

## ON HERMITE–HADAMARD TYPE INEQUALITIES FOR HARMONICAL $h$ -CONVEX INTERVAL-VALUED FUNCTIONS

DAFANG ZHAO, TIANQING AN, GUOJU YE AND DELFIM F. M. TORRES\*

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*Abstract.* We introduce and investigate the concept of harmonical  $h$ -convexity for interval-valued functions. Under this new concept, we prove some new Hermite–Hadamard type inequalities for the interval Riemann integral.

### 1. Introduction

The following inequality is known in the literature as the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  and  $a, b \in I$  with  $a < b$ . For various interesting extensions and generalizations of this inequality, see [7, 23, 25]. In 2014, İşcan introduced the concept of harmonical convexity and established some Hermite–Hadamard type inequalities for this class of functions [9]. Some further refinements of such inequalities, for harmonical convex functions, have been studied in [10, 12, 22]. In 2015, Noor et al. introduced the class of harmonical  $h$ -convex functions and established some Hermite–Hadamard type inequalities [21]. For some recent investigations on harmonical  $h$ -convexity, we refer the interested readers to [1, 15, 16].

On the other hand, interval analysis and interval-valued functions were initially introduced in numerical analysis by Moore in the celebrated book [18]. Because of its wide applications in various fields, interval analysis has emerged as a very useful research area over the last fifty years: see, e.g., [4, 5, 19] and references therein. Recently, several classical integral inequalities have been extended not only to the context of interval-valued functions by Chalco-Cano et al. [2, 3], Román-Flores et al. [24], Flores-Franulić et al. [8], Costa and Román-Flores [6], but also to more general set-valued maps by Klaričić Bakula and Nikodem [11], Matkowski and Nikodem [14], Mitroi et al. [17], and Nikodem et al. [20].

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\* Corresponding author.

Our research is mainly motivated by the results of İşcan [9] and Noor et al. [21]. We begin by introducing the notion of harmonical  $h$ -convexity for interval-valued functions. Then we prove some new Hermite–Hadamard type inequalities for the introduced class of functions. Our inequalities are interval-valued counterparts of the results from [9, 21].

The paper is organized as follows. After Section 2 of preliminaries, in Section 3 the harmonical  $h$ -convexity concept for interval-valued functions is given and new Hermite–Hadamard type inequalities are proved. We end with Section 4 of conclusions and future work.

## 2. Preliminaries

We begin by recalling some basic definitions, notation and properties, which are used throughout the paper. A real interval  $[u]$  is the bounded, closed subset of  $\mathbb{R}$  defined by

$$[u] = [\underline{u}, \bar{u}] = \{x \in \mathbb{R} \mid \underline{u} \leq x \leq \bar{u}\},$$

where  $\underline{u}, \bar{u} \in \mathbb{R}$  and  $\underline{u} \leq \bar{u}$ . The numbers  $\underline{u}$  and  $\bar{u}$  are called the left and right endpoints of  $[\underline{u}, \bar{u}]$ , respectively. When  $\underline{u}$  and  $\bar{u}$  are equal, the interval  $[u]$  is said to be degenerated. In this paper, the term interval will mean a nonempty interval. We call  $[u]$  positive if  $\underline{u} > 0$  or negative if  $\bar{u} < 0$ . The inclusion “ $\subseteq$ ” is defined by

$$[\underline{u}, \bar{u}] \subseteq [\underline{v}, \bar{v}] \Leftrightarrow \underline{v} \leq \underline{u}, \bar{u} \leq \bar{v}.$$

For an arbitrary real number  $\lambda$  and  $[u]$ , the interval  $\lambda[u]$  is given by

$$\lambda[\underline{u}, \bar{u}] = \begin{cases} [\lambda\underline{u}, \lambda\bar{u}] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ [\lambda\bar{u}, \lambda\underline{u}] & \text{if } \lambda < 0. \end{cases}$$

For  $[u] = [\underline{u}, \bar{u}]$  and  $[v] = [\underline{v}, \bar{v}]$ , the four arithmetic operators are defined by

$$\begin{aligned} [u] + [v] &= [\underline{u} + \underline{v}, \bar{u} + \bar{v}], \\ [u] - [v] &= [\underline{u} - \bar{v}, \bar{u} - \underline{v}], \\ [u] \cdot [v] &= [\min\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}, \max\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}], \\ [u]/[v] &= [\min\{\underline{u}/\underline{v}, \underline{u}/\bar{v}, \bar{u}/\underline{v}, \bar{u}/\bar{v}\}, \\ &\quad \max\{\underline{u}/\underline{v}, \underline{u}/\bar{v}, \bar{u}/\underline{v}, \bar{u}/\bar{v}\}], \text{ where } 0 \notin [\underline{v}, \bar{v}]. \end{aligned}$$

We denote by  $\mathbb{R}_{\mathcal{I}}$  the set of all intervals of  $\mathbb{R}$ , and by  $\mathbb{R}_{\mathcal{I}}^+$  and  $\mathbb{R}_{\mathcal{I}}^-$  the set of all positive intervals and negative intervals of  $\mathbb{R}$ , respectively. The Hausdorff–Pompeiu distance between intervals  $[\underline{u}, \bar{u}]$  and  $[\underline{v}, \bar{v}]$  is defined by

$$d([\underline{u}, \bar{u}], [\underline{v}, \bar{v}]) = \max\{|\underline{u} - \underline{v}|, |\bar{u} - \bar{v}|\}.$$

It is well known that  $(\mathbb{R}_{\mathcal{I}}, d)$  is a complete metric space.

DEFINITION 1. (See [13]) Let  $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{I}}$  be such that  $f(t) = [\underline{f}(t), \overline{f}(t)]$  for each  $t \in [a, b]$ , and  $\underline{f}, \overline{f}$  are Riemann integrable on  $[a, b]$ . Then we say that  $f$  is Riemann integrable on  $[\underline{a}, \overline{b}]$  and denote

$$\int_a^b f(t)dt = \left[ \int_a^b \underline{f}(t)dt, \int_a^b \overline{f}(t)dt \right].$$

The collection of all interval-valued functions that are  $R$ -integrable on  $[a, b]$  will be denoted by  $\mathcal{I}\mathcal{R}_{([a,b])}$ .

We end this section of preliminaries by recalling some useful known concepts.

DEFINITION 2. (See [9]) We say that  $K_h \subset \mathbb{R} \setminus \{0\}$  is a harmonical convex set if

$$\frac{xy}{tx + (1-t)y} \in K_h$$

for all  $x, y \in K_h$  and  $t \in [0, 1]$ .

DEFINITION 3. (See [21]) Let  $h : [0, 1] \subseteq J \rightarrow \mathbb{R}$  be a non-negative function with  $h \not\equiv 0$ , and  $K_h$  a harmonical convex set. We say that  $f : K_h \rightarrow \mathbb{R}$  is a harmonical  $h$ -convex function if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h(t)f(x) + h(1-t)f(y)$$

for all  $x, y \in K_h$  and  $t \in [0, 1]$ .

Note that if  $h(t) = t$ , then function  $f$  is called a harmonical convex function [9]; if  $h(t) = 1$ , then  $f$  is called a harmonical  $P$ -convex function [21]; while, if  $h(t) = t^s$ , then  $f$  is called a harmonical  $s$ -convex function [21].

### 3. Main results: new Hermite–Hadamard type inequalities

In this section, we prove new Hermite–Hadamard type inequalities for harmonical  $h$ -convex interval-valued functions.

DEFINITION 4. Let  $h : [0, 1] \subseteq J \rightarrow \mathbb{R}$  be a non-negative function such that  $h \not\equiv 0$ , and  $K_h$  a harmonical convex set. We say that  $f : K_h \rightarrow \mathbb{R}_{\mathcal{I}}^+$  is a harmonical  $h$ -convex interval-valued function if

$$h(t)f(x) + h(1-t)f(y) \subseteq f\left(\frac{xy}{tx + (1-t)y}\right) \quad (1)$$

for all  $x, y \in K_h$  and  $t \in [0, 1]$ . If the set inclusion (1) is reversed, then  $f$  is said to be a harmonical  $h$ -concave interval-valued function. The set of all harmonical  $h$ -convex and harmonical  $h$ -concave interval-valued functions are denoted by  $SX(h, K_h, \mathbb{R}_{\mathcal{I}}^+)$  and  $SV(h, K_h, \mathbb{R}_{\mathcal{I}}^+)$ , respectively.

The next theorem is an interval-valued counterpart of [21, Theorem 3.2].

**THEOREM 1.** *Let  $f : K_h \rightarrow \mathbb{R}_{\mathcal{G}}^+$  be an interval-valued function with  $a < b$  and  $a, b \in K_h$ ,  $f \in \mathcal{I}\mathcal{R}_{([a,b])}$ , and let  $h : [0, 1] \rightarrow (0, \infty)$  be a continuous function. If  $f \in SX(h, K_h, \mathbb{R}_{\mathcal{G}}^+)$ , then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq [f(a) + f(b)] \int_0^1 h(t) dt.$$

If  $f \in SV(h, K_h, \mathbb{R}_{\mathcal{G}}^+)$ , then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right) \subseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \subseteq [f(a) + f(b)] \int_0^1 h(t) dt.$$

*Proof.* We first assume that  $f \in SX(h, K_h, \mathbb{R}_{\mathcal{G}}^+)$ . Then one has

$$h\left(\frac{1}{2}\right)f(x) + h\left(\frac{1}{2}\right)f(y) \subseteq f\left(\frac{xy}{\frac{1}{2}x + \frac{1}{2}y}\right) = f\left(\frac{2xy}{x+y}\right).$$

Let

$$x = \frac{ab}{ta + (1-t)b}, \quad y = \frac{ab}{tb + (1-t)a}.$$

Then,

$$h\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta + (1-t)b}\right) + f\left(\frac{ab}{tb + (1-t)a}\right) \right] \subseteq f\left(\frac{2ab}{a+b}\right). \quad (2)$$

Integrating both sides of inequality (2) over  $[0, 1]$ , we have

$$\begin{aligned} \int_0^1 f\left(\frac{2ab}{a+b}\right) dt &= \left[ \int_0^1 \underline{f}\left(\frac{2ab}{a+b}\right) dt, \int_0^1 \bar{f}\left(\frac{2ab}{a+b}\right) dt \right] \\ &= f\left(\frac{2ab}{a+b}\right) \\ &\supseteq h\left(\frac{1}{2}\right) \int_0^1 \left[ f\left(\frac{ab}{ta + (1-t)b}\right) + f\left(\frac{ab}{tb + (1-t)a}\right) \right] dt \\ &= h\left(\frac{1}{2}\right) \left[ \int_0^1 \left( \underline{f}\left(\frac{ab}{ta + (1-t)b}\right) + \underline{f}\left(\frac{ab}{tb + (1-t)a}\right) \right) dt, \right. \\ &\quad \left. \int_0^1 \left( \bar{f}\left(\frac{ab}{ta + (1-t)b}\right) + \bar{f}\left(\frac{ab}{tb + (1-t)a}\right) \right) dt \right] \\ &= h\left(\frac{1}{2}\right) \left[ \frac{2ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx, \frac{2ab}{b-a} \int_a^b \frac{\bar{f}(x)}{x^2} dx \right] \\ &= 2h\left(\frac{1}{2}\right) \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx. \end{aligned}$$

This implies that

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx.$$

The proof of the second relation follows by using (1) with  $x = a$  and  $y = b$  and integrating with respect to  $t$  over  $[0, 1]$ , that is,

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq [f(a) + f(b)] \int_0^1 h(t) dt.$$

The intended result follows. If  $f \in SV(h, K_h, \mathbb{R}_{\mathcal{J}}^+)$ , then the proof is similar and is left to the reader.  $\square$

REMARK 1. If  $h(t) = t^s$ , then Theorem 1 gives a result for harmonical  $s$ -functions:

$$2^{s-1} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq \frac{1}{s+1} [f(a) + f(b)]. \quad (3)$$

If  $h(t) = t$ , then Theorem 1 gives a result for harmonical convex functions:

$$f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq \frac{f(a) + f(b)}{2}. \quad (4)$$

If  $h(t) = 1$ , then Theorem 1 gives a result for harmonical  $P$ -functions:

$$\frac{1}{2} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq f(a) + f(b). \quad (5)$$

THEOREM 2. Let  $f : K_h \rightarrow \mathbb{R}_{\mathcal{J}}^+$  be an interval-valued function with  $a < b$  and  $a, b \in K_h$ ,  $f \in \mathcal{I} \mathcal{R}([a, b])$ , and let  $h : [0, 1] \rightarrow (0, \infty)$  be a continuous function. If  $f \in SX(h, K_h, \mathbb{R}_{\mathcal{J}}^+)$ , then

$$\begin{aligned} \frac{1}{4 \left[ h\left(\frac{1}{2}\right) \right]^2} f\left(\frac{2ab}{a+b}\right) &\supseteq \Delta_1 \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq \Delta_2 \\ &\supseteq [f(a) + f(b)] \left[ \frac{1}{2} + h\left(\frac{1}{2}\right) \right] \int_0^1 h(t) dt, \end{aligned}$$

where

$$\Delta_1 = \frac{1}{4h\left(\frac{1}{2}\right)} \left[ f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right) \right]$$

and

$$\Delta_2 = \left[ \frac{f(a) + f(b)}{2} + f\left(\frac{2ab}{a+b}\right) \right] \int_0^1 h(t) dt.$$

If  $f \in SV(h, K_h, \mathbb{R}_{\mathcal{J}}^+)$ , then the opposite signs of inclusion are valid in the above formulas.

*Proof.* We only give the proof of the first part of Theorem 2. Since  $f \in SX(h, K_h, \mathbb{R}_{\mathcal{G}}^+)$ , we have

$$f\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right) [f(x) + f(y)]$$

for all  $x, y \in K_h$  and  $t = \frac{1}{2}$ . Choosing

$$x = \frac{a \frac{2ab}{a+b}}{ta + (1-t) \frac{2ab}{a+b}}, \quad y = \frac{a \frac{2ab}{a+b}}{t \frac{2ab}{a+b} + (1-t)a},$$

we get

$$f\left(\frac{4ab}{a+3b}\right) \supseteq h\left(\frac{1}{2}\right) \left[ f\left(\frac{a \frac{2ab}{a+b}}{ta + (1-t) \frac{2ab}{a+b}}\right) + f\left(\frac{a \frac{2ab}{a+b}}{t \frac{2ab}{a+b} + (1-t)a}\right) \right].$$

Integrating both sides of the above inequality over  $[0, 1]$ , we have

$$\begin{aligned} f\left(\frac{4ab}{a+3b}\right) &\supseteq h\left(\frac{1}{2}\right) \left[ \int_0^1 \underline{f}\left(\frac{a \frac{2ab}{a+b}}{ta + (1-t) \frac{2ab}{a+b}}\right) + \underline{f}\left(\frac{a \frac{2ab}{a+b}}{t \frac{2ab}{a+b} + (1-t)a}\right) dt, \right. \\ &\quad \left. \int_0^1 \overline{f}\left(\frac{a \frac{2ab}{a+b}}{ta + (1-t) \frac{2ab}{a+b}}\right) + \overline{f}\left(\frac{a \frac{2ab}{a+b}}{t \frac{2ab}{a+b} + (1-t)a}\right) dt \right] \\ &= h\left(\frac{1}{2}\right) \frac{4ab}{b-a} \left[ \int_a^{\frac{2ab}{a+b}} \frac{\underline{f}(x)}{x^2} dx, \int_a^{\frac{2ab}{a+b}} \frac{\overline{f}(x)}{x^2} dx \right] \\ &= h\left(\frac{1}{2}\right) \frac{4ab}{b-a} \int_a^{\frac{2ab}{a+b}} \frac{f(x)}{x^2} dx. \end{aligned}$$

Similarly, we have

$$f\left(\frac{4ab}{3a+b}\right) \supseteq h\left(\frac{1}{2}\right) \frac{4ab}{b-a} \int_{\frac{2ab}{a+b}}^b \frac{f(x)}{x^2} dx.$$

Consequently, we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\supseteq h\left(\frac{1}{2}\right) \left[ f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right) \right] = 4 \left[ h\left(\frac{1}{2}\right) \right]^2 \Delta_1 \\ &\supseteq 4 \left[ h\left(\frac{1}{2}\right) \right]^2 \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx. \end{aligned}$$

Thanks to Theorem 1,

$$\begin{aligned} \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2} f\left(\frac{2ab}{a+b}\right) &\supseteq \Delta_1 \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ &= \frac{1}{2} \left[ \frac{2ab}{b-a} \int_a^{\frac{2ab}{a+b}} \frac{f(x)}{x^2} dx + \frac{2ab}{b-a} \int_{\frac{2ab}{a+b}}^b \frac{f(x)}{x^2} dx \right] \\ &\supseteq \frac{1}{2} \left[ f(a) + f(b) + 2f\left(\frac{2ab}{a+b}\right) \right] \int_0^1 h(t) dt = \Delta_2 \\ &\supseteq \left[ \frac{f(a)+f(b)}{2} + h\left(\frac{1}{2}\right)f(a) + h\left(\frac{1}{2}\right)f(b) \right] \int_0^1 h(t) dt \\ &= [f(a) + f(b)] \left[ \frac{1}{2} + h\left(\frac{1}{2}\right) \right] \int_0^1 h(t) dt \end{aligned}$$

and the result follows.  $\square$

REMARK 2. Like in Remark 1, from Theorem 2 we obtain particular results for harmonical convex, harmonical  $P$ -convex, and harmonical  $s$ -convex functions.

THEOREM 3. Let  $f, g : K_h \rightarrow \mathbb{R}_{\mathcal{F}}^+$  be interval-valued functions with  $a < b$  and  $a, b \in K_h$ ,  $fg \in \mathcal{S}\mathcal{R}_{([a,b])}$ , and  $h_1, h_2 : [0, 1] \rightarrow (0, \infty)$  be continuous functions. If  $f \in SX(h_1, K_h, \mathbb{R}_{\mathcal{F}}^+)$ ,  $g \in SX(h_2, K_h, \mathbb{R}_{\mathcal{F}}^+)$ , then

$$\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \supseteq M(a, b) \int_0^1 h_1(t)h_2(t) dt + N(a, b) \int_0^1 h_1(t)h_2(1-t) dt,$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b), \quad N(a, b) = f(a)g(b) + f(b)g(a).$$

If  $f \in SV(h_1, K_h, \mathbb{R}_{\mathcal{F}}^+)$ ,  $g \in SV(h_2, K_h, \mathbb{R}_{\mathcal{F}}^+)$ , then

$$\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \subseteq M(a, b) \int_0^1 h_1(t)h_2(t) dt + N(a, b) \int_0^1 h_1(t)h_2(1-t) dt.$$

*Proof.* By hypothesis, one has

$$\begin{aligned} f\left(\frac{ab}{ta+(1-t)b}\right) &\supseteq h_1(t)f(a) + h_1(1-t)f(b), \\ g\left(\frac{ab}{ta+(1-t)b}\right) &\supseteq h_2(t)g(a) + h_2(1-t)g(b). \end{aligned}$$

Then,

$$\begin{aligned} &f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\supseteq h_1(t)h_2(t)f(a)g(a) + h_1(t)h_2(1-t)f(a)g(b) \\ &\quad + h_1(1-t)h_2(t)f(b)g(a) + h_1(1-t)h_2(1-t)f(b)g(b). \end{aligned}$$

Integrating both sides of the above inequality over  $[0, 1]$ , we have

$$\begin{aligned}
 & \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)dt \\
 &= \int_0^1 \left[ \underline{f}\left(\frac{ab}{ta+(1-t)b}\right)\underline{g}\left(\frac{ab}{ta+(1-t)b}\right), \overline{f}\left(\frac{ab}{ta+(1-t)b}\right)\overline{g}\left(\frac{ab}{ta+(1-t)b}\right) \right] dt \\
 &= \left[ \int_0^1 \underline{f}\left(\frac{ab}{ta+(1-t)b}\right)\underline{g}\left(\frac{ab}{ta+(1-t)b}\right)dt, \int_0^1 \overline{f}\left(\frac{ab}{ta+(1-t)b}\right)\overline{g}\left(\frac{ab}{ta+(1-t)b}\right)dt \right] \\
 &= \left[ \frac{ab}{b-a} \int_a^b \frac{\underline{f}(x)\underline{g}(x)}{x^2} dx, \frac{ab}{b-a} \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{x^2} dx \right] \\
 &= \frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \\
 &\supseteq f(a)g(a) \int_0^1 h_1(t)h_2(t)dt + f(a)g(b) \int_0^1 h_1(t)h_2(1-t)dt \\
 &\quad + f(b)g(a) \int_0^1 h_1(1-t)h_2(t)dt + f(b)g(b) \int_0^1 h_1(1-t)h_2(1-t)dt \\
 &= M(a, b) \int_0^1 h_1(t)h_2(t)dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)dt.
 \end{aligned}$$

This concludes the proof.  $\square$

REMARK 3. Similarly as before, from Theorem 3 we obtain particular results for harmonical convex, harmonical  $P$ -convex, and harmonical  $s$ -convex functions.

THEOREM 4. Let  $f, g : K_h \rightarrow \mathbb{R}_{\mathcal{F}}^+$  be interval-valued functions with  $a < b$ , where  $a, b \in K_h$  and  $fg \in \mathcal{S}\mathcal{R}_{([a,b])}$ , and  $h_1, h_2 : [0, 1] \rightarrow (0, \infty)$  be continuous functions. If  $f \in SX(h_1, K_h, \mathbb{R}_{\mathcal{F}}^+)$  and  $g \in SX(h_2, K_h, \mathbb{R}_{\mathcal{F}}^+)$ , then

$$\begin{aligned}
 \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) &\supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \\
 &\quad + M(a, b) \int_0^1 h_1(t)h_2(1-t)dt + N(a, b) \int_0^1 h_1(t)h_2(t)dt. \quad (6)
 \end{aligned}$$

If  $f \in SX(h_1, K_h, \mathbb{R}_{\mathcal{F}}^+)$  and  $g \in SX(h_2, K_h, \mathbb{R}_{\mathcal{F}}^+)$ , then previous formula (6) holds with the opposite sign of inclusion.

*Proof.* Let  $\xi = \frac{2ab}{a+b}$ . By hypothesis, one has

$$\begin{aligned}
 f(\xi) &\supseteq h_1\left(\frac{1}{2}\right)f\left(\frac{ab}{ta+(1-t)b}\right) + h_1\left(\frac{1}{2}\right)f\left(\frac{ab}{tb+(1-t)a}\right), \\
 g(\xi) &\supseteq h_2\left(\frac{1}{2}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + h_2\left(\frac{1}{2}\right)g\left(\frac{ab}{tb+(1-t)a}\right).
 \end{aligned}$$

Then,

$$\begin{aligned}
 & f(\xi)g(\xi) \\
 \geq & h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right] \\
 & +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\right] \\
 \geq & h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right] \\
 & +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[(h_1(t)f(a)+h_1(1-t)f(b))(h_2(1-t)g(a)+h_2(t)g(b))\right. \\
 & \left.+ (h_1(1-t)f(a)+h_1(t)f(b))(h_2(t)g(a)+h_2(1-t)g(b))\right] \\
 = & h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)\right] \\
 & +h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[(h_1(t)h_2(1-t)+h_1(1-t)h_2(t))M(a,b)\right. \\
 & \left.+ (h_1(t)h_2(t)+h_1(1-t)h_2(1-t))N(a,b)\right].
 \end{aligned}$$

Integrating over  $[0, 1]$ , we have

$$\begin{aligned}
 \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f(\xi)g(\xi) \geq & \frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}dx \\
 & +M(a,b)\int_0^1h_1(t)h_2(1-t)dt+N(a,b)\int_0^1h_1(t)h_2(t)dt.
 \end{aligned}$$

This concludes the proof.  $\square$

REMARK 4. From Theorem 4, we obtain particular results for harmonical convex, harmonical  $P$ -convex, and harmonical  $s$ -convex functions.

#### 4. Conclusions

We introduced the new concept of harmonical  $h$ -convexity for interval-valued functions. Some interesting Hermite–Hadamard type inequalities for harmonical  $h$ -convex interval-valued functions have then been proved. Our results give interval-valued counterparts of the inequalities presented by İşcan and Noor et al., respectively in [9] and [21].

Further developments are possible. As a future research direction, we intend to investigate Hermite–Hadamard type inequalities for harmonical  $h$ -convex interval-valued functions on arbitrary time scales.

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Dafang Zhao  
College of Science  
Hohai University  
Nanjing, Jiangsu 210098, China  
School of Mathematics and Statistics  
Hubei Normal University  
Huangshi, Hubei 435002, China  
e-mail: dafangzhao@163.com

Tianqing An  
College of Science  
Hohai University  
Nanjing, Jiangsu 210098, China  
e-mail: antq@hhu.edu.cn

Guoju Ye  
College of Science  
Hohai University  
Nanjing, Jiangsu 210098, China  
e-mail: yegj@hhu.edu.cn

Delfim F. M. Torres  
Center for Research and Development in Mathematics and  
Applications (CIDMA)  
Department of Mathematics, University of Aveiro  
3810-193 Aveiro, Portugal  
e-mail: delfim@ua.pt