

A NEW IMPROVED FORM OF THE HILBERT INEQUALITY AND ITS APPLICATIONS

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(Communicated by M. Praljak)

Abstract. In this paper, it shows a new improved form of the Hilbert inequality by introducing a proper weight function $\Omega(\lambda, x)$ with a parameter $\lambda (\lambda > \frac{1}{2})$. As applications, a new refinement of Widder's inequality and an extension of Hardy-Littlewood's inequality are given.

1. Introduction

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, then we have the following Hilbert's integral inequality (see [14]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \quad (1.1)$$

where the constant factor π is the best possible. In 1925, by introducing one pair of conjugate exponents (p, q) , Hardy [3] gave an extension of (1.1) as follows:

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.2)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.2) are important in analysis and its applications (see [3, 13]).

In 1934, Hardy gave an extension of (1.2) as follows:

If $k_1(x, y)$ is a non-negative homogeneous function of degree -1 ,

$$k_p = \int_0^\infty k_1(u, 1)u^{-\frac{1}{p}} du \in R_+ = (0, +\infty),$$

Mathematics subject classification (2010): 26D15.

Keywords and phrases: Hilbert inequality, parameter, Widder inequality, Hardy-Littlewood inequality.

This research is supported by the Foundation of Hunan Educational Department (No.18C0592).

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then

$$\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)dx dy < k_p \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor k_p is the best possible (see [3, Theorem 319]). Additionally, a Hilbert-type integral inequality with the non-homogeneous kernel is provided (see [3, Theorem 350]) as follows:

if $h(u) > 0$, $\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in R_+$, then

$$\int_0^\infty \int_0^\infty h(x,y)f(x)g(y)dx dy < \phi\left(\frac{1}{p}\right) \left(\int_0^\infty x^{p-2}f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $\phi\left(\frac{1}{p}\right)$ is still the best possible.

By introducing an independent parameter $\lambda \in (0, \infty)$ and the beta function, in 1998, Yang [1] gave an extension of (1.1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty y^{1-\lambda} g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.5}$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible, and

$$B(u, v) := \int_0^\infty \frac{t^{v-1}}{(1+t)^{u+v}} dt, \quad (u, v > 0)$$

is the Beta function.

In 1999, by introducing matrix method, Gao [10] gave another extension of (1.1) as follows:

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, the inner product (f, g) and the norm $\|f\|$ of f are defined $(f, g) = \int_0^\infty f(t)g(t)dt$ and $\|f\| = \left(\int_0^\infty f^2(t)dt \right)^{\frac{1}{2}}$, respectively. Then

$$\left(\int_0^\infty \int_0^\infty \frac{f(s)g(t)}{s+t} ds dt \right)^2 < \pi^2(1-A)\|f\|^2\|g\|^2, \tag{1.6}$$

where $A = \frac{1}{\pi} \left(\frac{x}{\|g\|} - \frac{y}{\|f\|} \right)^2$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(g, e)$ and $y = (2\pi)^{\frac{1}{2}}(f, e^{-s})$, e is exponential integral with parameter.

In recent years, there has been increasing interest in extending of (1.1) and (1.2); see, for example, [2, 4, 5, 7, 8, 9, 11, 12, 15] and the references cited therein.

In this paper, by using a new method with a parameter $\lambda (\lambda > \frac{1}{2})$, applying the weight functions and the technique of real analysis, we establish a improvement of the Hilbert inequality. As application, we give a new refinement of Widder’s inequality and an extension of Hardy-Littlewood’s inequality to illustrate the main results.

2. Some Lemmas

In what follows, we use the following notation for the convenience of the reader.

$$H(\lambda, x, y) = \frac{f(x)f(y)}{x^\lambda + y^\lambda}, \tag{1}$$

$$F(x, y) = 1 - c(x) + c(y), \tag{2}$$

and

$$J_1 = \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^\lambda + y^\lambda} \left(\frac{x}{y}\right)^{\frac{1}{2}} F(x, y) dx dy, J_2 = \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^\lambda + y^\lambda} \left(\frac{y}{x}\right)^{\frac{1}{2}} F(x, y) dx dy, \tag{3}$$

where $\lambda > \frac{1}{2}$, $f : [0, \infty) \rightarrow R^+$ is a real function and $c : [0, +\infty) \rightarrow R$ is a non-negative function.

In order to prove the main results, we need the following four lemmas.

LEMMA 2.1. *Let $H(\lambda, x, y)$ and $F(x, y)$ be defined as (1) and (2), respectively. Then*

$$\int_0^\infty \int_0^\infty H(\lambda, x, y) dx dy = \int_0^\infty \int_0^\infty H(\lambda, x, y) F(x, y) dx dy.$$

Proof. It is obvious that

$$\int_0^\infty \int_0^\infty H(\lambda, x, y) dx dy = \int_0^\infty \int_0^\infty H(\lambda, x, y) F(x, y) dx dy.$$

Hence, Lemma 2.1 holds. \square

LEMMA 2.2. *Let $\lambda > \frac{1}{2}$, then*

$$\int_0^\infty \frac{t^{m-1}}{1+t^\lambda} dt = \frac{\pi}{\lambda \sin \frac{m\pi}{\lambda}}. \tag{2.1}$$

Proof. By the integral formula(see, [6, page 591])

$$\int_0^\infty \frac{t^{m-1}}{(1+bt^a)^{m+n}} dt = a^{-1} b^{-\frac{m}{a}} B\left(\frac{m}{a}, m+n-\frac{m}{a}\right),$$

where $a, b > 0$, m and n are real numbers, $B(p, q)$ is Beta function. We have equality(2.1) immediately. \square

LEMMA 2.3. *Let J_1, J_2 be defined as (3). Then*

$$J_1 J_2 = \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}}\right)^2 \left(\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty \Omega(\lambda, x) f^2(x) dx\right)^2, \tag{2.2}$$

where the weight function $\Omega(\lambda, x)$ is defined by

$$\Omega(\lambda, x) = x^{1-\lambda} \left(\frac{\pi c(x)}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}} du \right). \tag{2.3}$$

Proof. Let $u = \frac{y}{x}$. Then

$$\begin{aligned} J_1 &= \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^\lambda + y^\lambda} \left(\frac{x}{y}\right)^{\frac{1}{2}} F(x, y) dx dy \\ &= \int_0^\infty \left(x^{1-\lambda} \int_0^\infty \frac{1}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}} (1 - c(x) + c(xu)) du \right) f^2(x) dx. \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} J_1 &= \int_0^\infty \left(x^{1-\lambda} (1 - c(x)) \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} + x^{1-\lambda} \int_0^\infty \frac{c(xu)}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}} du \right) f^2(x) dx \\ &= \int_0^\infty \left(\frac{x^{1-\lambda} \pi}{\lambda \sin \frac{\pi}{2\lambda}} - x^{1-\lambda} \left(\frac{c(x)\pi}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}} du \right) \right) f^2(x) dx \\ &= \int_0^\infty \frac{x^{1-\lambda} \pi}{\lambda \sin \frac{\pi}{2\lambda}} f^2(x) dx - \int_0^\infty x^{1-\lambda} \left(\frac{\pi c(x)}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{1}{2}} du \right) f^2(x) dx \\ &= \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx - \int_0^\infty \Omega(\lambda, x) f^2(x) dx, \end{aligned} \tag{2.4}$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Similarly, we have

$$\begin{aligned} J_2 &= \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^\lambda + y^\lambda} \left(\frac{y}{x}\right)^{\frac{1}{2}} F(x, y) dx dy \\ &= \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx + \int_0^\infty \Omega(\lambda, x) f^2(x) dx. \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that (2.2) holds. \square

REMARK 2.1. let $\lambda = 1$, we get from (2.2)

$$J_1 J_2 = \pi^2 \left(\int_0^\infty f^2(x) dx \right)^2 - \left(\int_0^\infty \tilde{\Omega}(x) f^2(x) dx \right)^2, \tag{2.6}$$

where

$$\tilde{\Omega}(x) = \pi c(x) - \int_0^\infty \frac{c(xu)}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du.$$

Let $u = t^2$, we have

$$\tilde{\Omega}(x) = \pi c(x) - 2 \int_0^\infty \frac{c(xt^2)}{1+t^2} dt. \tag{2.7}$$

LEMMA 2.4. Let $c(x) = \frac{1}{1+x}$ ($x \geq 0$), then (2.7) becomes

$$\tilde{\Omega}(x) = \frac{\pi}{1+x} - \frac{\pi}{1+\sqrt{x}}. \tag{2.8}$$

Proof. By the integral formula(see, [16, page 158, formula 97])

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)},$$

it is ease to deduce that

$$\int_0^\infty \frac{c(xt^2)}{1+t^2} dt = \int_0^\infty \frac{dt}{(1+t^2)(1+xt^2)} = \frac{\pi}{1+\sqrt{x}}. \tag{2.9}$$

Substituting (2.9) into (2.7), we obtain (2.8). \square

3. Main results

We may now state and prove our main results.

THEOREM 3.1. Let $H(\lambda, x, y)$ and $F(x, y)$ be defined as (1) and (2), respectively. Then

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda} dx dy \right)^2 < \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \right)^2 - \left(\int_0^\infty \Omega(\lambda, x) f^2(x) dx \right)^2, \tag{3.1}$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Proof. By Schwarz’s inequality and Lemma 2.1, we have

$$\begin{aligned} \left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda} dx dy \right)^2 &= \left(\int_0^\infty \int_0^\infty H(\lambda, x, y) dx dy \right)^2 \\ &= \left(\int_0^\infty \int_0^\infty H(\lambda, x, y) F(x, y) \right)^2 \\ &= \left\{ \int_0^\infty \int_0^\infty \left(\frac{f(x)}{(x^\lambda + y^\lambda)^{\frac{1}{2}}} \left(\frac{x}{y} \right)^{\frac{1}{4}} (F(x, y))^{\frac{1}{2}} \right) \left(\frac{f(y)}{(x^\lambda + y^\lambda)^{\frac{1}{2}}} \left(\frac{y}{x} \right)^{\frac{1}{4}} (F(x, y))^{\frac{1}{2}} \right) dx dy \right\}^2 \\ &\leq \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^\lambda + y^\lambda} \left(\frac{x}{y} \right)^{\frac{1}{2}} F(x, y) dx dy \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^\lambda + y^\lambda} \left(\frac{y}{x} \right)^{\frac{1}{2}} F(x, y) dx dy. \end{aligned} \tag{3.2}$$

Since $f(x) \neq 0$, it is impossible to take equality in (3.2), we obtain

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda} dx dy \right)^2 < J_1 J_2.$$

By Lemma 2.3, we get (3.1). This completes the proof. \square

THEOREM 3.2. *Let $F(x,y)$ be defined as (2) and let $f, g : [0, +\infty) \rightarrow R^+$ be real function. Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy\right)^4 < \left\{ \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}}\right)^2 \left(\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty \Omega(\lambda, x) f^2(x) dx\right)^2 \right\} \times \left\{ \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}}\right)^2 \left(\int_0^\infty x^{1-\lambda} g^2(x) dx\right)^2 - \left(\int_0^\infty \Omega(\lambda, x) g^2(x) dx\right)^2 \right\}, \tag{3.3}$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Proof. By Schwarz’s inequality, we have

$$\begin{aligned} \left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy\right)^4 &= \left\{ \left[\int_0^1 \left(\int_0^\infty t^{x^\lambda - \frac{1}{2}} f(x) dx\right) \left(\int_0^\infty t^{y^\lambda - \frac{1}{2}} g(y) dy\right) dt \right]^2 \right\}^2 \\ &\leq \left\{ \int_0^1 \left(\int_0^\infty t^{x^\lambda - \frac{1}{2}} f(x) dx\right)^2 dt \right\}^2 \left\{ \int_0^1 \left(\int_0^\infty t^{y^\lambda - \frac{1}{2}} g(y) dy\right)^2 dt \right\}^2 \\ &= \left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda} dx dy\right)^2 \left(\int_0^\infty \int_0^\infty \frac{g(x)g(y)}{x^\lambda + y^\lambda} dx dy\right)^2. \end{aligned} \tag{3.4}$$

Similarly to the proof of Theorem 3.1, we obtain

$$\left(\int_0^\infty \int_0^\infty \frac{g(x)g(y)}{x^\lambda + y^\lambda} dx dy\right)^2 \leq \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}}\right)^2 \left(\int_0^\infty x^{1-\lambda} g^2(x) dx\right)^2 - \left(\int_0^\infty \Omega(\lambda, x) g^2(x) dx\right)^2. \tag{3.5}$$

Since $g(x) \neq 0$, it is impossible to take equality in (3.5).

Substituting (3.1) and (3.5) into (3.4), we get (3.3). This completes the proof. \square

In particular, let $\lambda = 1$, we have a new improvement of the Hilbert inequality as following corollary.

COROLLARY 3.1. *Let $F(x,y)$ be defined as (2) and let $f, g : [0, +\infty) \rightarrow R^+$ be real function. Then*

$$\begin{aligned} \left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}\right)^4 &< \left\{ \pi^2 \left(\int_0^\infty f^2(x) dx\right)^2 - \left(\int_0^\infty \tilde{\Omega}(x) f^2(x) dx\right)^2 \right\} \times \\ &\times \left\{ \pi^2 \left(\int_0^\infty g^2(x) dx\right)^2 - \left(\int_0^\infty \tilde{\Omega}(x) g^2(x) dx\right)^2 \right\}, \end{aligned} \tag{3.6}$$

where the weight function $\tilde{\Omega}(x)$ is defined by (2.7).

If we choose $c(x) = \frac{1}{1+x}$ ($x \geq 0$), by (2.8), then (3.6) becomes

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} \right)^4 < \pi^4 \left\{ \left(\int_0^\infty f^2(x)dx \right)^2 - \left(\int_0^\infty \widehat{\Omega}(x)f^2(x)dx \right)^2 \right\} \times \\ \times \left\{ \left(\int_0^\infty g^2(x)dx \right)^2 - \left(\int_0^\infty \widehat{\Omega}(x)g^2(x)dx \right)^2 \right\},$$

where $\widehat{\Omega}(x) = \frac{1}{1+x} - \frac{1}{1+\sqrt{x}}$.

REMARK 3.1. The non-negative function $c(x)$ is chosen for maximum flexibility, because it only satisfies condition: $F(x, y) = 1 - c(x) + c(y) \geq 0$. Therefore, if we choose $c(x) = \frac{1}{2} \cos \sqrt{x}$, then by the integral formula(see, [16, page 189, formula 534])

$$\int_0^\infty \frac{\cos ax}{b^2 + x^2} dt = \frac{\pi}{2b} e^{-ab}, \quad (a \geq 0, Re b > 0).$$

It is easy to calculate that

$$2 \int_0^\infty \frac{c(xt^2)}{1+t^2} dt = \int_0^\infty \frac{\cos \sqrt{xt}}{1+t^2} dt = \frac{\pi}{2} e^{-\sqrt{x}},$$

hence, we get

$$\widehat{\Omega}(x) = \frac{1}{2} (\cos \sqrt{x} - e^{-\sqrt{x}}).$$

COROLLARY 3.2. With the same assumptions as Corollary 3.1, let $\lambda = 2$, then

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^2 + y^2} dx dy \right)^4 < \left\{ \frac{\pi^2}{2} \left(\int_0^\infty \frac{1}{x} f^2(x) dx \right)^2 - \left(\int_0^\infty \bar{\Omega}(2, x) f^2(x) dx \right)^2 \right\} \times \\ \times \left\{ \frac{\pi^2}{2} \left(\int_0^\infty \frac{1}{x} g^2(x) dx \right)^2 - \left(\int_0^\infty \bar{\Omega}(2, x) g^2(x) dx \right)^2 \right\},$$

where the weight function $\bar{\Omega}(2, x)$ is defined by

$$\bar{\Omega}(2, x) = \frac{1}{x} \left(\frac{\pi c(x)}{\sqrt{2}} - \int_0^\infty \frac{c(xu)}{1+u^2} \left(\frac{1}{u} \right)^{\frac{1}{2}} du \right).$$

4. Applications

In this section, we give a new refinement of Widder’s inequality and an extension of Hardy-Littlewood’s inequality as follows.

4.1. A new refinement of Widder’s inequality

The following inequality is Widder’s inequality (see [15]):

Let $a_n \geq 0 (n = 0, 1, 2, \dots)$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$. If $A(x) \neq 0$, then

$$\int_0^1 A^2(x) dx < \pi \int_0^{\infty} \left(e^{-x} A^*(x) \right)^2 dx \tag{4.1}$$

We give a new refinement of (4.1) as follows:

THEOREM 4.1. *Let $a_n \geq 0 (n = 0, 1, 2, \dots)$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $A(x) \neq 0$ and $A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$. Let $F(x, y)$ be defined as (3). then*

$$\left(\int_0^1 A^2(x) dx \right)^2 < \pi^2 \left(\int_0^{\infty} \left(e^{-x} A^*(x) \right)^2 dx \right)^2 - \left(\int_0^{\infty} \tilde{\Omega}(x) \left(e^{-x} A^*(x) \right)^2 dx \right)^2 \tag{4.2}$$

where the weight function $\tilde{\Omega}(x)$ is defined by (2.7).

Proof. First, we have following relation:

$$\begin{aligned} \int_0^{\infty} e^{-t} A^*(tx) dt &= \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n (xt)^n}{n!} dt = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^{\infty} t^n e^{-t} dt \\ &= \sum_{n=0}^{\infty} a_n x^n = A(x). \end{aligned}$$

Squaring and integrating both sides of aforementioned equality from 0 to 1, we obtain

$$\int_0^1 A^2(x) dx = \int_0^1 \left(\int_0^{\infty} e^{-t} A^*(tx) dt \right)^2 dx.$$

Let $tx = s$, from right side of aforementioned equality, we have

$$\begin{aligned} \int_0^1 A^2(x) dx &= \int_0^1 \left(\int_0^{\infty} e^{-t} A^*(tx) dt \right)^2 dx \\ &= \int_0^1 \left(\int_0^{\infty} e^{-\frac{s}{x}} A^*(s) ds \right)^2 \frac{1}{x^2} dx. \end{aligned}$$

Let $y = \frac{1}{x}$, we get

$$\int_0^1 A^2(x) dx = \int_1^{\infty} \left(\int_0^{\infty} e^{-sy} A^*(s) ds \right)^2 dy.$$

Let $u = y - 1$, then

$$\int_0^1 A^2(x) dx = \int_0^{\infty} \left(\int_0^{\infty} e^{-su-s} A^*(s) ds \right)^2 du$$

$$\begin{aligned}
 &= \int_0^\infty \left(\int_0^\infty e^{-su} f(s) ds \right)^2 du \\
 &= \int_0^\infty \int_0^\infty \frac{f(s)f(t)}{s+t} ds dt,
 \end{aligned} \tag{4.3}$$

where $f(x) = e^{-x}A^*(x)$.

Thus, by Theorem (3.1) (choose $\lambda = 1$), we get (4.2) from (4.3). This completes the proof. \square

4.2. An extension of Hardy-Littlewood’s inequality

Hardy-Littlewood proved the following inequality(see [3]):

Let $f(x), x \in [0, 1)$, be a non-negative real function, $a_n = \int_0^1 x^n f(x) dx, n = 0, 1, 2, \dots$

Then

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x) dx, \tag{4.4}$$

where π is the best constant.

In 1997, Gao [9] extended the inequality (4.4) and established the following inequality, named the Hardy-Littlewood integral inequality.

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 h^2(x) dx, \tag{4.5}$$

where $f(x) = \int_0^1 t^x h(t) dt, x \in [0, +\infty)$.

In 1999, Gao [11] refined (4.5) as following inequality:

$$\int_0^\infty f^2(x) dx < \pi \int_0^1 t h^2(t) dt. \tag{4.6}$$

In this paper, we will further extend the inequality (4.6). For notational simplicity, define

$$f(x) = \int_0^1 t^{x\lambda} |h(t)| dt, (x \geq 0, \lambda > \frac{1}{2}) \tag{4.7}$$

where $h(t) \neq 0, (t \in [0, 1))$, is a real function.

We are in position to state and show the following theorem.

THEOREM 4.2. *Suppose that $f(x)$ be defined as (4.7), and $F(x, y)$ be defined as (2). Then*

$$\begin{aligned}
 \left(\int_0^\infty f^2(x) dx \right)^4 &< \left\{ \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx \right)^2 - \left(\int_0^\infty \Omega(\lambda, x) f^2(x) dx \right)^2 \right\} \times \\
 &\times \left(\int_0^1 t h^2(t) dt \right)^2,
 \end{aligned} \tag{4.8}$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Proof. In view of definition of $f(x)$, we rewrite $f^2(x)$ as

$$f^2(x) = \int_0^1 f(x)t^{x^\lambda} |h(t)| dt.$$

By Schwarz's inequality and Theorem 3.1, we have

$$\begin{aligned} \left(\int_0^\infty f^2(x) dx \right)^2 &= \left\{ \int_0^\infty \left(\int_0^1 f(x)t^{x^\lambda} |h(t)| dt \right) dx \right\}^2 \\ &= \left\{ \int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}} dx \right) t^{\frac{1}{2}} |h(t)| dt \right\}^2 \\ &\leq \int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}} dx \right)^2 dt \int_0^1 t h^2(t) dt \\ &= \int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}} dx \right) \left(\int_0^\infty f(y)t^{y^\lambda - \frac{1}{2}} dy \right) dt \int_0^1 t h^2(t) dt \\ &= \int_0^1 \left(\int_0^\infty \int_0^\infty f(x)f(y)t^{x^\lambda + y^\lambda - 1} dx dy \right) dt \int_0^1 t h^2(t) dt \\ &= \left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda} dx dy \right) \int_0^1 t h^2(t) dt \\ &\leq \left\{ \left(\frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx \right)^2 - \left(\int_0^\infty \Omega(\lambda, x) f^2(x) dx \right)^2 \right\}^{\frac{1}{2}} \int_0^1 t h^2(t) dt, \end{aligned} \tag{4.9}$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Since $h(t) \neq 0$, $f(x) \neq 0$, it is impossible to take equality in (4.9). Thus, we get inequality (4.8). This completes the proof. \square

Acknowledgements. We are very grateful to the anonymous referees for their careful reading and helpful suggestions which led to an improvement of our original manuscript. The authors also wish to express their sincere thanks to Professor Yang Qiaoshun for his assistance.

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(Received July 11, 2018)

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