

## LOWER BOUNDS FOR THE FIRST EIGENVALUES OF THE $p$ -LAPLACIAN AND THE WEIGHTED $p$ -LAPLACIAN

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(Communicated by S. Varošanec)

*Abstract.* In this paper, we investigate the  $p$ -Laplacian  $\Delta_p$  on a complete noncompact submanifold of a Riemannian manifold with sectional curvature bounded above by a negative constant. Moreover, we study the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  on an  $n$ -dimensional complete noncompact smooth metric measure space  $(M, g, e^{-\varphi} dv)$  with  $M$  being a submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ . We obtain some estimates for their first eigenvalues. They reflect the relations between the first eigenvalues of these two kinds of nonlinear operators and the geometrical data of manifolds. Our results cover a result derived by Lin (Nonlinear Anal., **148** (2017), 126-137) for the Laplacian, some results of Du and Mao (J. Math. Anal. Appl., **456** (2017), 787-795) for the drifting Laplacian and the  $p$ -Laplacian.

### 1. Introduction

In this paper, we are concerned with lower bounds of the first eigenvalues of the  $p$ -Laplacian and the weighted  $p$ -Laplacian. This question is related to the first eigenvalue  $\lambda_1(M)$  of the Laplacian  $\Delta$  on  $M$ . According to Schoen and Yau [15], it is an important question to find conditions which implies  $\lambda_1(M) > 0$ . For an  $n$ -dimensional, complete noncompact, simply connected Riemannian manifold whose sectional curvature bounded above by  $-c^2$ , McKean [12] showed that

$$\lambda_1(M) \geq \frac{(n-1)^2 c^2}{4}. \tag{1.1}$$

It is sharp in the sense that the equality holds for the hyperbolic space  $\mathbb{H}^n(-c^2)$  with the constant curvature  $-c^2$ . The hyperbolic space is an important kind of space form. In 2001, for an  $n$ -dimensional complete noncompact submanifold  $M$  in the hyperbolic space  $\mathbb{H}^m(-1)$  with the mean curvature vector  $H$ , Cheung and Leung [2] proved that

*Mathematics subject classification* (2010): 35P30, 47J10, 53C21.

*Keywords and phrases:* Eigenvalue,  $p$ -Laplacian, weighted  $p$ -Laplacian, hyperbolic space, metric measure space.

The first author was supported by the National Natural Science Foundation of China (Grant No.11001130) and the Fundamental Research Funds for the Central Universities (Grant No.30917011335). The third author was supported by the National Natural Science Foundation of China (Grant Nos.11861036, 11826213) and the Natural Science Foundation of Jiangxi Province (Grant No.20171ACB21023).

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if  $|H| \leq \kappa$  for some constant  $\kappa < n - 1$ , then the first eigenvalue of the Laplacian on  $M$  (with the Dirichlet boundary condition) satisfies

$$\lambda_1(M) \geq \frac{1}{4}(n - 1 - \kappa)^2. \tag{1.2}$$

It implies that if  $M^n$  be a complete minimal submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ , then

$$\lambda_1(M) \geq \frac{1}{4}(n - 1)^2. \tag{1.3}$$

Moreover, (1.3) is sharp because the equality holds when  $M$  is totally geodesic. In 2017, Lin [11] proved that the first eigenvalue of the Laplacian of  $M$  satisfies

$$\lambda_1(M) \geq \frac{(n - 1)^2 c^2}{4} \left( 1 - nD(n)\|H\|_n \right)^2, \tag{1.4}$$

where  $M$  is an  $n$  dimensional complete noncompact submanifold in  $N$ , a complete simply connected Riemannian manifold with sectional curvature  $K_N$  satisfying  $K_N \leq -c^2$  for a positive constant  $c > 0$ , and the mean curvature vector  $H$  of  $M$  in  $N$  satisfies  $\|H\|_n < \frac{1}{nD(n)}$ , being  $D(n) = 2^n(1+n)^{\frac{n+1}{2}}(n-1)^{-1}\sigma_n^{-1/n}$ ,  $\|H\|_n = (\int_M |H|^n dv)^{\frac{1}{n}}$  and  $\sigma_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . In fact,  $D(n)$  is the constant which governs a  $L^1$ -Sobolev inequality due to Hoffman and Spruck [6]. In particular, if  $M$  is a complete noncompact minimal submanifold in  $N$ , it holds

$$\lambda_1(M) \geq \frac{(n - 1)^2 c^2}{4}. \tag{1.5}$$

In the last few years, as a quasilinear elliptic partial differential operator, the  $p$ -Laplacian has emerged from some physical problems (cf. [7]). The definition of the  $p$ -Laplacian is as follows. Let  $\Omega$  be a bounded domain of an  $n$ -dimensional Riemannian manifold  $M$ . For  $1 < p < \infty$  and any  $u \in W_0^{1,p}(\Omega)$ , the  $p$ -Laplacian  $\Delta_p$  is defined by

$$\Delta_p u = \operatorname{div} (|\nabla u|^{p-2} \nabla u),$$

where  $\operatorname{div}$  is the divergence operator and  $\nabla$  is the gradient operator. The  $p$ -Laplacian has some important applications in non-Newtonian fluids, glaciology, turbulence theory, climatology, nonlinear diffusion, flow through porous media and so on. For example, the following equation

$$-\Delta_p u(x) = \lambda u(x) + kf(|u|)u(x) \tag{1.6}$$

is described dilatant fluids when  $p > 2$ , pseudoplastics when  $p < 2$ , whereas  $p = 2$  corresponds to Newtonian fluids.

Some scholars gave some results for eigenvalue estimate of the  $p$ -Laplacian  $\Delta_p$ . For example, for an  $n$ -dimensional, complete noncompact, simply connected Riemannian manifold  $M$  with sectional curvature  $K \leq -c^2 < 0$ , Lima, Montenegro and Santos [10] proved that the  $p$ -fundamental tone of the  $p$ -Laplacian satisfies

$$\lambda_{1,p}(M) \geq \frac{(n - 1)^p c^p}{p^p}. \tag{1.7}$$

This result shows an interesting connection between the nonlinear  $p$ -Laplacian and the linear Laplacian. In fact, when  $p = 2$ , (1.7) becomes the famous result (1.1) of Mckean[12] for the Laplacian. Du and Mao [3] gave several upper bounds in terms of the norm of the mean curvature vector of  $M$  for the first non-zero eigenvalue of the  $p$ -Laplacian.

In this paper, we obtain the following results for the  $p$ -Laplacian:

**THEOREM 1.** *Let  $M$  be an  $n$ -dimensional complete noncompact submanifold in  $N$  with the mean curvature vector  $H$ , where  $n > 1$  and  $N$  is a complete simply connected Riemannian manifold with sectional curvature  $K_N$  satisfying  $K_N \leq -c^2$  for a positive constant  $c > 0$ . Set*

$$\|H\|_n = \left( \int_M |H|^n dv \right)^{\frac{1}{n}} \quad \text{and} \quad D(n) = 2^n(1+n)^{\frac{n+1}{2}}(n-1)^{-1}\sigma_n^{\frac{1}{n}},$$

where  $\sigma_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . If  $\|H\|_n < \frac{1}{nD(n)}$ , then the first eigenvalue of the  $p$ -Laplacian satisfies

$$\lambda_{1,p}(M) \geq \frac{(n-1)^p c^p \left( 1 - nD(n)\|H\|_n \right)^p}{p^p}. \tag{1.8}$$

**REMARK 1.** When  $p = 2$ , the nonlinear  $p$ -Laplacian  $\Delta_p$  degenerates into the linear Laplacian  $\Delta$ . Correspondingly, the inequality (1.8) becomes (1.4) derived by Lin [11]. Hence Theorem 1 covers the result (1.4) of Lin [11] for the Laplacian.

From Theorem 1, we can get the following corollary which generalizes the result (1.5) of [11] for the Laplacian to the  $p$ -Laplacian.

**COROLLARY 1.** *Let  $M$  be an  $n$ -dimensional complete minimal submanifold in  $N$ , where  $n > 1$  and  $N$  is a complete simply connected Riemannian manifold with sectional curvature  $K_N$  satisfying  $K_N \leq -c^2$  for a positive constant  $c > 0$ . Then the following estimate for the first eigenvalue of the  $p$ -Laplacian holds*

$$\lambda_{1,p}(M) \geq \frac{(n-1)^p c^p}{p^p}. \tag{1.9}$$

Furthermore, we study the weighted  $p$ -Laplacian on a smooth metric measure space. For a given complete Riemannian manifold  $(M, g)$  with the metric  $g$ , the triple  $(M, g, e^{-\varphi} dv)$  is called a smooth metric measured space, where  $\varphi$  is a smooth real-valued function on  $M$  and  $dv$  is the Riemannian volume element associated with  $g$ . In recent years, the metric measure space has been studied extensively in the geometry and analysis (cf. [16]). The most remarkable example is Perelman’s entropy formula for the Ricci flow in [14]. On  $(M, g, e^{-\varphi} dv)$ , we can define the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  as follows

$$\Delta_{p,\varphi} u = \operatorname{div}(\|\nabla u\|^{p-2} \nabla u) - \|\nabla u\|^{p-2} g(\nabla \varphi, \nabla u).$$

On the one hand, when  $p = 2$ , the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  becomes the drifting Laplacian  $\Delta_\varphi$  defined by

$$\Delta_\varphi u = \Delta u - g(\nabla\varphi, \nabla u).$$

The drifting Laplacian is also called the weighted Laplacian, the  $\varphi$ -Laplacian or the Witten-Laplacian in the previous literature. Futaki, Li and Li [5], Li and Wei [9] gave some lower bounds for the first nonzero eigenvalue of the drifting Laplacian on a compact smooth metric measure space without boundary or with boundary. In 2017, for an  $n$ -dimensional complete noncompact smooth metric measure space with  $M$  being a submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ , Du and Mao[4] proved that if the mean curvature vector  $H$  of  $M$  in  $\mathbb{H}^m(-1)$  satisfies  $|H| \leq \kappa$  for some constant  $\kappa < n - 1$ , and  $|\nabla\varphi| \leq C$  for some constant  $C$ , then the first eigenvalue  $\lambda_{1,\varphi}(M)$  of the drifting Laplacian on  $M$  satisfies

$$\lambda_{1,\varphi}(M) \geq \frac{(n - 1 - \kappa - C)^2}{4}. \tag{1.10}$$

Notice that when  $\varphi$  is a constant, (1.10) for the weighted Laplacian becomes (1.2) of Cheung and Leung for the Laplacian. On the other hand, if  $\varphi$  is a constant, the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  becomes the  $p$ -Laplacian. For an  $n$ -dimensional complete noncompact submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ , Du and Mao[4] also proved that if  $|H| \leq \kappa$  for some constant  $\kappa < n - 1$ , then the first eigenvalue of the  $p$ -Laplacian on  $M$  satisfies

$$\lambda_{1,p}(M) \geq \left(\frac{n - 1 - \kappa}{p}\right)^p. \tag{1.11}$$

Contrary to the Laplacian, there is little knowledge about the first eigenvalue of the weighted  $p$ -Laplacian. In 2016, Wang and Li [17] gave an estimate for lower bound of the first eigenvalue of the weighted  $p$ -Laplacian on a closed smooth metric measure space  $(M, g, e^{-\varphi} dv)$  under the assumption of  $m$ -dimensional Bakry-Émery curvature on  $M$  bounded by  $K$ . Moreover, they showed that the above bound still holds for the weighted  $p$ -Laplacian under the Dirichlet boundary condition or the Neumann boundary condition. In this paper, we get the following results for the weighted  $p$ -Laplacian.

**THEOREM 2.** *Let  $(M, g, e^{-\varphi} dv)$  be an  $n$ -dimensional complete noncompact smooth metric measure space with  $n > 1$ ,  $|\nabla\varphi| \leq C$  for some constant  $C$  and  $M$  being a submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ . Denote  $H$  by the mean curvature vector of  $M$  in  $\mathbb{H}^m(-1)$ . If  $|H| \leq \kappa$  for some constant  $\kappa < n - 1 - C$ , then the first eigenvalue  $\lambda_{1,p,\varphi}(M)$  of the weighted  $p$ -Laplacian on  $M$  satisfies*

$$\lambda_{1,p,\varphi}(M) \geq \left(\frac{n - 1 - \kappa - C}{p}\right)^p. \tag{1.12}$$

**REMARK 2.** It is easy to find that (1.12) becomes (1.10) when  $p = 2$ . Hence we generalizes (1.10) of [4] for the drifting Laplacian to the weighted  $p$ -Laplacian. Moreover, if  $\varphi$  is a constant, then the constant  $C$  in Theorem 2 can be chosen to be

$C = 0$ . Correspondingly, the estimate (1.12) becomes (1.11). Therefore, (1.11) of Du and Mao [4] is included by Theorem 2 as a special case.

If  $M$  is minimal, then  $\kappa = 0$ . Hence we have the following corollary:

**COROLLARY 2.** *Let  $(M, g, e^{-\varphi} dv)$  be an  $n$ -dimensional smooth metric measure space with  $n > 1$ ,  $|\nabla\varphi| \leq C$  for some constant  $C$  and  $M$  being a complete minimal submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ . Then the first eigenvalue  $\lambda_{1,p}(M)$  of the weighted  $p$ -Laplacian on  $M$  satisfies*

$$\lambda_{1,p,\varphi}(M) \geq \frac{1}{p^p}(n - 1 - C)^p. \tag{1.13}$$

**REMARK 3.** According to McKean’s results in [12], we know that the first eigenvalue of the Laplacian on  $\mathbb{H}^n(-1)$  satisfies  $\lambda_1(\mathbb{H}^n(-1)) = \frac{1}{4}(n - 1)^2$ . This shows that the estimate (1.13) is sharp for the totally geodesic submanifold  $\mathbb{H}^n(-1)$  in  $\mathbb{H}^m(-1)$  for  $p = 2$  and  $\varphi$  is a constant.

### 2. Proof of Theorem 1 for the $p$ -Laplacian

In this section, we investigate eigenvalue estimate for the  $p$ -Laplacian and give the proof of Theorem 1. For the convenience of reader, we first give some knowledge about the  $p$ -fundamental tone and the first nonzero eigenvalue of the  $p$ -Laplacian.

Let  $\Omega$  be a bounded domain in a smooth Riemannian manifold  $M$ . Denote by  $C_0^\infty(\Omega)$  the set of smooth functions with compact support in  $\Omega$ . The Sobolev space  $W_0^{1,p}(\Omega)$  is given by the closure of  $C_0^\infty(\Omega)$  with the norm

$$\|u\|_{1,p} = \left( \int_\Omega |u|^p dv + \int_\Omega \|\nabla u\|^p dv \right)^{\frac{1}{p}}, \tag{2.1}$$

where  $dv$  is the Riemannian volume element on  $M$  and  $\|\cdot\|$  denotes the norm of some prescribed vector field on  $M$  with respect to the metric of  $M$ .

Because the solutions of the following nonlinear eigenvalue problem

$$\Delta_p u = -\lambda |u|^{p-2} u \tag{2.2}$$

are only locally  $C^{1,\alpha}(\Omega)$  for any  $p > 1$  (exceptions for the case  $p = 2$ ), they must be described in the sense of distribution. That is to say,  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$  is an eigenfunction associated to the eigenvalue  $\lambda$ , if

$$\int_\Omega \|\nabla u\|^{p-2} \langle \nabla u, \nabla \psi \rangle dv = \lambda \int_\Omega |u|^{p-2} u \psi dv, \tag{2.3}$$

for any test function  $\psi \in C_0^\infty(\Omega)$ . It is known that the set  $\sigma_p(M)$  of eigenvalues of problem (2.2) is an unbounded subset of  $[0, \infty)$  whose infimum  $\inf \sigma_p = \mu_{1,p}(\Omega)$  is an eigenvalue. It is also known that for geodesic balls of space forms the first eigenvalue is simple and the first eigenfunction is radial (cf. [13]).

For a bounded domain  $\Omega$  with smooth boundary in a Riemannian manifold, the Dirichlet, the Neumann and some other classes of non-linear eigenvalue problems associated to the  $p$ -Laplacian have the following properties: There exists a nondecreasing sequence if nonnegative eigenvalues obtained by the Ljusternik-Schnirelman principle,  $(\lambda_n)_n$ , tending to  $\infty$  as  $n \rightarrow \infty$ . The first eigenvalue is simple and isolated. Only eigenfunctions associated with the first eigenvalue do not change sign.

Let  $\Omega \subset M$  a domain in a Riemannian manifold. The  $p$ -fundamental tone of  $\Omega$ , denoted by  $\lambda_p^*(\Omega)$  is defined as follows:

$$\lambda_p^*(\Omega) = \inf \left\{ \frac{\int_{\Omega} \|\nabla f\|^p dv}{\int_{\Omega} |f|^p dv} \mid f \in W_0^{1,p}(\Omega), f \neq 0 \right\}. \tag{2.4}$$

When  $\Omega$  is a relatively compact domain with smooth smooth boundary, the  $p$ -fundamental tone  $\lambda_p^*(\Omega)$  coincides with the first eigenvalue  $\lambda_{1,p}(\Omega) = \inf \sigma_p$  of the following non-linear Dirichlet eigenvalue problem of the  $p$ -Laplacian on  $\Omega$

$$\begin{cases} \Delta_p u = -\lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let  $M$  be a complete noncompact Riemannian manifold. Denote by  $B(q, r)$  a geodesic ball, with center  $q$  and radius  $r$ , on  $M$ . According to Rayleigh’s Theorem and the Max-min principle, the first eigenvalue  $\lambda_{1,p}(B(q, r))$  of problem (2.5) on  $B(q, r)$  can be characterized by

$$\lambda_{1,p}(B(q, r)) = \inf \left\{ \frac{\int_{B(q,r)} \|\nabla f\|^p dv}{\int_{B(q,r)} |f|^p dv} \mid f \in W_0^{1,p}(B(q, r)), f \neq 0 \right\}. \tag{2.6}$$

It is known (cf. [4]) that  $\lambda_{1,p}(B(q, r))$  decreases as  $r$  increases and then it has a limit independent of the choice of the center  $q$ . Thus the first eigenvalue of the  $p$ -Laplacian on  $M$  is defined by

$$\lambda_{1,p}(M) = \lim_{r \rightarrow \infty} \lambda_{1,p}(B(q, r)). \tag{2.7}$$

**Proof of Theorem 1** Let  $\phi : M \hookrightarrow N$  be an isometric immersion,  $\rho(x)$  be the geodesic measured function on  $N$  from a fixed point  $x_0 \in N \setminus M$  to  $x$ . From the proof of [1], we know that

$$\Delta(\rho \circ \phi) \geq (n - 1)c - n\|H\|. \tag{2.8}$$

Set  $r = \rho \circ \phi$ . Taking  $f \in C_0^\infty(M)$ , we have

$$\begin{aligned} \operatorname{div}(|f|^p \nabla r) &= \langle \nabla |f|^p, \nabla r \rangle + |f|^p \Delta r \\ &\geq -p|f|^{p-1} \|\nabla f\| + (n - 1)c|f|^p - n|f|^p \|H\| \end{aligned} \tag{2.9}$$

by using the Cauchy-Bunyakovsky-Schwarz inequality. Integrating both sides of the inequality (2.9) over  $M$ , and using the divergence theorem, we derive

$$0 \geq -p \int_M |f|^{p-1} \|\nabla f\| dv + (n - 1)c \int_M |f|^p dv - n \int_M |f|^p \|H\| dv. \tag{2.10}$$

Hoffman and Spruck [6] showed that if the ambient manifold has non-positive sectional curvature, then the following  $L^1$ -Sobolev inequality

$$\left( \int_M h^{\frac{n}{n-1}} dv \right)^{\frac{n-1}{n}} \leq D(n) \int_M (\|\nabla h\| + n\|H\|h) dv \tag{2.11}$$

holds for any  $h \in C_0^1(M)$ . Hence, using (2.11), we deduce

$$\begin{aligned} \int_M |f|^p \|H\| dv &\leq \left( \int_M \|H\|^n dv \right)^{\frac{1}{n}} \left( \int_M |f|^{\frac{pn}{n-1}} dv \right)^{\frac{n-1}{n}} \\ &\leq D(n) \|H\|_n \int_M (p|f|^{p-1} \|\nabla f\| + n|f|^p \|H\|) dv, \end{aligned} \tag{2.12}$$

where  $\|H\|_n = \left( \int_M \|H\|^n dv \right)^{\frac{1}{n}}$ . It can be rewritten as

$$\int_M |f|^p \|H\| dv \leq \frac{pD(n)\|H\|_n}{1-nD(n)\|H\|_n} \int_M |f|^{p-1} \|\nabla f\| dv. \tag{2.13}$$

Thereby, using (2.13), the inequality (2.10) can be converted to

$$\begin{aligned} 0 &\geq -p \int_M |f|^{p-1} \|\nabla f\| dv - \frac{npD(n)\|H\|_n}{1-nD(n)\|H\|_n} \int_M |f|^{p-1} \|\nabla f\| dv \\ &\quad + (n-1)c \int_M |f|^p dv \\ &= -\frac{p}{1-nD(n)\|H\|_n} \int_M |f|^{p-1} \|\nabla f\| dv + (n-1)c \int_M |f|^p dv. \end{aligned} \tag{2.14}$$

The Young inequality shows: for any  $\alpha, \beta > 0$ , if  $\frac{1}{s} + \frac{1}{t} = 1$ , then it holds

$$\alpha\beta \leq \frac{\alpha^s}{s} + \frac{\beta^t}{t}. \tag{2.15}$$

It implies that for any  $\varepsilon > 0$ , the next inequality holds:

$$\alpha\beta \leq \varepsilon\alpha^s + (\varepsilon s)^{-\frac{1}{s}} \frac{1}{t} \beta^t. \tag{2.16}$$

Therefore, taking  $\alpha = |f|^{p-1}$ ,  $\beta = \|\nabla f\|$ ,  $s = \frac{p}{p-1}$  and  $t = p$  in (2.16), we obtain

$$|f|^{p-1} \|\nabla f\| \leq \varepsilon(|f|^{p-1})^{\frac{p}{p-1}} + \left(\varepsilon \frac{p}{p-1}\right)^{1-p} \frac{1}{p} \|\nabla f\|^p, \tag{2.17}$$

where  $\varepsilon$  is a positive constant to be determined later. Then it follows from (2.14) and (2.17) that

$$\begin{aligned} 0 &\geq -\frac{1}{1-nD(n)\|H\|_n} \int_M \left[ p\varepsilon|f|^p + \left(\varepsilon \frac{p}{p-1}\right)^{1-p} \|\nabla f\|^p \right] dv \\ &\quad + (n-1)c \int_M |f|^p dv. \end{aligned} \tag{2.18}$$

It implies that

$$\int_M \|\nabla f\|^p dv \geq \left[ (n-1)c \left( 1 - nD(n)\|H\|_n \right) - p\varepsilon \right] \left( \frac{\varepsilon}{p-1} \right)^{p-1} \int_M |f|^p dv. \tag{2.19}$$

Consider the function

$$\zeta(\varepsilon) = (n-1)c \left( 1 - nD(n)\|H\|_n \right) \varepsilon^{p-1} - p\varepsilon^p.$$

Now we calculate the maximum of this function. According to

$$\zeta'(\varepsilon) = \varepsilon^{p-2} \left[ (p-1)(n-1)c \left( 1 - nD(n)\|H\|_n \right) - p^2\varepsilon \right], \tag{2.20}$$

the critical points of  $\zeta$  are given by

$$\varepsilon_1 = 0 \quad \text{and} \quad \varepsilon_2 = \frac{p-1}{p^2} (n-1)c \left( 1 - nD(n)\|H\|_n \right).$$

Moreover, we have

$$\zeta''(\varepsilon) = (p-1)\varepsilon^{p-3} \left[ (p-2)(n-1)c \left( 1 - nD(n)\|H\|_n \right) - p^2\varepsilon \right]. \tag{2.21}$$

It is from a straightforward calculation that the function  $\zeta(\varepsilon)$  takes the maximum

$$\zeta(\varepsilon_2) = \frac{(p-1)^{p-1} (n-1)^p c^p \left( 1 - nD(n)\|H\|_n \right)^p}{p^{2p-1}} \tag{2.22}$$

at the point  $\varepsilon_2$ . Taking  $\varepsilon = \varepsilon_2$  in (2.19), we find that the Rayleigh quotient satisfies

$$\frac{\int_M \|\nabla f\|^p dv}{\int_M |f|^p dv} \geq \frac{(n-1)^p c^p \left( 1 - nD(n)\|H\|_n \right)^p}{p^p}. \tag{2.23}$$

According to (2.6), (2.7) and (2.23), we know that (1.8) is true. This completes the proof of Theorem 1. □

### 3. Proof of Theorem 2 for the weighted $p$ -Laplacian

In this section, we study eigenvalue estimate for the weighted  $p$ -Laplacian on a smooth metric measure space and give the proof of Theorem 2.

A smooth metric measure space  $(M, g, e^{-\varphi} dv)$  is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. For every smooth function  $u$  on  $M$ , the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  can also be defined by

$$\Delta_{p,\varphi} u = e^{-\varphi} \operatorname{div} (e^\varphi \|\nabla u\|^{p-2} \nabla u).$$

Consider the following nonlinear Dirichlet eigenvalue problem of the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  on  $B(q,r)$

$$\begin{cases} \Delta_{p,\varphi}u = -\lambda|u|^{p-2}u, & \text{in } B(q,r), \\ u = 0, & \text{on } \partial B(q,r). \end{cases} \tag{3.1}$$

Similar to the case of the  $p$ -Laplacian, the first Dirichlet eigenvalue of problem (3.1) of the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  on  $B(q,r)$  can be characterized by

$$\lambda_{1,p,\varphi}(B(q,r)) = \inf \left\{ \frac{\int_{B(q,r)} \|\nabla f\|^p e^{-\varphi} dv}{\int_{B(q,r)} |f|^p e^{-\varphi} dv} \mid f \in W_0^{1,p}(B(q,r)) \right\}. \tag{3.2}$$

Then the first eigenvalue of the weighted  $p$ -Laplacian  $\Delta_{p,\varphi}$  on  $M$  can be defined by

$$\lambda_{1,p,\varphi}(M) = \lim_{r \rightarrow \infty} \lambda_{1,p,\varphi}(B(q,r)). \tag{3.3}$$

It was proved in [8] that the infimum

$$\begin{aligned} \varsigma_{p,\varphi}(M) = \inf \left\{ \int_M \|\nabla f\|^p e^{-\varphi} dv \mid f \in W^{1,p}(M), \int_M |f|^p e^{-\varphi} dv = 1, \right. \\ \left. \int_M |f|^{p-2} u e^{-\varphi} dv = 0 \right\} \end{aligned}$$

is achieved by a  $C^{1,\alpha}$  eigenfunction  $u$  which satisfies the Euler-Lagrange equation

$$\Delta_{p,\varphi}u = -\varsigma_{p,\varphi}(M)|u|^{p-2}u.$$

In order to prove Theorem 2, we need the following lemma obtained by Du and Mao [4].

LEMMA 1. Assume that  $(M, g, e^{-\varphi} dv)$  is an  $n$ -dimensional smooth metric measure space with  $M$  being a submanifold in the hyperbolic space  $\mathbb{H}^m(-1)$ . Then we have

$$\Delta_\varphi \cosh r = n \cosh r + \langle H, \bar{\nabla} r \rangle|_M \cdot \sinh r - \sinh r \cdot g(\nabla r, \nabla \varphi) \tag{3.4}$$

and

$$\Delta_\varphi r = (n - \|\nabla r\|^2) \coth r + \langle H, \bar{\nabla} r \rangle|_M - g(\nabla r, \nabla \varphi), \tag{3.5}$$

where  $r$  is measured from a fixed point in  $\mathbb{H}^m(-1) \setminus M$ .

Now we give the proof of Theorem 2.

**Proof of Theorem 2** Let  $r$  denote the distant measured function from a fixed point in  $\mathbb{H}^m(-1) \setminus M$ . Taking  $f \in C_0^\infty(\Omega)$ , we have

$$\operatorname{div}(|f|^p e^{-\varphi} \nabla r) = g(\nabla(|f|^p), \nabla r) e^{-\varphi} + |f|^p e^{-\varphi} \Delta r - |f|^p e^{-\varphi} g(\nabla r, \nabla \varphi). \tag{3.6}$$

Denote by  $\bar{\nabla}$  the gradient operator on  $\mathbb{H}^m(-1)$ . Then we know that  $\|\bar{\nabla}r\| = 1$ . Hence it implies  $|\nabla r| \leq 1$ . Due to Lemma 1 of [4], we have

$$\begin{aligned} \Delta_\varphi r &= (n - \|\nabla r\|^2) \operatorname{coth} r + \langle H, \bar{\nabla} r \rangle|_M - g(\nabla r, \nabla \varphi) \\ &\geq n - 1 - \|H\| \cdot \|\bar{\nabla} r\| - \|\nabla \varphi\| \cdot \|\nabla r\| \\ &\geq n - 1 - \kappa - C. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), and using the Cauchy-Bunyakovsky-Schwarz inequality, we can obtain

$$\begin{aligned} \operatorname{div}(|f|^p e^{-\varphi} \nabla r) &= g(\nabla |f|^p, \nabla r) e^{-\varphi} + |f|^p e^{-\varphi} \Delta_\varphi r \\ &\geq \left[ -p|f|^{p-1} \|\nabla f\| + (n - 1 - \kappa - C)|f|^p \right] e^{-\varphi}. \end{aligned} \tag{3.8}$$

Similar to (2.17), it follows from the Young inequality that

$$-p|f|^{p-1} \|\nabla f\| \geq -p\varepsilon|f|^p - \left( \varepsilon \frac{p}{p-1} \right)^{1-p} \|\nabla f\|^p \tag{3.9}$$

for all  $\varepsilon > 0$ . Substituting (3.9) into (3.8), we get

$$\begin{aligned} \operatorname{div}(|f|^p \nabla r e^{-\varphi}) &\geq \left[ -p\varepsilon|f|^p - \left( \varepsilon \frac{p}{p-1} \right)^{1-p} \|\nabla f\|^p \right. \\ &\quad \left. + (n - 1 - \kappa - C)|f|^p \right] e^{-\varphi}, \end{aligned} \tag{3.10}$$

where  $\varepsilon$  is a positive constant to be determined later. Integrating both sides of inequality (3.10) on  $M$ , and using the divergence theorem, we derive

$$\int_M \left[ p\varepsilon|f|^p + \left( \varepsilon \frac{p}{p-1} \right)^{1-p} \|\nabla f\|^p \right] e^{-\varphi} dv \geq \int_M (n - 1 - \kappa - C)|f|^p e^{-\varphi} dv. \tag{3.11}$$

It implies

$$\int_M \|\nabla f\|^p e^{-\varphi} dv \geq (n - 1 - \kappa - C - p\varepsilon) \left( \varepsilon \frac{p}{p-1} \right)^{p-1} \int_M |f|^p e^{-\varphi} dv. \tag{3.12}$$

In order to choose a suitable constant  $\varepsilon$ , we consider the maximum of the function

$$\xi(\varepsilon) = (n - 1 - \kappa - C - p\varepsilon) \left( \varepsilon \frac{p}{p-1} \right)^{p-1}.$$

Then the derivative of the function  $\xi(\varepsilon)$  is

$$\xi'(\varepsilon) = p \left( \varepsilon \frac{p}{p-1} \right)^{p-2} \left( n - 1 - \kappa - C - \frac{p^2}{p-1} \varepsilon \right).$$

Hence we can get two critical points of the function  $\xi(\varepsilon)$  as follows

$$\varepsilon_1 = 0 \quad \text{and} \quad \varepsilon_2 = \frac{p-1}{p^2}(n-1-\kappa-C).$$

By a direct calculation, we have

$$\xi''(\varepsilon) = \frac{p^2}{p-1} \left( \varepsilon \frac{p}{p-1} \right)^{p-3} [(p-2)(n-1-\kappa-C) - p^2\varepsilon].$$

Hence, noticing that  $\kappa < n-1-C$  and  $p > 1$ , we find that

$$\xi''(\varepsilon_2) = -\frac{p^3}{p-1} \left( \frac{n-1-\kappa-C}{p} \right)^{p-2} < 0.$$

Consequently, we find that the function  $\xi(\varepsilon)$  takes the maximum

$$\xi(\varepsilon_2) = \left( \frac{n-1-\kappa-C}{p} \right)^p.$$

Taking  $\varepsilon = \varepsilon_2$  in (3.12), we derive

$$\frac{\int_M \|\nabla f\|^p e^{-\varphi} dv}{\int_M |f|^p e^{-\varphi} dv} \geq \frac{1}{p^p} (n-1-\kappa-C)^p. \tag{3.13}$$

Therefore, we know that (1.12) holds according to (3.2), (3.3) and (3.13). This finishes the proof of Theorem 2. □

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(Received March 28, 2019)

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