y w(t)dt}{2} - \frac{w(y) \int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} \right] \cdot f'(y) \geq \left[\frac{\int_x^y w(t)dt}{2} - \frac{w(x) \int_x^y (t-x)w(t)dt}{\int_x^y w(t)dt} \right] \cdot f'(x),$$

and

$$\frac{\int_x^y w(t)dt}{w(x) + w(y)} \cdot \frac{f'(y) - f'(x)}{2} \geq \frac{w(y)f'(y) \int_x^y (y-t)w(t)dt - w(x)f'(x) \int_x^y (t-x)w(t)dt}{(w(x) + w(y)) \cdot \int_x^y w(t)dt}.$$

On the other hand, if we apply (13) to $z = x$ and multiply by $\frac{w(x)}{w(x)+w(y)}$, and also to $z = y$ and multiply by $\frac{w(y)}{w(x)+w(y)}$, and then add those two identities, we obtain:

$$\begin{aligned} & \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{w(x)f(x) + w(y)f(y)}{w(x) + w(y)} \\ & + \frac{w(y)f'(y) \int_x^y (y-t)w(t)dt - w(x)f'(x) \int_x^y (t-x)w(t)dt}{(w(x) + w(y)) \cdot \int_x^y w(t)dt} \\ & = \frac{\int_x^y [w(x) \cdot \int_t^y (s-t)w(s)ds + w(y) \cdot \int_x^t (t-s)w(s)ds] f''(t)dt}{(w(x) + w(y)) \cdot \int_x^y w(t)dt}, \end{aligned} \tag{24}$$

So, we have proved that the last term in brackets in (22) is greater or equal

$$\frac{w(y) \int_x^y (y-s)w(s)dsf'(y)}{(w(x) + w(y)) \int_x^y w(s)ds} - \frac{w(x) \int_x^y (s-x)w(s)dsf'(x)}{(w(x) + w(y)) \int_x^y w(s)ds},$$

and therefore,

$$\begin{aligned} & (y-x) \left(\frac{\partial T_w}{\partial y} - \frac{\partial T_w}{\partial x} \right) \\ & \geq \frac{(y-x)(w(x) + w(y)) \cdot \int_x^y [w(x) \cdot \int_t^y (s-t)w(s)ds + w(y) \cdot \int_x^t (t-s)w(s)ds] f''(t)dt}{\int_x^y w(t)dt \cdot (w(x) + w(y)) \cdot \int_x^y w(t)dt} \\ & = \frac{(y-x) \cdot \int_x^y [w(x) \cdot \int_t^y (s-t)w(s)ds + w(y) \cdot \int_x^t (t-s)w(s)ds] f''(t)dt}{(\int_x^y w(t)dt)^2}. \end{aligned} \tag{25}$$

Since f is convex and the integrals in the brackets are non negative, we have proved $(y-x) \left(\frac{\partial T_w}{\partial y} - \frac{\partial T_w}{\partial x} \right) \geq 0$, for all $x, y \in I, x < y$, so the function T_w is Schur-convex. \square

REMARK 4. For $w(t) = \frac{1}{y-x}, t \in [x, y]$, we get the result obtained in [1], so Theorem 6 can be considered as generalised version of it.

COROLLARY 2. If $T_w(x, y)$ is Schur-convex function on I^2 , then for all $x, y \in I$

$$\frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt \leq \frac{f(x) + f(y)}{2}.$$

Proof. Since $T_w(x, y)$ is Schur-convex, then for $x, y \in I$ we have $(\frac{x+y}{2}, \frac{x+y}{2}) \prec (x, y)$, so

$$T_w\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \leq T_w(x, y)$$

$$\frac{1}{\int_x^y w(t)dt} \int_x^y w(t)f(t)dt \leq \frac{f(x)+f(y)}{2},$$

which is the right-hand side of the Fejér-Hermite-Hadamard inequality. \square

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(Received November 1, 2018)

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