

WEIGHTED ESTIMATES OF MULTILINEAR FRACTIONAL INTEGRAL OPERATORS FOR RADIAL FUNCTIONS

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Abstract. Moen (2009) proved weighted estimates for multilinear fractional integral operators. We consider weighted estimates of these operators for radial functions and power weights and obtain a better result. Our result is a multilinear variant of the one by De Napoli, Drelichman and Durán (2011). As applications, we get improvements of the bilinear Caffarelli-Kohn-Nirenberg's inequality.

1. Introduction

Consider the fractional integral operator

$$I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Weighted estimates for this operator were proved by Stein and Weiss [10].

THEOREM A. *If* $0 < \alpha < n, 1 < p \leq q < \infty, A < n/p', B < n/q,$

$$A + B \geq 0 \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha - A - B}{n},$$

then

$$\| |x|^{-B} I_{\alpha}f \|_{L^q} \leq C \| |x|^A f \|_{L^p}.$$

De Napoli, Drelichman and Durán [3] proved the following.

THEOREM B. *If* $0 < \alpha < n, 1 < p \leq q < \infty, A < n/p', B < n/q,$

$$A + B \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} \right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha - A - B}{n},$$

then

$$\| |x|^{-B} I_{\alpha}f \|_{L^q} \leq C \| |x|^A f \|_{L^p} \quad \text{for any radial function } f.$$

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It will be very important to notice that when we restrict our attention to functions with radial symmetry, we get a better result. Indeed, such results have many applications to PDE. For example, in [4] the authors proved a radial improvement of Caffarelli-Kohn-Nirenberg’s theorem [1]; see also [2].

In this paper we consider multilinear fractional integrals restricted to radial functions and prove weighted estimates. As applications, we show some bilinear variants of Caffarelli-Kohn-Nirenberg’s inequality ; see Section 5. Our result cannot be obtained by the general theory for multilinear fractional integral operators by Moen [8]; see Theorem C. Our proof is not an analogy of the linear case [3]. The proof in [3] is technical, and not applicable to multilinear case. For simplicity of notation, we consider bilinear fractional integral operators.

DEFINITION 1.

$$I_\alpha(f, g)(x) = \iint_{\mathbb{R}^{2n}} \frac{f(y)g(z)}{(|x - y| + |x - z|)^{2n-\alpha}} dydz, \quad 0 < \alpha < 2n.$$

Our result is the following.

THEOREM 1. *Let*

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad A := A_1 + A_2.$$

Assume that $0 < \alpha < 2n, 1 < p_1, p_2 < \infty, p \leq q < \infty, A_1 < n/p'_1, A_2 < n/p'_2, B < n/q,$

$$A + B \geq (n - 1) \left(\frac{1}{q} - \frac{1}{p} \right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha - A - B}{n}.$$

Then for any radial functions f and $g,$

$$\| |x|^{-B} I_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}}.$$

Throughout this paper we will let C denote a positive constant whose value may change from line to line, but which is independent of essential parameters.

REMARK 1. Note that

$$|x|^{-B} I_\alpha(|f|, |g|)(x) \leq C |x|^{-B_1} I_{\alpha_1}|f|(x) \cdot |x|^{-B_2} I_{\alpha_2}|g|(x),$$

where $\alpha = \alpha_1 + \alpha_2$ and $B = B_1 + B_2$. However our theorem cannot be obtained from Theorem B and Hölder’s inequality.

Moen proved the following.

THEOREM C ([8, Corollary 2.4]). *Let $1/p := 1/p_1 + 1/p_2$ and $A := A_1 + A_2$. Assume that $0 < \alpha < 2n, 1 < p_1, p_2 < \infty, p \leq q < \infty, A_1 < n/p'_1, A_2 < n/p'_2$ and $B < n/q$. If*

$$\sup_{Q:\text{balls}} |Q|^{\alpha/n+1/q-1/p} \left(\frac{1}{|Q|} \int_Q |x|^{-Bq} dx \right)^{1/q} \left(\prod_{i=1}^2 \frac{1}{|Q|} \int_Q |x|^{-A_i p_i} dx \right)^{1/p_i} < \infty,$$

then

$$\| |x|^{-B} I_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}}.$$

As a corollary of Theorem C we can only prove Theorem 1 with $A + B \geq 0$ (cf. [6, p.285]). This is a bilinear variant of Theorem A.

We use the following notation. For any measurable set E , $|E| = \int_E dx$, $w(E) = \int_E w(x) dx$ and χ_E is the characteristic function of E . We also denote $\|f\|_{L^p(w)} = (\int_{\mathbb{R}^n} |f(x)|^p w(x) dx)^{1/p}$.

2. Proof of Theorem 1

We define maximal operators introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González [7] and Moen [8].

DEFINITION 2. Let $\alpha \geq 0$.

$$M_\alpha(f, g)(x) = \sup_{x \in Q} |Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \left(\frac{1}{|Q|} \int_Q |g(y)| dy \right),$$

where the supremum is taken over all balls containing x .

We define some variants of these maximal operators.

DEFINITION 3. For $0 < \alpha < 2n$,

$$m_\alpha(f, g)(x) = \sup_{R > |x|} R^\alpha \left(\frac{1}{R^n} \int_{|y| < R} |f(y)| dy \right) \left(\frac{1}{R^n} \int_{|y| < R} |g(y)| dy \right).$$

Let $0 \leq \alpha < 2$. For radial functions $f(x) = f_0(|x|)$ and $g(x) = g_0(|x|)$,

$$\tilde{m}_\alpha(f, g)(x) = \sup_{0 < R < \frac{|x|}{2}} R^\alpha \left(\frac{1}{R} \int_{|x|-R}^{|x|+R} |f_0(s)| ds \right) \left(\frac{1}{R} \int_{|x|-R}^{|x|+R} |g_0(s)| ds \right).$$

We know the following lemma by Duoandikoetxea [5].

LEMMA 1. *If $R < |x|/2$, then for any radial function $h(x) = h_0(|x|)$,*

$$\frac{C}{R^n} \int_{|x-y| < R} |h(y)| dy \leq \frac{1}{R} \int_{|x|-R}^{|x|+R} |h_0(s)| ds \leq \frac{C'}{R^n} \int_{|x-y| < 2R} |h(y)| dy.$$

By Lemma 1, we have the following lemma.

LEMMA 2. For any radial functions f and g ,

$$\tilde{m}_\alpha(f, g)(x) \leq CM_0(f, g)(x).$$

The next lemma is essentially proved by Duoandikoetxea [5].

LEMMA 3. For any radial functions $f(x) = f_0(|x|)$ and $g(x) = g_0(|x|)$,

$$M_\alpha(f, g)(x) \leq Cm_\alpha(f, g)(x) \quad \text{if } \alpha \geq 2, \tag{1}$$

$$M_\alpha(f, g)(x) \leq C(m_\alpha(f, g)(x) + \tilde{m}_\alpha(f, g)(x)) \quad \text{if } 0 < \alpha < 2. \tag{2}$$

Proof. Let x be fixed and let $Q = \{y : |x - y| < R\}$.

If $\alpha > 0$ and $R \geq |x|/2$,

$$\begin{aligned} & |Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \left(\frac{1}{|Q|} \int_Q |g(y)| dy \right) \\ & \leq CR^\alpha \left(\frac{1}{R^n} \int_{|y| < 3R} |f(y)| dy \right) \left(\frac{1}{R^n} \int_{|y| < 3R} |g(y)| dy \right) \leq Cm_\alpha(f, g)(x). \end{aligned} \tag{3}$$

If $0 < \alpha < 2$ and $R < |x|/2$, we have by Lemma 1,

$$|Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \left(\frac{1}{|Q|} \int_Q |g(y)| dy \right) \leq C\tilde{m}_\alpha(f, g)(x). \tag{4}$$

By (3) and (4), we obtain (2).

If $\alpha \geq 2$ and $R < |x|/2$,

$$\begin{aligned} & R^\alpha \left(\frac{1}{R} \int_{|x-R}^{|x|+R} |f_0(s)| ds \right) \left(\frac{1}{R} \int_{|x-R}^{|x|+R} |g_0(s)| ds \right) \\ & \leq R^{\alpha-2} \int_{|x|/2}^{2|x|} |f_0(s)| ds \cdot \int_{|x|/2}^{2|x|} |g_0(s)| ds \\ & \leq C|x|^{\alpha-2}|x|^{1-n} \int_{|y| \leq 2|x|} |f(y)| dy \cdot |x|^{1-n} \int_{|y| \leq 2|x|} |g(y)| dy \\ & \leq Cm_\alpha(f, g)(x). \end{aligned} \tag{5}$$

By (3) and (5), we obtain (1). \square

The following lemma is essential for our proof.

LEMMA 4. Let $1/p := 1/p_1 + 1/p_2$ and $A := A_1 + A_2$. Assume that $A + B \geq (n - 1)(1/q - 1/p)$, $1/q = 1/p - (\alpha - A - B)/n$ and $p \leq q < \infty$. Then for any radial functions $f(x) = f_0(|x|)$ and $g(x) = g_0(|x|)$,

$$\tilde{m}_\alpha(f, g)(x) \leq C \left\| \cdot \right\|_{L^{p_1}}^{|A_1|} \left\| \cdot \right\|_{L^{p_2}}^{|A_2|} \cdot |x|^{B+Ap/q} \cdot M_0(f, g)(x)^\varepsilon$$

where $\varepsilon = p/q$.

Proof. Let x be fixed and for any $0 < R < |x|/2$, we have

$$\begin{aligned}
 I &:= R^\alpha \left(\frac{1}{R} \int_{|x|-R}^{|x|+R} |f_0(s)| ds \right) \left(\frac{1}{R} \int_{|x|-R}^{|x|+R} |g_0(s)| ds \right) \\
 &\leq CR^{\alpha-2} \cdot R^{1/p'_1+1/p'_2} \\
 &\quad \times \left(\int_{|x|-R}^{|x|+R} |f_0(s)|^{p_1} ds \right)^{1/p_1} \left(\int_{|x|-R}^{|x|+R} |g_0(s)|^{p_2} ds \right)^{1/p_2} \\
 &\leq CR^{\alpha-1/p} |x|^{-A_1-(n-1)/p_1} \left(\int_{|x|-R}^{|x|+R} |f_0(s)s^{A_1}|^{p_1} s^{n-1} ds \right)^{1/p_1} \\
 &\quad \times |x|^{-A_2-(n-1)/p_2} \left(\int_{|x|-R}^{|x|+R} |g_0(s)s^{A_2}|^{p_2} s^{n-1} ds \right)^{1/p_2} \\
 &\leq CR^{\alpha-1/p} |x|^{-A-(n-1)/p} \left\| \cdot \right\|_{L^{p_1}}^{|A_1|} \left\| \cdot \right\|_{L^{p_2}}^{|A_2|}. \tag{6}
 \end{aligned}$$

Also

$$I \leq CR^\alpha M_0(f, g)(x). \tag{7}$$

By (6) and (7), we obtain

$$\begin{aligned}
 I &\leq CR^{(\alpha-1/p)(1-\varepsilon)} |x|^{(-A-(n-1)/p)(1-\varepsilon)} \left\| \cdot \right\|_{L^{p_1}}^{|A_1|} \left\| \cdot \right\|_{L^{p_2}}^{|A_2|} \\
 &\quad \times R^{\alpha\varepsilon} M_0(f, g)(x)^\varepsilon \\
 &\leq C \left\| \cdot \right\|_{L^{p_1}}^{|A_1|} \left\| \cdot \right\|_{L^{p_2}}^{|A_2|} R^{1/q-1/p+\alpha} |x|^{(-A-(n-1)/p)(1-\varepsilon)} M_0(f, g)(x)^\varepsilon.
 \end{aligned}$$

Since $1/q - 1/p = -\alpha/n + (A+B)/n \geq -\alpha/n + (1-1/n)(1/q - 1/p)$, we have $1/q - 1/p + \alpha \geq 0$. Therefore

$$I \leq C \left\| \cdot \right\|_{L^{p_1}}^{|A_1|} \left\| \cdot \right\|_{L^{p_2}}^{|A_2|} |x|^{B+Ap/q} M_0(f, g)(x)^\varepsilon. \quad \square$$

The next two propositions are corollaries of the theorems proved by Lerner et al. [7] and Moen [8].

PROPOSITION 1. *Assume the same conditions as in Theorem 1. Then*

$$\left\| |x|^A M_0(f, g) \right\|_{L^p} \leq C \left\| |x|^{A_1} f \right\|_{L^{p_1}} \left\| |x|^{A_2} g \right\|_{L^{p_2}}.$$

PROPOSITION 2. *If $0 < q < \infty$ and $B < n/q$, then*

$$\left\| |x|^{-B} I_\alpha(f, g) \right\|_{L^q} \leq C \left\| |x|^{-B} M_\alpha(f, g) \right\|_{L^q}.$$

The next proposition is a bilinear version of the theorem by Moen [9].

PROPOSITION 3. *Assume the same conditions as in Theorem 1. Then*

$$\| |x|^{-B} m_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}}.$$

Assuming these propositions temporarily, we prove Theorem 1. We shall prove them in the following sections.

Proof of Theorem 1. By Proposition 2 and Lemma 3, it suffices to show the following:

$$\| |x|^{-B} m_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}} \quad \text{when } \alpha > 0, \tag{8}$$

$$\| |x|^{-B} \tilde{m}_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}} \quad \text{when } 0 < \alpha < 2. \tag{9}$$

Inequality (8) is proved in Proposition 3.

Let $\varepsilon = p/q$. By Lemma 4 and Proposition 1, we have

$$\begin{aligned} \| |x|^{-B} \tilde{m}_\alpha(f, g) \|_{L^q} &\leq C \| |x|^{A_1} f \|_{L^{p_1}}^{1-\varepsilon} \cdot \| |x|^{A_2} g \|_{L^{p_2}}^{1-\varepsilon} \cdot \| |x|^A M_0(f, g) \|_{L^p}^\varepsilon \\ &\leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}}. \quad \square \end{aligned}$$

3. Proofs of Propositions 1 and 2

We recall some important properties for weight functions. Let $1/p = 1/p_1 + 1/p_2$. In this section, w_1 and w_2 are nonnegative functions and $\sigma = w_1^{p/p_1} \cdot w_2^{p/p_2}$. Following Lerner et al. [7], we define some weight classes.

DEFINITION 4. We say that $(w_1, w_2) \in \mathbb{A}_{(p_1, p_2)}$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \sigma(x) dx \right)^{1/p} \prod_{i=1}^2 \left(\frac{1}{|Q|} \int_Q w_i(x)^{1-p'_i} dx \right)^{1/p_i} < \infty,$$

where the supremum is taken over all balls.

PROPOSITION 4. ([7, Theorem 3.6]) *A pair of weights $(w_1, w_2) \in \mathbb{A}_{(p_1, p_2)}$ if and only if*

$$w_1^{1-p'_1} \in \mathbb{A}_{2p'_1}, \quad w_2^{1-p'_2} \in \mathbb{A}_{2p'_2} \quad \text{and} \quad \sigma \in \mathbb{A}_{2p},$$

where \mathbb{A}_s is the Muckenhoupt weight class.

REMARK 2. Power weights $|x|^a \in \mathbb{A}_s$ if and only if $-n < a < n(s-1)$. For elementary properties of the Muckenhoupt weight classes, see for example [6].

THEOREM D ([7, Theorem 3.7]). *If $(w_1, w_2) \in \mathbb{A}_{(p_1, p_2)}$, then*

$$\| M_0(f, g) \|_{L^p(\sigma)} \leq C \| f \|_{L^{p_1}(w_1)} \| g \|_{L^{p_2}(w_2)}.$$

As a corollary of Theorem D we can prove Proposition 1.

Proof of Proposition 1. By Theorem D, it suffices to show that

$$(|x|^{A_1 p_1}, |x|^{A_2 p_2}) \in \mathbb{A}_{(p_1, p_2)}.$$

By Proposition 4 we need to show that

$$|x|^{A_1 p_1(1-p'_1)} \in \mathbb{A}_{2p'_1}, \quad |x|^{A_2 p_2(1-p'_2)} \in \mathbb{A}_{2p'_2} \quad \text{and} \quad |x|^{A p} \in \mathbb{A}_{2p}.$$

By Remark 2, it is sufficient to show the following:

$$-n < A_1 p_1(1-p'_1) < n(2p'_1-1), \quad (10)$$

$$-n < A_2 p_2(1-p'_2) < n(2p'_2-1), \quad (11)$$

$$-n < A p < n(2p-1). \quad (12)$$

These inequalities are easily proved by the assumptions for the indices. We prove (10) and (12).

The inequalities in (10) are equivalent to

$$\frac{n}{p'_1} > A_1 > -n \left(1 + \frac{1}{p_1} \right).$$

Since $A \geq (n-1)(1/q - 1/p) - B > (-n+1)/p - 1/q$, we have

$$\begin{aligned} A_1 &> \frac{-n+1}{p} - \frac{1}{q} - A_2 > -n \left(\frac{1}{p_1} + \frac{1}{p_2} \right) + \frac{1}{p} - \frac{1}{q} - \frac{n}{p'_2} \\ &= -\frac{n}{p_1} - n + \frac{1}{p} - \frac{1}{q} \geq -n \left(1 + \frac{1}{p_1} \right). \end{aligned}$$

We prove (12) as follows.

$$A \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} \right) - B > (n-1) \left(\frac{1}{q} - \frac{1}{p} \right) - \frac{n}{q} = -\frac{n}{p} + \frac{1}{p} - \frac{1}{q} \geq -\frac{n}{p}$$

and

$$A = A_1 + A_2 < \frac{n}{p'_1} + \frac{n}{p'_2} = n \left(2 - \frac{1}{p} \right). \quad \square$$

Moen [8] proved the following theorem.

THEOREM E ([8, Theorem 3.1]). *If a nonnegative function $w(x) \in \bigcup_{s \geq 1} \mathbb{A}_s$, then*

$$\|I_\alpha(f, g)\|_{L^q(w)} \leq C \|M_\alpha(f, g)\|_{L^q(w)}$$

where $0 < q < \infty$.

As a corollary of Theorem E we can prove Proposition 2.

Proof of Proposition 2. By Remark 2, $|x|^{-Bq} \in \bigcup_{s \geq 1} \mathbb{A}_s$ if and only if $B < n/q$. \square

4. Proof of Proposition 3

Following an idea of Moen [9], we define other maximal operators and weight classes. Let \mathcal{B} be the set of all open balls centered at the origin.

DEFINITION 5. Let σ_1 and σ_2 be nonnegative locally integrable functions.

$$M_{(\sigma_1, \sigma_2)}^{\mathcal{B}}(f, g)(x) = \sup_{x \in Q \in \mathcal{B}} \left(\frac{1}{\sigma_1(Q)} \int_Q |f(y)| \sigma_1(y) dy \right) \left(\frac{1}{\sigma_2(Q)} \int_Q |g(y)| \sigma_2(y) dy \right),$$

where the supremum is taken over all balls $Q \in \mathcal{B}$ containing x .

DEFINITION 6. Let $0 \leq \alpha < 2n$.

$$M_{\alpha}^{\mathcal{B}}(f, g)(x) = \sup_{x \in Q \in \mathcal{B}} |Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)| dy \right) \left(\frac{1}{|Q|} \int_Q |g(y)| dy \right).$$

Note that $M_{\alpha}^{\mathcal{B}}(f, g)(x) \approx m_{\alpha}(f, g)(x)$; see Definition 3.

LEMMA 5. Let $1 < p_1, p_2 < \infty, 1/p = 1/p_1 + 1/p_2$ and $\sigma = \sigma_1^{p/p_1} \cdot \sigma_2^{p/p_2}$. Then

$$\|M_{(\sigma_1, \sigma_2)}^{\mathcal{B}}(f, g)\|_{L^p(\sigma)} \leq C \|f\|_{L^{p_1}(\sigma_1)} \|g\|_{L^{p_2}(\sigma_2)}.$$

Proof. Consider the following maximal operators

$$M_{\sigma_i}^{\mathcal{B}}h(x) := \sup_{x \in Q \in \mathcal{B}} \frac{1}{\sigma_i(Q)} \int_Q |h(y)| \sigma_i(y) dy, \quad i = 1, 2.$$

Since $M_{\sigma_i}^{\mathcal{B}}$ is bounded on $L^{p_i}(\sigma_i)$ (cf. [5, p.559]), the lemma is proved by Hölder’s inequality. \square

DEFINITION 7. Let $1 < p_1, p_2 < \infty, 1/p = 1/p_1 + 1/p_2, 0 < q < \infty$ and $0 \leq \alpha < 2n$. For nonnegative functions u, w_1 and w_2 , let $\sigma_1 = w_1^{1-p'_1}, \sigma_2 = w_2^{1-p'_2}$ and $\sigma = \sigma_1^{p/p_1} \cdot \sigma_2^{p/p_2}$. We say $(u, w_1, w_2) \in S_{p_1, p_2, q, \alpha}^{\mathcal{B}}$ if $\sigma_1, \sigma_2 \in L^1_{loc}(\mathbb{R}^n)$ and

$$[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} := \sup_{Q \in \mathcal{B}} \frac{\left(\int_Q [m_{\alpha}(\chi_Q \sigma_1, \chi_Q \sigma_2)(x)]^q u(x) dx \right)^{1/q}}{\sigma(Q)^{1/p}} < \infty.$$

The next lemma is a bilinear variant of Theorem 4.1 in [9].

LEMMA 6. Let $1 < p_1, p_2 < \infty, 1/p = 1/p_1 + 1/p_2, p \leq q < \infty$ and $0 \leq \alpha < 2n$. If $(u, w_1, w_2) \in S_{p_1, p_2, q, \alpha}^{\mathcal{B}}$, then

$$\|m_{\alpha}(f, g)\|_{L^q(u)} \leq C \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)}.$$

Proof. The proof is almost the same as the one in [9] (see also [5, Theorem 3.5]). We show only an outline of the proof. Since $M_\alpha^{\mathcal{B}}(f, g)(x) \approx m_\alpha(f, g)(x)$ we prove the claim for $M_\alpha^{\mathcal{B}}(f, g)$.

Let $\Omega_k = \{x : 2^k < M_\alpha^{\mathcal{B}}(f, g)(x) \leq 2^{k+1}\}$. For any $x \in \Omega_k$, there exists a ball $Q_x \in \mathcal{B}$ such that

$$|Q_x|^{\alpha/n} \frac{1}{|Q_x|} \int_{Q_x} |f(y)| dy \frac{1}{|Q_x|} \int_{Q_x} |g(y)| dy > 2^k.$$

Let $B_k = \bigcup_{x \in \Omega_k} Q_x$. Then $B_k \in \mathcal{B}$ and

$$|B_k|^{\alpha/n} \frac{1}{|B_k|} \int_{B_k} |f(y)| dy \frac{1}{|B_k|} \int_{B_k} |g(y)| dy > 2^k \quad \text{and} \quad \Omega_k \subset B_k.$$

By this property, our proof is easier than the one in [9]. We write

$$\begin{aligned} & \int_{\mathbb{R}^n} M_\alpha^{\mathcal{B}}(f, g)(x)^q u(x) dx \\ & \leq C \sum_k u(\Omega_k) \left(|B_k|^{\alpha/n} \frac{1}{|B_k|} \int_{B_k} |f(y)| dy \frac{1}{|B_k|} \int_{B_k} |g(y)| dy \right)^q \\ & = \int_{\mathbb{Z}} g(k) d\mu(k), \end{aligned}$$

where

$$g(k) = \left(\frac{1}{\sigma_1(B_k)} \int_{B_k} |f(y)| \sigma_1(y)^{-1} \sigma_1(y) dy \frac{1}{\sigma_2(B_k)} \int_{B_k} |g(y)| \sigma_2(y)^{-1} \sigma_2(y) dy \right)^q$$

and μ is a discrete measure on \mathbb{Z} with

$$\mu(k) = u(\Omega_k) \left(|B_k|^{\alpha/n} \frac{\sigma_1(B_k)}{|B_k|} \frac{\sigma_2(B_k)}{|B_k|} \right)^q.$$

Let $\Gamma_\lambda = \{k \in \mathbb{Z} : g(k) > \lambda\}$ and $G_\lambda = \bigcup_{k \in \Gamma_\lambda} B_k$. Note that $G_\lambda \in \mathcal{B}$. Since

$$\begin{aligned} \mu(\Gamma_\lambda) &= \sum_{k \in \Gamma_\lambda} u(\Omega_k) \left(|B_k|^{\alpha/n} \frac{\sigma_1(B_k)}{|B_k|} \frac{\sigma_2(B_k)}{|B_k|} \right)^q \\ &\leq \int_{G_\lambda} M_\alpha^{\mathcal{B}}(\sigma_1 \chi_{G_\lambda}, \sigma_2 \chi_{G_\lambda})(x)^q u(x) dx \\ &\leq [u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \sigma(G_\lambda)^{q/p} \\ &\leq C [u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \sigma(\{x : M_{\sigma_1, \sigma_2}^{\mathcal{B}}(f/\sigma_1, g/\sigma_2)(x)^q > \lambda\})^{q/p}, \end{aligned}$$

we have

$$\begin{aligned}
 & \int_{\mathbb{Z}} g \, d\mu = \int_0^\infty \mu(\Gamma_\lambda) \, d\lambda \\
 & \leq C[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \int_0^\infty \sigma(\{x : M_{\sigma_1, \sigma_2}^{\mathcal{B}}(f/\sigma_1, g/\sigma_2)(x)^q > \lambda\})^{q/p} \, d\lambda \\
 & = C[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \int_0^\infty (t\sigma(\{x : M_{\sigma_1, \sigma_2}^{\mathcal{B}}(f/\sigma_1, g/\sigma_2)(x)^p > t\}))^{q/p} \frac{dt}{t} \\
 & \leq C[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \left(\int_{\mathbb{R}^n} M_{\sigma_1, \sigma_2}^{\mathcal{B}}(f/\sigma_1, g/\sigma_2)(x)^p \sigma(x) \, dx \right)^{q/p} \\
 & \leq C[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \|f/\sigma_1\|_{L^{p_1}(\sigma_1)}^q \|g/\sigma_2\|_{L^{p_2}(\sigma_2)}^q \\
 & = C[u, w_1, w_2]_{S_{p_1, p_2, q, \alpha}^{\mathcal{B}}} \|f\|_{L^{p_1}(w_1)}^q \|g\|_{L^{p_2}(w_2)}^q
 \end{aligned}$$

by Lemma 5. \square

Proof of Proposition 3.

By Lemma 6, it suffices to show that $(|x|^{-Bq}, |x|^{A_1 p_1}, |x|^{A_2 p_2}) \in S_{p_1, p_2, q, \alpha}^{\mathcal{B}}$. Let $\sigma_1 = |x|^{A_1 p_1(1-p'_1)}$, $\sigma_2 = |x|^{A_2 p_2(1-p'_2)}$, $\sigma = \sigma_1^{p/p_1} \cdot \sigma_2^{p/p_2}$, $u = |x|^{-Bq}$ and $Q = \{y : |y| < R\}$. Since $(A_1(1-p'_1) + A_2(1-p'_2))p > -n$, we have

$$\sigma(Q)^{1/p} = CR^{A_1(1-p'_1) + A_2(1-p'_2) + n/p}. \tag{13}$$

Since $A_1 p_1(1-p'_1) > -n$ and $A_2 p_2(1-p'_2) > -n$, we have for $x \in Q$,

$$\begin{aligned}
 & m_\alpha(\chi_Q \sigma_1, \chi_Q \sigma_2)(x) = \\
 & \max \left\{ \sup_{|x| \leq S \leq R} S^\alpha \left(\frac{1}{S^n} \int_{|y| < S} |y|^{A_1 p_1(1-p'_1)} \, dy \right) \left(\frac{1}{S^n} \int_{|y| < S} |y|^{A_2 p_2(1-p'_2)} \, dy \right), \right. \\
 & \quad \left. \sup_{S > R} S^\alpha \left(\frac{1}{S^n} \int_{|y| < R} |y|^{A_1 p_1(1-p'_1)} \, dy \right) \left(\frac{1}{S^n} \int_{|y| < R} |y|^{A_2 p_2(1-p'_2)} \, dy \right) \right\} \\
 & \leq C \max(|x|^{\alpha + A_1 p_1(1-p'_1) + A_2 p_2(1-p'_2)}, R^{\alpha + A_1 p_1(1-p'_1) + A_2 p_2(1-p'_2)}).
 \end{aligned}$$

Since $(\alpha + A_1 p_1(1-p'_1) + A_2 p_2(1-p'_2) - B)q > -n$ and $-Bq > -n$, we have

$$\left(\int_Q [m_\alpha(\chi_Q \sigma_1, \chi_Q \sigma_2)(x)]^q u(x) \, dx \right)^{1/q} \leq CR^{\alpha + A_1 p_1(1-p'_1) + A_2 p_2(1-p'_2) + B + n/q}. \tag{14}$$

By (13) and (14), we obtain the desired result. \square

5. Applications

We show some bilinear variants of Caffarelli-Kohn-Nirenberg’s theorem. Our results improve the ones in [8, Section 7]. First we rewrite Theorem 1 as follows. Let $1/p = 1/p_1 + 1/p_2$ and $A = A_1 + A_2$.

THEOREM 2. Assume that $1 < p_1, p_2 < \infty, p \leq q < \infty, A_1 < n/p'_1, A_2 < n/p'_2, 0 < \alpha < 2n$,

$$\alpha - \frac{n}{p} < A \quad \text{and} \quad \frac{1}{p} - \alpha \leq \frac{1}{q}.$$

Then for any radial functions f and g ,

$$\| |x|^B I_\alpha(f, g) \|_{L^q} \leq C \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}},$$

where $B = A + n(1/p - 1/q) - \alpha$.

By using this theorem, we obtain the next theorems.

THEOREM 3. Assume that $1 < p_2, p_2 < \infty, p \leq q < \infty, A_1 < n/p'_1, A_2 < n/p'_2$,

$$1 - \frac{n}{p} < A \quad \text{and} \quad \frac{1}{p} - 1 \leq \frac{1}{q}.$$

Then for any radial functions $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} & \| |x|^B f \cdot g \|_{L^q} \\ & \leq C \left(\| |x|^{A_1} |\nabla f| \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}} + \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} |\nabla g| \|_{L^{p_2}} \right), \end{aligned}$$

where $B = A + n(1/p - 1/q) - 1$.

Proof. Let ∇_{2n} be the gradient in \mathbb{R}^{2n} . By the argument in Stein [11, p.125],

$$\begin{aligned} |f(x)g(x)| & \leq C \iint_{\mathbb{R}^{2n}} \frac{|\nabla_{2n}(f(x-y)g(x-z))|}{(|y|+|z|)^{2n-1}} dydz \\ & \leq C(I_1(|\nabla f|, |g|)(x) + I_1(|f|, |\nabla g|)(x)). \quad \square \end{aligned}$$

THEOREM 4. Assume that $n \geq 2, 1 < p_2, p_2 < \infty, p \leq q < \infty, A_1 < n/p'_1, A_2 < n/p'_2$ and

$$2 - \frac{n}{p} < A.$$

Then for any radial functions $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$\| |x|^B f \cdot g \|_{L^q} \leq C \| |x|^{A_1} |\Delta f| \|_{L^{p_1}} \| |x|^{A_2} g \|_{L^{p_2}} + \| |x|^{A_1} f \|_{L^{p_1}} \| |x|^{A_2} |\Delta g| \|_{L^{p_2}},$$

where $B = A + n(1/p - 1/q) - 2$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Proof. Let Δ_{2n} be the Laplacian in \mathbb{R}^{2n} . Since

$$f(x)g(x) = C \iint_{\mathbb{R}^{2n}} \frac{\Delta_{2n}(f(y)g(z))}{(|x-y|^2 + |x-z|^2)^{(2n-2)/2}} dydz,$$

we have

$$|f(x)g(x)| \leq C(I_2(|\Delta f|, |g|)(x) + I_2(|f|, |\Delta g|)(x)). \quad \square$$

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