

## SHARP BERNSTEIN INEQUALITIES USING CONVEX ANALYSIS TECHNIQUES

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*Dedicated to Professor Luis Bernal González  
on the occasion of his 60th birthday*

*(Communicated by T. Erdélyi)*

*Abstract.* In this paper we consider the space of polynomials of degree at most three in the real line endowed with the sup norm over the unit interval. We provide, explicitly, all the extreme points of the unit ball of this space. Using the previous geometrical description, we obtain the Bernstein function for the first and second derivative of the polynomials of degree at most 3.

### 1. Introduction and preliminaries

Let  $\mathcal{P}_n(\mathbb{R})$  stand for the space of real polynomials in the real line of degree at most  $n$  endowed with the norm

$$\|P\| = \max\{|P(x)| : x \in [-1, 1]\},$$

and let

$$B_n = \{P \in \mathcal{P}_n(\mathbb{R}) : \|P\| \leq 1\},$$

be the closed unit ball of  $\mathcal{P}_n(\mathbb{R})$ . The first results of this paper are devoted to the study of the geometry of  $B_n$ . We are interested in providing an explicit description of the set  $\text{ext}(B_n)$  of the extreme points of  $B_n$ . We recall that  $e$  is an extreme point of a convex set  $C$  in a linear space if  $e$  is not an interior point of any segment with endpoints in  $C$ . The case where  $n = 2$  was solved in [2]. The solution of this problem for general  $n$  is perhaps a far too complicated problem. However, we show in Section 2 that for  $n = 3$  a reasonable characterization of  $\text{ext}(B_3)$  can be found. Our approach to solve the problem for  $n = 3$  is based on an interesting result by Konheim and Rivlin (see [21]) where the authors find a simple property that characterizes the elements of

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*Mathematics subject classification* (2010): 41A17, 26D05.

*Keywords and phrases:* Polynomial inequalities, Bernstein and Markov inequalities, extreme points.

G. Araújo was supported by Grant 2019/0014 Paraíba State Research Foundation (FAPESQ). G. A. Muñoz-Fernández, Daniel L. Rodríguez-Vidanes and J. B. Seoane-Sepúlveda were supported by Grant PGC2018-097286-B-I00.

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$\text{ext}(B_n)$  for all  $n \in \mathbb{N}$  (see Section 2 for details). Unfortunately, Konheim and Rivlin's characterization does not include an explicit description of  $\text{ext}(B_n)$ .

The study of the extreme points of finite dimensional polynomial spaces has raised the interest of a great many authors in the past. We mention below just a few of the publications that preceded this work. In [2] Aron and Klimek provide the set  $\text{ext}(B_2)$  explicitly. Additionally, they also find a complete description of the real polynomials of degree at most 2 endowed with the sup norm over the unit disk in the complex plane  $\mathbb{D}$ . In [31] the authors extended the study done in [2] to the space of all the real trinomials with independent term on the real line with the sup norm on symmetric intervals. The same question was solved by Neuwirth [32] for trinomials on  $\mathbb{D}$ . In a similar direction, Choi and Kim [7, 8, 9] considered the same problem for scalar-valued 2-homogeneous polynomials on the real spaces  $\ell_1^2$ ,  $\ell_2^2$  and  $\ell_\infty^2$  whereas Grecu [15] treated the case of scalar-valued 2-homogeneous polynomials on the real spaces  $\ell_p^2$  with  $1 < p < \infty$ . See also [13, 14, 15, 16, 17, 18] for related questions concerning real or complex homogeneous polynomials of degree 2 or 3.

The geometry of finite dimensional spaces of polynomials on non-symmetric convex bodies has also been studied. Recall that a convex body is a closed bounded convex set with nonempty interior and therefore, a convex body in a finite dimensional space is a compact convex set with nonempty interior. For instance, in [28] it can be found a full description of the extreme points of the space of 2-homogeneous polynomials on  $\mathbb{R}^2$  with the sup norm over the simplex  $\Delta$  (the triangle of vertices  $(0, 0)$ ,  $(0, 1)$  and  $(1, 0)$ ). An explicit description of the extreme 2-homogeneous polynomials on the square  $\square$  with vertices  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$  and  $(1, 1)$  can be found in [12]. Also, the extreme polynomials of degree at most 2 on  $\Delta$  have also been characterized in [24], [25] and [26]. The extreme 2-polynomials on a sector  $D(\beta)$  of amplitude  $\beta$  in  $\mathbb{R}^2$ , namely  $D(\beta) = \{re^{i\theta} : \theta \in [0, \beta] \text{ and } r \in [0, 1]\}$ , can be found in [27] for  $\beta = \frac{\pi}{4}$ ,  $\beta = \frac{\pi}{2}$ ,  $\beta = \frac{3\pi}{4}$  and  $\beta \geq \pi$ . For an arbitrary  $\beta \in [0, 2\pi]$  see [3].

A deep understanding of the geometrical properties of a polynomial space may be of help in order to obtain sharp polynomial inequalities (see for instance [19, 1]). As a matter of fact, in Section 3 we use the geometrical results obtained in Section 2 to obtain sharp Bernstein inequalities for the first and second derivatives of polynomials of degree at most 3. This question was addressed already by A. A. Markov at the end of the 19<sup>th</sup> century. Voronovskaja [33] solved the problem for the first derivative of polynomial of arbitrary degree whereas V. A. Gusev [20] (see also the appendix of [33] for an English translation) completed the solution of the Bernstein problem for higher derivatives. However, both Gusev and Voronovskaja's results are not explicit. To motivate the study of Bernstein type inequalities let us introduce a few historical results. Firstly, recall that Markov and Bernstein inequalities are estimates on the derivative of a polynomial.

A sharp estimate on the norm of the derivative of a polynomial in  $\mathcal{P}_n(\mathbb{R})$  for all  $n$  is due to one of the brothers Markov:

**THEOREM 1.** (A. A. Markov, 1889) *If  $P$  is a polynomial of degree at most  $n \in \mathbb{N}$ , then  $|P'(x)| \leq n^2 \|P\|$  for all  $x \in [-1, 1]$ . Equality is attained at the end points of  $[-1, 1]$  for the  $n$ -th Chebyshev polynomial of the first kind, defined by  $T_n(x) = \cos(n \arccos x)$*

for  $x \in [-1, 1]$ .

In particular, it follows that  $\|P'\| \leq n^2 \|P\|$  for all  $P \in \mathcal{P}_n(\mathbb{R})$  and  $n^2$  cannot be replaced by a smaller constant. Interestingly, it is said in [6] that the first to solve successfully a Markov type problem was the famous chemist D. Mendeleev (the author of the Periodic Table of the Elements), who proved Markov's inequality for  $n = 2$ . Markov's original paper ([22]) is written in old Russian, but an English translation can be found in [23]. For a modern proof we refer to [6], where many other interesting results and comments are presented in connection with the problem of estimating the derivatives of a polynomial. An important generalization of Theorem 1 was published just three years later by the other brother Markov, and it provides a sharp estimate for the norm of any derivative of a polynomial. In the following,  $P^{(k)}$  denotes the  $k$ -th derivative of  $P$ .

**THEOREM 2.** (V. A. Markov, 1892) *If  $P$  is a polynomial of degree at most  $n \in \mathbb{N}$ , then*

$$|P^{(k)}(x)| \leq |T_n^{(k)}(\pm 1)| = \frac{n^2(n^2 - 1^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdots (2k-1)} \|P\|,$$

for all  $x \in [-1, 1]$ . Obviously, equality is attained at the end points of  $[-1, 1]$  for  $T_n$ .

Although Markov's estimates are sharp since they are achieved for any derivative for  $T_n$  at  $x = \pm 1$ , they could be improved for any fixed point in  $(-1, 1)$ . If  $x \in [-1, 1]$  is fixed, then we define  $\mathcal{B}_{n,k}(x)$  as the best (smallest) constant in the following inequality:

$$|P^{(k)}(x)| \leq \mathcal{B}_{n,k}(x) \cdot \|P\|, \quad (1)$$

for every  $P \in \mathcal{P}_n(\mathbb{R})$ . For simplicity, we set  $\mathcal{B}_{n,1} = \mathcal{B}_n$  for all  $n \in \mathbb{N}$ . An estimate on  $\mathcal{B}_n(x)$  can be easily derived from the complex version of Markov's Theorem due to S. Bernstein ([4], [5]):

**THEOREM 3.** (S. Bernstein, 1912) *If  $P \in \mathcal{P}_n(\mathbb{R})$  then*

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}} \cdot \|P\|,$$

for every  $x \in (-1, 1)$ . In other words,

$$\mathcal{B}_n(x) \leq \frac{n}{\sqrt{1-x^2}} \quad (2)$$

in  $(-1, 1)$ .

Bernstein's estimate coincides with  $\mathcal{B}_n(x)$  in  $n$  points in  $[-1, 1]$ , but it is far from being optimal in most of the interval  $[-1, 1]$ . The importance of Bernstein's estimate rests on the fact that it has been used in modern proofs of Markov's Theorem in order to simplify the original proof. In order to obtain a pointwise estimate of the  $k$ -th derivative

of a polynomial in  $\mathcal{P}_n(\mathbb{R})$ , we may simply iterate Bernstein's Theorem. Doing so, we find

$$\mathcal{B}_{n,k}(x) \leq \frac{n(n-1)\cdots(n-k+1)}{\sqrt{(1-x^2)^k}},$$

for all  $x \in (-1, 1)$ . Duffin and Schaeffer [11] improved the previous estimate:

**THEOREM 4.** (R. Duffin & A. C. Schaeffer, 1938) *If  $P \in \mathcal{P}_n(\mathbb{R})$  and  $x \in (-1, 1)$ , then for  $1 \leq k \leq n$  we have*

$$|P^{(k)}(x)| \leq \sqrt{[T_n^{(k)}(x)]^2 + [S_n^{(k)}(x)]^2} \cdot \|P\|,$$

where  $S_n(x) = \sin(n \arccos x)$ , for all  $x \in [-1, 1]$ . Thus

$$\mathcal{B}_{n,k}(x) \leq \mathcal{M}_{n,k}(x), \tag{3}$$

where  $\mathcal{M}_{n,k}(x) := \sqrt{[T_n^{(k)}(x)]^2 + [S_n^{(k)}(x)]^2}$  in  $(-1, 1)$ .

Observe that (2) coincides with (3) for  $k = 1$ . Also, from Theorem 2 we deduce that  $\mathcal{B}_{n,k}(x) \leq |T^{(k)}(\pm 1)|$  for all  $x \in [-1, 1]$ . Inequality (3) improves the previous estimate in most of the interval  $[-1, 1]$ , although it is not good in any neighborhood of  $\pm 1$ . Interestingly, Duffin and Schaeffer [11] used Theorem 4 to provide an alternative (and simpler) proof of V. A. Markov's estimate, Theorem 2.

In Section 3 we will find an explicit formula for  $\mathcal{B}_2(x)$ ,  $\mathcal{B}_3(x)$  and  $\mathcal{B}_{3,2}(x)$  for all  $x \in \mathbb{R}$ . The technique we will use in order to obtain those Bernstein functions relies on the following easily verified consequence of the Steinitz' Theorem, which in its turn, is nothing but a finite dimensional version of the Krein-Milman's Theorem:

**REMARK 1.** If  $C$  is a convex body in a finite dimensional Banach space and  $f : C \rightarrow \mathbb{R}$  is a convex function that attains its maximum, then there is an extreme point  $e \in C$  so that  $f(e) = \max\{f(x) : x \in C\}$ .

In particular, notice that for a fixed  $x \in [-1, 1]$ ,  $\mathcal{B}_{n,k}(x)$  is the maximum of the convex function  $\mathcal{B}_n \ni P \mapsto |P^{(k)}(x)| \in \mathbb{R}$ . To optimize that function we just need to maximize it at the points in  $\text{ext}(\mathcal{B}_n)$ . This is what we do in Section 3 using the characterization of  $\text{ext}(\mathcal{B}_2)$  given in [2] and our own description of  $\text{ext}(\mathcal{B}_3)$  provided in Section 2, to find, exactly, the functions  $\mathcal{B}_2(x)$ ,  $\mathcal{B}_3(x)$  and  $\mathcal{B}_{3,2}(x)$  in  $[-1, 1]$ .

## 2. Extreme polynomials of the unit ball of $\mathcal{P}_3(\mathbb{R})$

We begin by defining the multiplicity of a polynomial  $P$  at a point  $y \in \mathbb{R}$ .

**DEFINITION 1.** Let  $P(x) = a_0 + a_1x + \cdots + a_nx^n$  be a polynomial in  $\mathbb{R}$  of degree at most  $n$  and let  $y \in \mathbb{R}$ . The multiplicity of  $P$  at  $y$ , denoted by  $N(P, y)$ , is defined as the sum of the multiplicities of the roots of the polynomial  $P - y$ . We say that  $P$  has  $q$  points of multiplicity  $p$  when  $P - 1$  or  $P + 1$  intersects  $y = 0$  with multiplicity  $p$  at  $q$  points. For simplicity, we let  $N(P) := N(P, 1) + N(P, -1)$ .

Now, we provide a formal definition of extreme polynomial.

**DEFINITION 2.** Let  $P \in \mathcal{P}_n(\mathbb{R})$ . We say that  $P$  is extreme if it is an extreme point of the convex set  $B_n$ , or in other words, if  $P$  does not lie in the interior of any segment joining two polynomials of  $B_n$ . Alternatively,  $P$  is extreme if  $P = \frac{1}{2}(P_1 + P_2)$  with  $P_1, P_2 \in \mathcal{P}_n(\mathbb{R})$ , implies  $P = P_1 = P_2$ .

Our main goal in this section is to obtain, explicitly, all the extreme polynomials of  $B_n$ . We will use many times the following simple lemma, whose simple proof is spared to the interested reader.

**LEMMA 1.** *If  $P(x)$  is an extreme polynomial of  $B_n$ , then  $-P(x)$ ,  $P(-x)$  and  $-P(-x)$  are also extreme polynomials of  $B_n$ .*

Also, the main result of this section depends on the following theorem by Konheim and Rivlin (see [21]) that provides a characterization of the extreme polynomials of  $B_n$  for any  $n$ .

**THEOREM 5.** *A polynomial  $P \in \mathcal{P}_n(\mathbb{R})$  is an extreme point of  $B_n$  if, and only if,  $N(P) > n$ .*

**REMARK 2.** Theorem 5 states, basically, that the condition of being an extreme polynomial relies, heavily, on the sum of the multiplicities of the roots of the polynomials  $P \pm 1$ . Thus, we only need to find all the polynomials  $P$  of degree at most 3 such that  $N(P) > 3$ . To do so, we will proceed in a constructive manner considering all possibilities.

Notice that we have the following four cases that give all the possible extreme polynomials depending on the multiplicities of the roots of  $P + 1$  and  $P - 1$  (for the sake of simplicity, we will be stating the cases of the multiplicities for  $P$ , instead of considering the multiplicities of the roots of  $P - 1$  and  $P + 1$ ):

- (1)  $P$  has infinite multiplicity. Notice that in this case  $P \equiv \pm 1$ .
- (2)  $P$  has one point of multiplicity 3 and one point of multiplicity 1. In particular,  $N(P) = 4$ .
- (3)  $P$  has one point of multiplicity 2 and two points of multiplicity 1. In particular,  $N(P) = 4$ .
- (4)  $P$  has two points of multiplicity 2. In this case notice that  $4 \leq N(P) \leq 6$ .

Next, we state the main result of this section. The proof will be divided into three lemmas.

**THEOREM 6.** *Let  $P \in B_3$ . Then,  $P$  is an extreme polynomial of  $B_3$  if, and only if,  $P$  is of one of the following forms:*

- (i)  $P(x) = \pm 1$ ;

$$(ii) P(x) = \pm \left[ 1 - \frac{1}{4}(\pm x + 1)^3 \right];$$

$$(iii) P(x) = \pm(2x^2 - 1);$$

$$(iv) P(x) = \pm \left[ 1 - \frac{1}{(1-q^2)^2}(x-q)^2(4qx+2+2q^2) \right] \text{ or}$$

$$P(x) = \pm \left[ 1 + \frac{1}{(1-q^2)^2}(x+q)^2(4qx-2-2q^2) \right],$$

for every  $q \in (-\frac{1}{3}, 0)$ ;

$$(v) P(x) = \pm \left[ 1 + \frac{1}{(1+t)^2}(x-t)^2(x-1) \right] \text{ or}$$

$$P(x) = \pm \left[ 1 - \frac{1}{(1+t)^2}(x+t)^2(x+1) \right],$$

for every  $t \in (-\frac{1}{2}, 1)$ ;

$$(vi) P(x) = \pm \left[ 1 + \frac{4}{(s-r)^3}(x-r)^2 \left( x - \frac{3s-r}{2} \right) \right] \text{ or}$$

$$P(x) = \pm \left[ 1 - \frac{4}{(s-r)^3}(x+r)^2 \left( x + \frac{3s-r}{2} \right) \right],$$

for every  $-1 \leq r < s \leq 1$  such that  $s \geq \min \left\{ 3r + 2, \frac{r+2}{3} \right\}$ .

We begin by studying case (2) in Remark 2 which proves case (ii) in Theorem 6.

LEMMA 2. Assume  $P \in \mathbb{B}_3$  satisfies one of the following conditions:

- (a)  $P - 1$  has one root of multiplicity 3 and  $P + 1$  has one root of multiplicity 1, both in  $[-1, 1]$ .
- (b)  $P + 1$  has one root of multiplicity 3 and  $P - 1$  has one root of multiplicity 1, both in  $[-1, 1]$ .

Then  $P$  is an extreme polynomial in  $\mathbb{B}_3$  and  $P$  is of the form

$$P(x) = \pm \left[ 1 - \frac{1}{4}(\pm x + 1)^3 \right].$$

*Proof.* First of all, notice that case (b) can be deduced from case (a) by multiplying by  $-1$ , so assume case (a). Observe that the only possible numbers such that  $P - 1$  has multiplicity 3 in a single root are 1 and  $-1$  since otherwise  $\|P\| > 1$ . Suppose, without loss of generality by Lemma 1, that  $-1$  is a root of  $P - 1$  with multiplicity 3. Therefore  $P(-1) = 1$  and  $P(1) = -1$ . Indeed, since  $P + 1$  has one root of multiplicity 1, there exists a point in  $(-1, 1]$  such that  $P + 1$  vanishes. But if it were not 1, then  $\|P\| > 1$ . An sketch of such polynomials can be seen in Figure 1.

Using these assumptions, notice that  $P$  satisfies

$$P(x) - 1 = \alpha(x+1)^3.$$

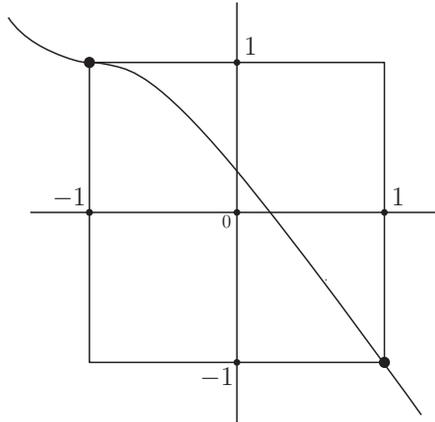


Figure 1: Sketch of a polynomial with a point of multiplicity 3 and a point of multiplicity 1.

Since  $P(1) = -1$ , we have

$$\alpha = -\frac{1}{4}.$$

Thus,

$$P(x) = -\frac{1}{4}(x+1)^3 + 1.$$

Now, we prove case (3) in Remark 2 which proves cases (iii), (iv) and (v) in Theorem 6.

LEMMA 3. Assume  $P \in \mathcal{B}_3$  satisfies one of the following conditions:

- (a)  $P - 1$  has one root of multiplicity 2 and  $P + 1$  has two roots of multiplicity 1, the three of them in  $[-1, 1]$ .
- (b)  $P + 1$  has one root of multiplicity 2 and  $P - 1$  has two roots of multiplicity 1, the three of them in  $[-1, 1]$ .

Then  $P$  is an extreme polynomial in  $\mathcal{B}_3$  and  $P$  has one of the following forms:

- (1)  $P(x) = \pm(-1 + 2x^2)$ .
- (2)  $P(x) = \pm \left[ 1 - \frac{2}{(1-q^2)^2} (\pm x - q)^2 (\pm 2qx + 1 + q^2) \right]$ , for every  $q \in (-\frac{1}{3}, 0)$ .
- (3)  $P(x) = \pm \left[ 1 + \frac{1}{(1+t)^2} (\pm x - t)^2 (\pm x - 1) \right]$ , for every  $t \in (-\frac{1}{2}, 1)$ .

*Proof.* Case (b) can be easily deduced from case (a) multiplying by -1. Assume that  $P + 1$  has two roots of multiplicity 1,  $P - 1$  has one root of multiplicity 2 and  $P$  is of degree 2, that is,  $P(x) = a + bx + cx^2$  with  $c \neq 0$ . Then  $P(1) = 1$  and  $P(-1) = 1$ .

Indeed, if there exists  $x_0 \in (-1, 1)$  such that  $P(x_0) = 1$ , then  $\|P\| > 1$ . Therefore, there exists  $x_0 \in (-1, 1)$  such that  $x_0$  is a root of  $P + 1$  with multiplicity 2. In particular,  $P$  achieves a local minimum at  $x_0$ . An sketch of such polynomials can be seen in Figure 2.

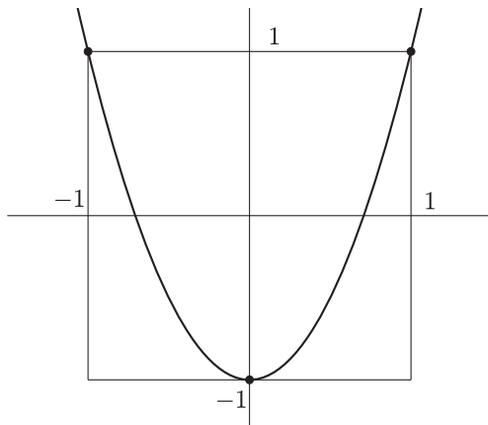


Figure 2: Sketch of a polynomial of degree 2 with a point of multiplicity 2 and two points of multiplicity 1.

Using these assumptions, we know that

$$1 = a + b + c,$$

$$1 = a - b + c.$$

Thus,  $b = 0$  and  $a + c = 1$ . We know that there exists  $x_0 \in (-1, 1)$  such that  $P(x_0) = -1 = a + cx_0^2$  is the absolute minimum of  $P$ . Thus, by applying the derivate  $P$ , we have

$$cx_0 = 0.$$

If  $x_0 \neq 0$ , then  $c = 0$  which is absurd. If  $x_0 = 0$ , then either  $c = 0$  (which is absurd) or  $c \neq 0$ . Thus,  $x_0 = 0$ , which implies that  $a = -1$  and  $c = 2$ , that is,

$$P(x) = -1 + 2x^2.$$

Now, assume that  $P$  is a polynomial of degree 3 such that  $P - 1$  has one root of multiplicity 2 and  $P + 1$  has two roots of multiplicity 1. Notice that if  $P + 1$  has two roots of multiplicity 1, then  $P(1) = -1$  and  $P(-1) = -1$ . Indeed, if there exists  $x_0 \in (-1, 1)$  such that  $P(x_0) = -1$ , then  $\|P\| > 1$ . Therefore, there exists  $x_1 \in (-1, 1)$  such that  $x_1$  is a root of  $P - 1$  with multiplicity 2. In particular,  $P$  has a local maximum at  $x_1$ , and this means that there exists  $x_2 \neq -1, 1, x_1$  such that  $P$  has a local minimum at  $x_2$ . Assume, without loss of generality, that  $x_2 \notin [-1, 1]$  and, in particular,  $x_2 > 1$  (the case where  $x_2$  is in the interior of  $[-1, 1]$  is treated at the end of this proof). Now, since  $P$  has a relative minimum at  $x_2$  and a relative maximum at  $x_1$ , there exists  $a > x_2$  such that  $P(a) = 1$ . An sketch of such polynomials can be seen in Figure 3.

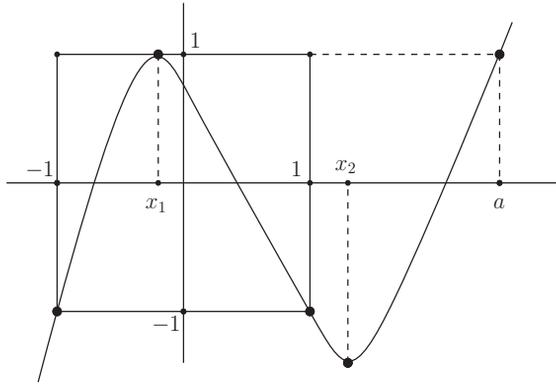


Figure 3: Sketch of a polynomial with a point of multiplicity 2 and two points of multiplicity 1 with a critical point outside  $[-1, 1]$ .

Using these assumptions,  $P$  satisfies

$$P(x) - 1 = \alpha(x - x_1)^2(x - a).$$

Since  $P(1) = -1$  and  $P(-1) = -1$ , we have

$$\begin{aligned} \alpha(1 - x_1)^2(1 - a) &= -2, \\ \alpha(-1 - x_1)^2(-1 - a) &= -2. \end{aligned} \quad (4)$$

Notice that  $\alpha > 0$ , because  $x - a \leq 0$  for every  $x \in [-1, 1]$  so that  $P(x) \leq 0$ . If we solve  $a$  in equations (4) in terms of  $x_1$  we have

$$a = -\frac{x_1^2 + 1}{2x_1}.$$

This implies that

$$-2 = P(1) - 1 = \alpha(1 - x_1)^2 \left( 1 + \frac{x_1^2 + 1}{2x_1} \right).$$

Therefore,

$$\alpha = -\frac{4x_1}{(1 - x_1^2)^2} > 0$$

if, and only if,  $x_1 < 0$ . Also,

$$a = -\frac{x_1^2 + 1}{2x_1} > 1,$$

for every  $x_1 \in (-1, 1)$ . Now, by applying the derivative to  $P$  and evaluating it at  $x_2$ , we have

$$0 = 2\alpha(x_2 - x_1)(x_2 - a) + \alpha(x_2 - x_1)^2. \quad (5)$$

Notice that we can solve  $x_2$  in terms of  $x_1$  and we have

$$x_2 = -\frac{1}{3x_1} > 1$$

if, and only if,  $x_1 > -\frac{1}{3}$ . Thus,

$$\begin{aligned} P(x) &= 1 - \frac{4x_1}{(1-x_1^2)^2} (x-x_1)^2 \left( x + \frac{x_1^2+1}{2x_1} \right) \\ &= 1 - \frac{2}{(1-x_1^2)^2} (x-x_1)^2 (2x_1x+1+x_1^2), \end{aligned}$$

for every  $x_1 \in (-\frac{1}{3}, 0)$ .

Finally, assume that  $P$  is a polynomial of degree 3 such that  $P+1$  has one root of multiplicity 2 and one root of multiplicity 1, and  $P-1$  has one root of multiplicity 1. Notice, using the same arguments as above, either  $P(1) = -1$  and  $P(-1) = 1$  or  $P(1) = 1$  and  $P(-1) = -1$ . By Lemma 1 we can assume that  $P(1) = -1$  and  $P(-1) = 1$  (the other case is proved changing  $x$  by  $-x$ ). Therefore, there exists  $x_1 \in (-1, 1)$  (the root of multiplicity 2 of  $P+1$ ) such that  $P$  has a local minimum at  $x_1$ . This construction also guarantees the existence of  $x_2 \neq -1, 1, x_1$  such that  $P$  has a local maximum at  $x_2$  which lies, necessarily in  $(x_1, 1)$ . In this case, notice that  $-1 < P(x_2) < 1$ . A sketch of such polynomials can be seen in Figure 4.

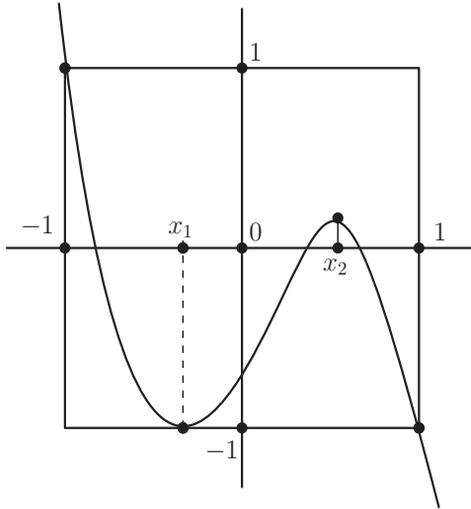


Figure 4: Sketch of a polynomial with a point of multiplicity 2 and two points of multiplicity 1, with a critical point inside  $(-1, 1)$ .

Using these assumptions,  $P$  is of the form

$$P(x) + 1 = \alpha(x-x_1)^2(x-1).$$

We begin by expressing  $x_2$  in terms of  $x_1$ . If we derive the previous formula, we have

$$P'(x) = 2\alpha(x - x_1)(x - 1) + \alpha(x - x_1)^2.$$

Using similar arguments as above, notice that  $\alpha < 0$ . Since  $P'(x_2) = 0$ ,  $\alpha < 0$  and  $x_1 < x_2$ , we have

$$x_2 = \frac{x_1 + 2}{3}.$$

Thus,

$$P(x_2) = -1 - \frac{4\alpha}{27}(1 - x_1)^3.$$

Since  $-1 < P(x_2) < 1$ , it can be checked that

$$-\frac{27}{2(1 - x_1)^3} < \alpha < 0.$$

Now, since  $P(-1) = 1$ , it can be proved that

$$\alpha = -\frac{1}{(1 + x_1)^2}.$$

So far we have found  $x_2$  and  $\alpha$  in terms of  $x_1$ . Now we are going to find the interval in which  $x_1$  lies. We know that

$$-\frac{27}{2(1 - x_1)^3} < -\frac{1}{(1 + x_1)^2} < 0.$$

Then it can be checked that

$$x_1 \in \left(-\frac{1}{2}, 1\right).$$

To sum it up, we have

$$P(x) = -1 - \frac{1}{(1 + x_1)^2}(x - x_1)^2(x - 1),$$

where  $x_1 \in \left(-\frac{1}{2}, 1\right)$ .

**REMARK 3.** It is interesting to investigate the limiting cases  $q = 0, -1/3$  and  $t = -1/2, 1$  in Lemma 3. If  $q = 0$  we obtain  $P(x) = \pm(1 - 2x^2)$ , which are the polynomials appearing in Lemma 3, case (1). If  $q = -1/3$ , we obtain the polynomials  $P(x) = \pm \left[1 + \frac{(\pm 3x + 1)(\pm 3x - 5)}{16}\right]$ , which satisfy  $N(P) = 5$  and will appear in Lemma 4 as extreme points with  $r = -1/3$  and  $s = 1$ . If  $t = -1/2$  then we obtain the Chebyshev polynomial  $P(x) = \pm(4x^3 - 3x)$ . The Chebyshev polynomial satisfies  $N(P) = 6$  and will appear as an extreme point in Lemma 4 with  $r = -1/2$  and  $s = 1/2$ . Finally, if  $t = 1$ , then we recover the polynomials  $P(x) = \pm \left[1 - \frac{(\pm x - 1)^3}{4}\right]$ , which appeared already in Lemma 2.

Finally, we tackle case (4) in Remark 2, which proves case (vi) in Theorem 6.

LEMMA 4. *If  $P \in \mathbb{B}_3$  is such that  $P - 1$  has one root of multiplicity 2 and  $P + 1$  has one root of multiplicity 2, both in  $[-1, 1]$ , then  $P$  is an extreme polynomial in  $\mathbb{B}_3$  and  $P$  is of the form*

$$P(x) = \pm \left[ 1 + \frac{4}{(s-r)^3} (\pm x - r)^2 \left( \pm x - \frac{3s-r}{2} \right) \right],$$

where  $-1 \leq r < s \leq 1$ ,  $3s - r \geq 2$  and  $3r - s \leq -2$ .

*Proof.* Without loss of generality, there exist  $-1 \leq x_1 < x_2 \leq 1$  such that  $P$  has a local maximum and a local minimum at  $x_1$  and at  $x_2$ , respectively, with  $P(x_1) = 1$  and  $P(x_2) = -1$ . The case where the minimum is attained before the maximum can be proved multiplying by  $-1$ . Therefore, there exist  $b \leq 1$  and  $a \geq 1$  such that  $P(b) = -1$  and  $P(a) = 1$ . Notice that  $a$  and  $b$  cannot be inside  $(-1, 1)$ , since otherwise we would have  $\|P\| > 1$ . An sketch of such polynomials can be seen in Figure 5.

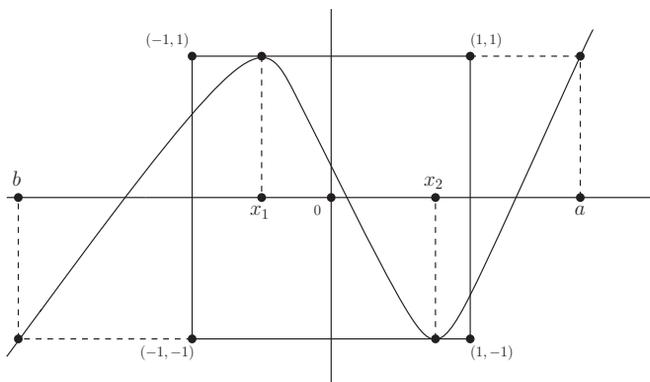


Figure 5: Sketch of a polynomial of degree 3 with two points of multiplicity 2.

Using the previous assumptions, notice that  $P$  satisfies

$$P(x) - 1 = \alpha(x - x_1)^2(x - a) \tag{6}$$

$$P(x) + 1 = \beta(x - x_2)^2(x - b). \tag{7}$$

We determine now  $\alpha$ ,  $\beta$ ,  $a$  and  $b$  in terms of  $x_1$  and  $x_2$ . If we derivate (6), we have

$$P'(x) = 2\alpha(x - x_1)(x - a) + \alpha(x - x_1)^2.$$

Thus,

$$0 = P'(x_2) = 2\alpha(x_2 - x_1)(x_2 - a) + \alpha(x_2 - x_1)^2.$$

Notice that  $x - a \leq 0$  for every  $x \in [-1, 1]$  for  $a \geq 1$ . Therefore  $\alpha > 0$  so that  $P(x) \leq 0$  for every  $x \in [-1, 1]$ . Also, since  $x_1 < x_2$ , we have

$$a = \frac{3x_2 - x_1}{2}.$$

The latter implies that  $3x_2 - x_1 \geq 2$ . On the other hand, using the same arguments for (7), we have

$$b = \frac{3x_1 - x_2}{2}.$$

Just as above we have  $3x_1 - x_2 \leq -2$ . Now, it is easy to see that

$$\begin{aligned} P(x) &= \alpha x^3 + (-2\alpha x_1 - \alpha a)x^2 + (\alpha x_1^2 + 2\alpha a x_1)x - \alpha a x_1^2 + 1, \\ P(x) &= \beta x^3 + (-2\beta x_2 - \beta b)x^2 + (\beta x_2^2 + 2\beta b x_2)x - \beta b x_2^2 - 1. \end{aligned}$$

Since it is the same polynomial, we have

$$\begin{aligned} \beta &= \alpha, \\ -2\alpha x_1 - \alpha a &= -2\beta x_2 - \beta b, \\ \alpha x_1^2 + 2\alpha a x_1 &= \beta x_2^2 + 2\beta b x_2, \\ -\alpha a x_1^2 + 1 &= -\beta b x_2^2 - 1. \end{aligned} \tag{8}$$

We are going to find  $\alpha$ . Since  $P(x_2) = -1$  and  $a = \frac{3x_2 - x_1}{2}$ , it can be easily seen that from

$$-1 = 1 + \alpha(x_2 - x_1)^2(x_2 - a),$$

we have

$$\alpha = \frac{4}{(x_2 - x_1)^3}.$$

It can also be checked by using these values of  $\alpha$ ,  $a$  and  $b$  in terms of  $x_1$  and  $x_2$  that the last three equations from (8) are satisfied. To sum it up,  $P$  is of the form

$$P(x) = 1 + \frac{4}{(x_2 - x_1)^3} (x - x_1)^2 \left( x - \frac{3x_2 - x_1}{2} \right),$$

where  $-1 \leq x_1 < x_2 \leq 1$ ,  $3x_2 - x_1 \geq 2$  and  $3x_1 - x_2 \leq -2$ .

Once we have obtained, constructively, an explicit formula for the extreme points of the unit ball of  $\mathcal{P}_3$ , in the following section we shall use this in order to obtain the Bernstein function for  $\mathcal{P}_3$ .

### 3. Bernstein's functions for $\mathcal{P}_3$

Using the description of  $\text{ext}(B_2)$  provided in [2] and the Krein-Milman approach, it is easy to prove (see for instance [30]) that

$$\mathcal{B}_2(x) = \begin{cases} \frac{1}{1-|x|} & \text{if } 0 \leq |x| \leq \frac{1}{2}, \\ 4|x| & \text{if } |x| \geq \frac{1}{2}. \end{cases} \tag{9}$$

Moreover, it can be easily verified that, for every  $x_0 \in \mathbb{R}$ , the following polynomials are extremal for  $\mathcal{B}_2$ :

1.  $P(x) = \pm \frac{1}{2(1-x_0)^2} [x^2 + 2(1-2x_0)x + 2x_0^2 - 1]$ , for  $0 \leq x_0 \leq 1/2$ .
2.  $P(x) = \pm \frac{1}{2(1+x_0)^2} [x^2 - 2(1+2x_0)x + 2x_0^2 - 1]$ , for  $-1/2 \leq x_0 \leq 0$ .
3.  $P(x) = 1 - 2x^2$  for  $|x_0| \geq 1/2$ .

Similarly, using Theorem 6:

**THEOREM 7.** *The Benstein's function for polynomials in  $\mathcal{P}_3(\mathbb{R})$  is given by*

$$\mathcal{B}_3(x) = \begin{cases} 3(1-4x^2) & \text{if } |x| \leq \frac{\sqrt{7}-2}{6}, \\ \frac{7\sqrt{7}+10}{9(|x|+1)} & \text{if } \frac{\sqrt{7}-2}{6} \leq |x| \leq \frac{2\sqrt{7}-1}{9}, \\ \frac{-16x^3}{(1-9x^2)(1-x^2)} & \text{if } \frac{2\sqrt{7}-1}{9} \leq |x| \leq \frac{1+2\sqrt{7}}{9}, \\ \frac{7\sqrt{7}-10}{9(1-|x|)} & \text{if } \frac{1+2\sqrt{7}}{9} \leq |x| \leq \frac{\sqrt{7}+2}{6}, \\ 3(4x^2-1) & \text{if } |x| \geq \frac{\sqrt{7}+2}{6}. \end{cases} \tag{10}$$

Moreover, for every  $x_0 \geq 0$ , the following polynomials are extremal for  $\mathcal{B}_3$ :

1.  $P(x) = \pm(4x^3 - 3x)$ , for  $x_0 \in \left[0, \frac{\sqrt{7}-2}{6}\right] \cup \left[\frac{\sqrt{7}+2}{6}, \infty\right)$ .

2.

$$P(x) = \pm \left\{ 1 + \frac{[3x - (4 - \sqrt{7})x_0 + \sqrt{7} - 1]^2 [3x + 4\sqrt{7} - 13 - (16 - 4\sqrt{7})x_0]}{2(4 - \sqrt{7})^3(1 + x_0)^3} \right\},$$

for  $x_0 \in \left[\frac{\sqrt{7}-2}{6}, \frac{2\sqrt{7}-1}{9}\right]$ .

3.

$$P(x) = \pm \left\{ 1 - \frac{2(2x_0x + 1 - 3x_0^2)^2 [(12x_0^2 - 4)x_0x + 9x_0^4 - 2x_0^2 + 1]}{(9x_0^4 - 10x_0^2 + 1)^2} \right\},$$

for  $x_0 \in \left[\frac{2\sqrt{7}-1}{9}, \frac{2\sqrt{7}+1}{9}\right]$ .

4.  $P(x) = \pm \left\{ 1 + \frac{27[x - (4 + \sqrt{7})x_0 + \sqrt{7} + 3]^2(x-1)}{2(4 + \sqrt{7})^3(1-x_0)^3} \right\}$ , for  $x_0 \in \left[\frac{2\sqrt{7}+1}{9}, \frac{\sqrt{7}+2}{6}\right]$ .

Observe that if  $x_0 \leq 0$  and  $P$  is extremal for  $|x_0|$ , then  $\pm P(-x)$  are extremal for  $x_0$ .

*Proof.* Since  $\mathcal{B}_3$  is clearly an even function, it suffices to calculate  $\mathcal{B}_3(x_0)$  for  $x_0 \geq 0$ . By direct inspection, it follows that the maximum of the absolute value of the derivatives with respect to  $x$  at  $x_0$  of the extreme polynomials of type (i), (ii) and (iii) in Theorem 6 is

$$\max \left\{ \frac{3}{4}(1+x_0)^2, 4x_0 \right\} = \begin{cases} \frac{3}{4}(1+x_0)^2 & \text{if } x_0 \in [0, 1/3], \\ 4x_0 & \text{if } x_0 \geq 1/3. \end{cases} \tag{11}$$

Now we study the polynomials of type (iv) in Theorem 6, whose derivatives (up to a sign) are given by the functions

$$f_{x_0}^+(q) = \frac{4(3q^2x_0 - 3qx_0^2 + q - x_0)}{(1-q^2)^2},$$

$$f_{x_0}^-(q) = \frac{4(3q^2x_0 + 3qx_0^2 + q - x_0)}{(1-q^2)^2}.$$

We now want to optimize  $|f_{x_0}^\pm(q)|$  as  $q \in (-1/3, 0)$  or, equivalently (by continuity), as  $q \in [-1/3, 0]$ . The critical points of  $f_{x_0}^\pm(q)$  are obtained from the equation

$$\frac{df_{x_0}^\pm(q)}{dq} = \frac{4(3q^2 + 1)(-3x_0^2 \pm 2qx_0 + 1)}{(1-q^2)^3} = 0,$$

which yields  $q_1 = \frac{3x_0^2-1}{2x_0}$  and  $q_2 = -\frac{3x_0^2-1}{2x_0}$  as the unique critical points of  $f_{x_0}^+$  and  $f_{x_0}^-$ , respectively. Notice that  $q_1 \in [-1/3, 0]$  if, and only if,  $x_0 \in \left[ \frac{-1+2\sqrt{7}}{9}, \frac{\sqrt{3}}{3} \right]$ , whereas  $q_2 \in [-1/3, 0]$  if, and only if,  $x_0 \in \left[ \frac{\sqrt{3}}{3}, \frac{1+2\sqrt{7}}{9} \right]$ . Hence, if  $x_0 \in \left[ 0, \frac{-1+2\sqrt{7}}{9} \right] \cup \left[ \frac{1+2\sqrt{7}}{9}, 1 \right]$ , we have

$$\begin{aligned} \sup_{q \in [-1/3, 0]} |f_{x_0}^\pm(q)| &= \max\{|f_{x_0}^\pm(0), |f_{x_0}^\pm(-1/3)|\} \\ &= \max \left\{ 4x_0, \left| -\frac{81}{16}x_0^2 \pm \frac{27}{8}x_0 + \frac{27}{16} \right| \right\} \\ &= \begin{cases} -\frac{81}{16}x_0^2 + \frac{27}{8}x_0 + \frac{27}{16} & \text{if } x_0 \in \left[ 0, \frac{-1+2\sqrt{7}}{9} \right], \\ \frac{81}{16}x_0^2 - \frac{27}{8}x_0 - \frac{27}{16} & \text{if } x_0 \in \left[ \frac{1+2\sqrt{7}}{9}, 1 \right]. \end{cases} \end{aligned}$$

On the other hand, it is easily checked that

$$\left| f_{x_0}^\pm \left( \pm \frac{3x_0^2-1}{2x_0} \right) \right| = \frac{-16x_0^3}{9x_0^4 - 10x_0^2 + 1} \geq \max \left\{ 4x_0, \left| -\frac{81}{16}x_0^2 \pm \frac{27}{8}x_0 + \frac{27}{16} \right| \right\}$$

whenever  $x_0 \in \left[ \frac{-1+2\sqrt{7}}{9}, \frac{1+2\sqrt{7}}{9} \right]$ , from which it follows that

$$\sup_{q \in [-1/3, 0]} |f_{x_0}^\pm(q)| = \begin{cases} -\frac{81}{16}x_0^2 + \frac{27}{8}x_0 + \frac{27}{16} & \text{if } x_0 \in \left[ 0, \frac{-1+2\sqrt{7}}{9} \right], \\ \frac{-16x_0^3}{9x_0^4 - 10x_0^2 + 1} & \text{if } x_0 \in \left[ \frac{-1+2\sqrt{7}}{9}, \frac{1+2\sqrt{7}}{9} \right], \\ \frac{81}{16}x_0^2 - \frac{27}{8}x_0 - \frac{27}{16} & \text{if } x_0 \in \left[ \frac{1+2\sqrt{7}}{9}, 1 \right]. \end{cases} \tag{12}$$

Now we are going to optimize the first derivatives at  $x_0$  of the polynomials of type (v) in Theorem 6, which are given by the functions:

$$g_{x_0}^+(t) = \frac{(t - x_0)(t - 3x_0 + 2)}{(1 + t)^2},$$

$$g_{x_0}^-(t) = \frac{(t + x_0)(t + 3x_0 + 2)}{(1 + t)^2},$$

for  $t \in (-1/2, 1)$ . Observe that, in order to maximize  $|g_{x_0}^\pm(t)|$  as  $t \in (-1/2, 1)$ , we might as well consider the interval  $[-1/2, 1]$  instead of  $(-1/2, 1)$ . The critical points of  $g_{x_0}^\pm(t)$  can be obtained from the equations

$$\frac{dg_{x_0}^+(t)}{dt} = \frac{2(-3x_0^2 + 2tx_0 + 1)}{(1 + t)^3} = 0,$$

$$\frac{dg_{x_0}^-(t)}{dt} = \frac{2(-3x_0^2 - 2tx_0 + 1)}{(1 + t)^3} = 0.$$

Hence,  $t_1 = \frac{3x_0^2 - 1}{2x_0}$  and  $t_2 = -\frac{3x_0^2 - 1}{2x_0}$  are the unique critical points of  $g_{x_0}^+(t)$  and  $g_{x_0}^-(t)$ , respectively. Also,  $t_1 \in [-1/2, 1]$  if, and only if,  $x_0 \in \left[\frac{-1 + \sqrt{13}}{6}, 1\right]$ , whereas  $t_2 \in [-1/2, 1]$  is equivalent to  $x_0 \in \left[\frac{1}{3}, \frac{1 + \sqrt{13}}{6}\right]$ . Treating  $|g_{x_0}^+(t)|$  and  $|g_{x_0}^-(t)|$  separately, we find

$$\sup_{t \in [-1/2, 1]} |g_{x_0}^+(t)| = \begin{cases} \max\{|g^+(-1/2)|, |g^+(1)|\} & \text{if } x_0 \in \left[0, \frac{-1 + \sqrt{13}}{6}\right], \\ \max\{|g^+(-1/2)|, |g^+(1)|, |g^+(t_1)|\} & \text{if } x_0 \in \left[\frac{-1 + \sqrt{13}}{6}, 1\right], \end{cases}$$

$$\sup_{t \in [-1/2, 1]} |g_{x_0}^-(t)| = \begin{cases} \max\{|g^-(-1/2)|, |g^-(1)|\} & \text{if } x_0 \in \left[0, \frac{1}{3}\right] \cup \left[\frac{1 + \sqrt{13}}{6}, 1\right], \\ \max\{|g^-(-1/2)|, |g^+(1)|, |g^-(t_2)|\} & \text{if } x_0 \in \left[\frac{1}{3}, \frac{1 + \sqrt{13}}{6}\right]. \end{cases}$$

From Figure 6 we immediately deduce that

$$\begin{aligned} \sup_{t \in [-1/2, 1]} |g_{x_0}^\pm(t)| &= \begin{cases} |g_{x_0}^\pm(-1/2)| & \text{if } x_0 \in [0, \lambda_0] \cup \left[\frac{1 + \sqrt{13}}{6}, 1\right], \\ |g_{x_0}^\pm(t_2)| & \text{if } x_0 \in \left[\lambda_0, \frac{1 + \sqrt{13}}{6}\right], \end{cases} \\ &= \begin{cases} -12x_0^2 + 3 & \text{if } x_0 \in [0, \lambda_0], \\ \frac{(x_0 + 1)^2}{-3x_0^2 + 2x_0 + 1} & \text{if } x_0 \in \left[\lambda_0, \frac{1 + \sqrt{13}}{6}\right], \\ 12x_0^2 - 3 & \text{if } x_0 \in \left[\frac{1 + \sqrt{13}}{6}, 1\right], \end{cases} \end{aligned} \tag{13}$$

where  $\lambda_0 \approx 0.3651$  is the smallest root of the equation  $|g_{x_0}^\pm(-1/2)| = |g_{x_0}^\pm(t_2)|$  in  $[0, 1]$  (see Figure 6). It is worth mentioning that  $\lambda_0$  does not play a crucial role in the final result.

We finally arrive at the analysis of the polynomials of type (vi), parametrized in the set

$$\mathcal{R} = \left\{ (r, s) : -1 \leq r < s \leq 1, s \geq \min \left\{ 3r + 2, \frac{r + 2}{3} \right\} \right\}.$$

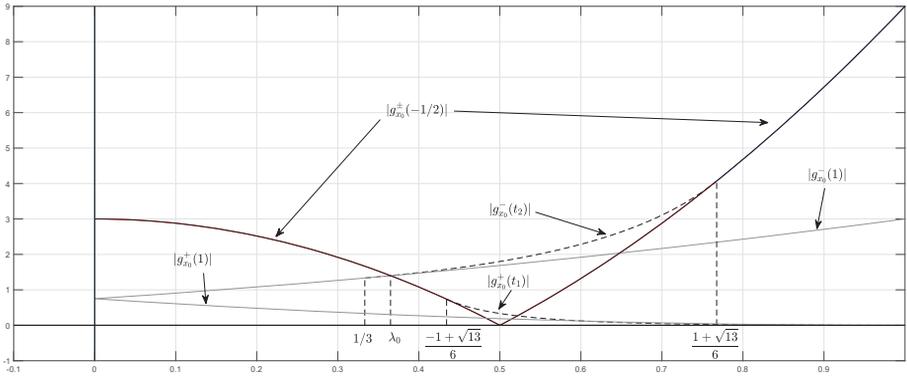


Figure 6: Visual calculation of  $\sup_{t \in [-1/2, 1]} |g_{x_0}^\pm(t)|$ .

See Figure 7 for a representation of  $\mathcal{R}$ . The first derivative of the polynomials of type

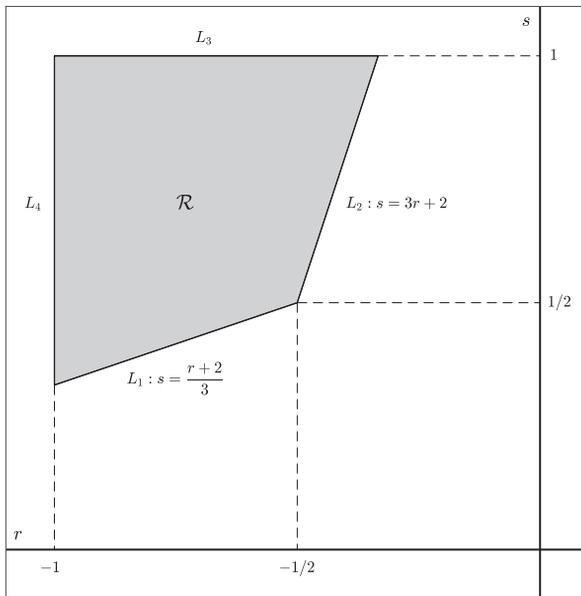


Figure 7: Representation of the parameter set  $\mathcal{R}$  in the  $(r, s)$  plane corresponding to the polynomials of type (vi) in Theorem 6.

(vi), up to a sign, are provided by the functions:

$$h_{x_0}^+(r, s) = \frac{-12(r - x_0)(s - x_0)}{(s - r)^3},$$

$$h_{x_0}^-(r, s) = \frac{12(r+x_0)(s+x_0)}{(s-r)^3},$$

for  $(r, s) \in \mathcal{R}$ .

We want to optimize  $|h_{x_0}^\pm(r, s)|$  in  $\mathcal{R}$ . In search for the critical points of  $h_{x_0}^+(r, s)$  and  $h_{x_0}^-(r, s)$ , we calculate the gradients:

$$\begin{aligned} \nabla h_{x_0}^+(r, s) &= \left( \frac{12(s-x_0)(2r+s-3x_0)}{(r-s)^4}, -\frac{(r-x_0)(r+2s-3x_0)}{(r-s)^4} \right), \\ \nabla h_{x_0}^-(r, s) &= \left( -\frac{12(s+x_0)(2r+s+3x_0)}{(r-s)^4}, \frac{(r+x_0)(r+2s+3x_0)}{(r-s)^4} \right). \end{aligned}$$

It is straightforward to see that  $\nabla h_{x_0}^\pm(r, s) \neq (0, 0)$  in  $\mathcal{R}$ . Hence, the maximum of  $|h_{x_0}^\pm(r, s)|$  in  $\mathcal{R}$  is attained in  $\partial\mathcal{R}$ . Due to the symmetry of  $\mathcal{R}$ ,  $h_{x_0}^+(r, s)$  and  $h_{x_0}^-(r, s)$  with respect to the line  $r = -s$ , it is enough to maximize  $|h_{x_0}^+(r, s)|$  in  $\partial\mathcal{R}$ . Notice that the restriction of  $|h_{x_0}^+(r, s)|$  to the segments  $L_3$  and  $L_4$  is a monotone function. However, the restrictions of  $h_{x_0}^+(r, s)$  to  $L_1$  and  $L_2$ , respectively,  $h_1$  and  $h_2$ , may have local extrema. We have

$$\begin{aligned} h_1(r) &= -\frac{27(r-x_0)(r-3x_0+2)}{2(r-1)^3} \quad \text{with } r \in \left[-1, -\frac{1}{2}\right], \\ h_2(r) &= \frac{3(r-x_0)(3r+2-x_0)}{2(r+1)^3} \quad \text{with } r \in \left[-\frac{1}{2}, -\frac{1}{3}\right], \end{aligned}$$

and

$$\begin{aligned} h_1'(r) &= \frac{27[r^2 + (-8x_0 + 6)r + 9x_0^2 - 10x_0 + 2]}{2(r-1)^4}, \\ h_2'(r) &= \frac{3[-3r^2 + (8x_0 + 2)r - 3x_0^2 + 2x_0 + 2]}{2(r+1)^4}. \end{aligned}$$

One checks easily that  $h_1(r)$  has two critical points,  $r_1^-$  and  $r_1^+$ , whereas  $h_2(r)$  has two critical points too,  $r_2^-$  and  $r_2^+$ , all given by

$$\begin{aligned} r_1^\pm &= (4 \pm \sqrt{7})x_0 \mp \sqrt{7} - 3, \\ r_2^\pm &= \frac{4 \pm \sqrt{7}}{3}x_0 + \frac{\pm\sqrt{7} + 1}{3}. \end{aligned}$$

Clearly,  $r_1^-$  is never in  $[-1, -\frac{1}{2}]$  and  $r_2^+$  never lies in  $[-\frac{1}{2}, -\frac{1}{3}]$ . Also,  $r_1^+ \in [-1, -\frac{1}{2}]$  if, and only if,  $x_0 \in I_1 := [\frac{1+2\sqrt{7}}{9}, \frac{2+\sqrt{7}}{6}]$ , and  $r_2^- \in [-\frac{1}{2}, -\frac{1}{3}]$  if, and only if,  $x_0 \in I_2 := [\frac{-2+\sqrt{7}}{6}, \frac{-1+2\sqrt{7}}{9}]$ . As a consequence of the latter,  $\sup_{(r,s) \in \mathcal{R}} |h_{x_0}^\pm(r, s)|$  is given by

$$\begin{cases} \max\{|h_{x_0}^+(-1, \frac{1}{3})|, |h_{x_0}^+(-1, 1)|, |h_{x_0}^+(-\frac{1}{3}, 1)|, |h_{x_0}^+(-\frac{1}{2}, \frac{1}{2})|\} & \text{if } x_0 \notin I_1 \cup I_2, \\ \max\{|h_{x_0}^+(-1, \frac{1}{3})|, |h_{x_0}^+(-1, 1)|, |h_{x_0}^+(-\frac{1}{3}, 1)|, |h_{x_0}^+(-\frac{1}{2}, \frac{1}{2})|, |h_2(r_2^-)|\} & \text{if } x_0 \in I_2, \\ \max\{|h_{x_0}^+(-1, \frac{1}{3})|, |h_{x_0}^+(-1, 1)|, |h_{x_0}^+(-\frac{1}{3}, 1)|, |h_{x_0}^+(-\frac{1}{2}, \frac{1}{2})|, |h_1(r_1^+)|\} & \text{if } x_0 \in I_1. \end{cases}$$

On the other hand

$$\begin{aligned} \left| h_{x_0}^+ \left( -1, \frac{1}{3} \right) \right| &= \left| \frac{27(x_0 + 1)(3x_0 - 1)}{16} \right|, \\ |h_{x_0}^+(-1, 1)| &= \left| \frac{3(x_0 - 1)(x_0 + 1)}{2} \right|, \\ \left| h_{x_0}^+ \left( -\frac{1}{3}, 1 \right) \right| &= \left| \frac{27(x_0 - 1)(3x_0 + 1)}{16} \right|, \\ \left| h_{x_0}^+ \left( -\frac{1}{2}, \frac{1}{2} \right) \right| &= |12x_0^2 - 3|, \\ |h_2(r_2^-)| &= \frac{7\sqrt{7} + 10}{9(1 + x_0)}, \\ |h_1(r_1^+)| &= \frac{7\sqrt{7} - 10}{9(1 - x_0)}. \end{aligned}$$

Comparing these functions we arrive at the following conclusion (see Figure 8):

$$\sup_{(r,s) \in \mathcal{R}} |h_{x_0}^\pm(r,s)| = \begin{cases} 3 - 12x_0^2 & \text{if } 0 \leq x_0 \leq \frac{\sqrt{7}-2}{6}, \\ \frac{7\sqrt{7}+10}{9(1+x_0)} & \text{if } \frac{\sqrt{7}-2}{6} \leq x_0 \leq \frac{-1+2\sqrt{7}}{9}, \\ \frac{27(1-x_0)(3x_0+1)}{16} & \text{if } \frac{-1+2\sqrt{7}}{9} \leq x_0 \leq \frac{\sqrt{3}}{3}, \\ \frac{27(1+x_0)(3x_0-1)}{16} & \text{if } \frac{\sqrt{3}}{3} \leq x_0 \leq \frac{1+2\sqrt{7}}{9}, \\ \frac{7\sqrt{7}-10}{9(1-x_0)} & \text{if } \frac{1+2\sqrt{7}}{9} \leq x_0 \leq \frac{\sqrt{7}+2}{6}, \\ 12x_0^2 - 3 & \text{if } \frac{\sqrt{7}+2}{6} \leq x_0 \leq 1. \end{cases} \quad (14)$$

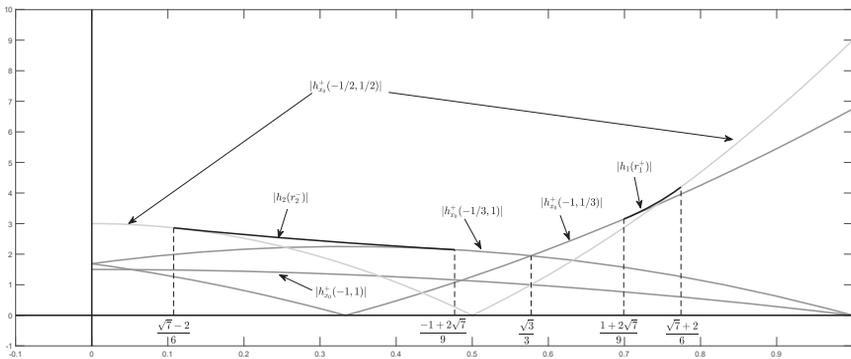


Figure 8: Visual calculation of  $\sup_{(r,s) \in \mathcal{R}} |h_{x_0}^\pm(r,s)|$ .

Finally, (10) follows by comparing the functions appearing in (11), (12), (13) and (14) (see Figure 9).

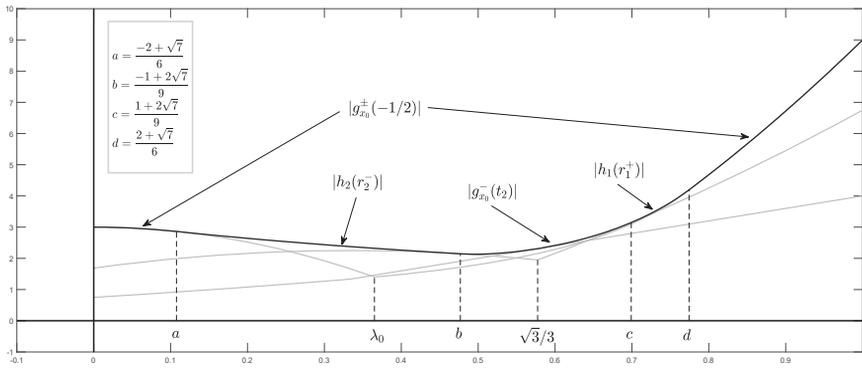


Figure 9: Comparison of the functions appearing in (11), (12), (13) and (14).

As for the extremal polynomials, we may assume that  $x_0 \geq 0$ . It is obvious that  $P(x) = \pm(4x^3 - 3x)$  are extremal for all  $x_0 \in [0, \frac{\sqrt{7}-2}{6}] \cup [\frac{\sqrt{7}+2}{6}, \infty)$ , which accounts for case (1). Now, if  $x_0 \in [\frac{\sqrt{7}-2}{6}, \frac{-1+2\sqrt{7}}{9}]$ , we have seen in this proof that  $\mathcal{B}_3(x_0)$  is attained for the polynomials of type (vi) of the form  $P(x) = \pm \left[ 1 + \frac{4}{(s-r)^3}(x-r)^2 \left( x - \frac{3s-r}{2} \right) \right]$ , with  $r = r_2^- = \frac{4-\sqrt{7}}{3}x_0 + \frac{-\sqrt{7}+1}{3}$  and  $s = 3r_2^- + 2 = (4 - \sqrt{7})x_0 - \sqrt{7} + 3$ . By substitution, we arrive at the polynomials in (2). If  $x_0 \in [\frac{2\sqrt{7}-1}{9}, \frac{2\sqrt{7}+1}{9}]$ , then we know already that  $\mathcal{B}_3(x_0)$  is attained for the polynomials of type (iv) given by  $P(x) = \pm \left[ 1 - \frac{1}{(1-q^2)^2}(x-q)^2(4qx+2+2q^2) \right]$  with  $q = q_2 = -\frac{3x_0^2-1}{2x_0}$  which are exactly the polynomials in case (3) of the statement. To finish, if  $x_0 \in [\frac{2\sqrt{7}+1}{9}, \frac{\sqrt{7}+2}{6}]$ , we have proved already that  $\mathcal{B}_3(x_0)$  is attained for the polynomials of type (vi) of the form  $P(x) = \pm \left[ 1 + \frac{4}{(s-r)^3}(x-r)^2 \left( x - \frac{3s-r}{2} \right) \right]$  with  $r = r_1^+ = (4 + \sqrt{7})x_0 - \sqrt{7} - 3$  and  $s = \frac{r_1^++2}{3} = \frac{4+\sqrt{7}}{3}x_0 - \frac{\sqrt{7}+1}{3}$ . Direct substitution leads us to the polynomials appearing in case (4) of the statement.

We refer to Figures 10 and 11 for the graphs of  $\mathcal{B}_2(x)$  and  $\mathcal{B}_3(x)$ , respectively, compared to the Bernstein’s estimate given in Theorem 3.

Using once more the Krein-Milman approach, we can also obtain  $\mathcal{B}_{3,2}(x)$ :

**THEOREM 8.** *The Benstein’s function corresponding to the second derivative of*

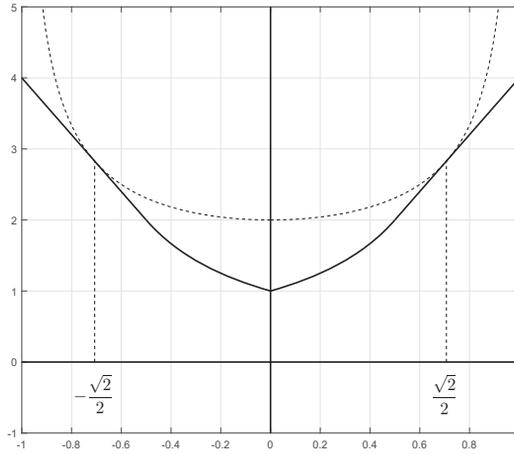


Figure 10: Sketch of  $\mathcal{B}_2(x)$  (solid line) compared to Bernstein's estimate  $2/\sqrt{1-x^2}$  (dashed line).

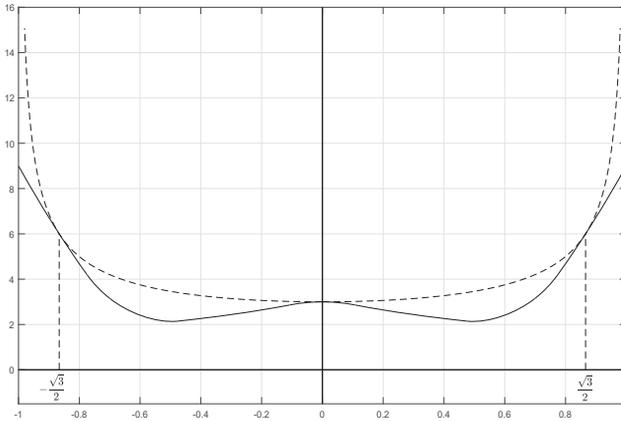


Figure 11: Sketch of  $\mathcal{B}_3(x)$  (solid line) compared to Bernstein's estimate  $9/\sqrt{1-x^2}$  (dashed line).

the polynomials in  $\mathcal{P}_3(\mathbb{R})$  is given by

$$\mathcal{B}_{3,2}(x) = \begin{cases} \frac{4}{1-9x^2} & \text{if } |x| \leq \frac{1}{9}, \\ \frac{9(|x|-1)^2}{24|x|} & \text{if } \frac{1}{9} \leq |x| \leq \frac{1}{3}, \\ 24|x| & \text{if } |x| \geq \frac{1}{3}. \end{cases} \tag{15}$$

Moreover, for every  $x_0 \in \mathbb{R}$ , the following polynomials are extremal for  $\mathcal{B}_{3,2}$ :

1.

$$P(x) = \pm \left[ 1 - \frac{2}{(1-9x_0^2)^2} (x-3x_0)^2 (6x_0x + 9x_0^2 + 1) \right], \quad \text{for } x_0 \in [0, 1/9],$$

$$P(x) = \pm \left[ 1 + \frac{2}{(1-9x_0^2)^2} (x+3x_0)^2 (6x_0x - 9x_0^2 - 1) \right], \quad \text{for } x_0 \in [-1/9, 0].$$

2.

$$P(x) = \pm \left[ 1 + \frac{2(4x-9x_0+5)^2(x-1)}{27(1-x_0)^3} \right], \quad \text{for } x_0 \in [1/9, 1/3],$$

$$P(x) = \pm \left[ 1 - \frac{2(4x+9x_0-5)^2(x+1)}{27(1-x_0)^3} \right], \quad \text{for } x_0 \in [-1/3, -1/9].$$

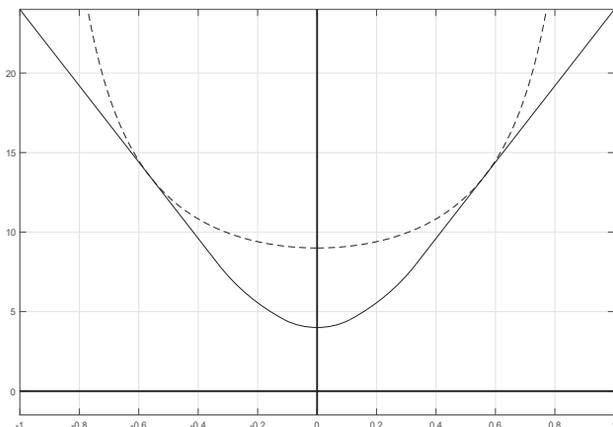
3.  $P(x) = \pm(4x^3 - 3x)$  for  $|x_0| \geq 1/3$ .

Figure 12: Sketch of  $\mathcal{B}_{3,2}(x)$  (solid line) compared to the Duffin and Schaeffer's estimate  $\mathcal{M}_{3,2}(x) = \frac{3\sqrt{-8x^2+9}}{\sqrt{(1-x^2)^3}}$  (dashed line).

*Proof.* Since  $\mathcal{B}_{3,2}$  is an even function, it suffices to calculate  $\mathcal{B}_{3,2}(x_0)$  for  $x_0 \geq 0$ . By direct inspection, it follows that the maximum of the absolute value of the derivatives with respect to  $x$  at  $x_0$  of the extreme polynomials of type (i), (ii) and (iii) in Theorem 6 is  $\max\{4, \frac{3}{2}(x_0+1)\} = 4$ . Now we turn our attention to the polynomials of type (iv) in Theorem 6. We find two kinds of polynomials. In order to maximize the absolute value of their second derivatives with respect to  $x$  at  $x_0$  we just need to optimize the functions

$$f_{x_0}^+(q) = \frac{4}{(1-q^2)^2} (3q^2 - 6qx_0 + 1),$$

$$f_{x_0}^-(q) = \frac{4}{(1-q^2)^2} (3q^2 + 6qx_0 + 1),$$

with  $q \in (-\frac{1}{3}, 0)$ . Since  $\frac{df_{x_0}^\pm}{dq}(q) = -\frac{8(3q^2+1)(q\pm 3x_0)}{(q^2-1)^3}$ , we have that  $3x_0$  and  $-3x_0$  are critical points of  $f_{x_0}^+(q)$  and  $f_{x_0}^-(q)$  respectively. Obviously,  $3x_0 \notin (-\frac{1}{3}, 0)$  since  $x_0 > 0$ , whereas  $-3x_0 \in (-\frac{1}{3}, 0)$  is equivalent to  $x_0 \in (0, \frac{1}{9})$ . Hence

$$\begin{aligned} \sup_{q \in (-1/3, 0)} |f_{x_0}^\pm(q)| &= \begin{cases} \max\{|f^\pm(0)|, |f^\pm(-1/3)|, |f^-( -3x_0)|\} & \text{if } x_0 \in [0, \frac{1}{9}], \\ \max\{|f^\pm(0)|, |f^\pm(-1/3)|\} & \text{if } x_0 \geq \frac{1}{9}. \end{cases} \\ &= \begin{cases} \frac{4}{1-9x_0^2} & \text{if } x_0 \in [0, \frac{1}{9}], \\ \frac{81}{8}x_0 + \frac{27}{8} & \text{if } x_0 \geq \frac{1}{9}. \end{cases} \end{aligned} \tag{16}$$

Observe that the function (16) is greater than 4, and hence the contribution of the polynomials of type (i), (ii) and (iii) to the calculation of  $\mathcal{B}(x_0)$  can be neglected.

Now we consider the polynomials of type (v) in Theorem 6. By taking their second derivative with respect to  $x$ , we need to optimize the functions

$$\begin{aligned} g_{x_0}^+(t) &= \frac{2(-2t + 3x_0 - 1)}{(t + 1)^2}, \\ g_{x_0}^-(t) &= \frac{2(-2t - 3x_0 - 1)}{(t + 1)^2}, \end{aligned}$$

for  $t \in (-1/2, 1)$ . Since  $\frac{dg_{x_0}^\pm}{dt}(t) = \frac{4(t \mp 3x_0)}{(t+1)^3}$ , it is obvious that  $g_{x_0}^+(t)$  and  $g_{x_0}^-$  have critical points at  $t = 3x_0$  and  $t = -3x_0$  respectively. Taking into consideration that  $x_0 \geq 0$ , we have that  $3x_0 \in (-1/2, 1)$  and  $-3x_0 \in (-1/2, 1)$  are equivalent, respectively, to  $x_0 \in (0, 1/3)$  and  $x_0 \in (0, 1/6)$ . Therefore

$$\begin{aligned} \sup_{t \in (-1/2, 1)} |g_{x_0}^\pm(t)| &= \begin{cases} \max\{|g_{x_0}^\pm(-1/2)|, |g_{x_0}^\pm(1)|, |g_{x_0}^\pm(\pm 3x_0)|\} & \text{if } x_0 \in [0, \frac{1}{6}], \\ \max\{|g_{x_0}^\pm(-1/2)|, |g_{x_0}^\pm(1)|, |g_{x_0}^+(3x_0)|\} & \text{if } x_0 \in [\frac{1}{6}, \frac{1}{3}], \\ \max\{|g_{x_0}^\pm(-1/2)|, |g_{x_0}^\pm(1)|\} & \text{if } x_0 \geq \frac{1}{3}. \end{cases} \\ &= \begin{cases} \max\left\{24x_0, \frac{3}{2}(1 \pm x_0), \frac{1}{1 \pm 3x_0}\right\} & \text{if } x_0 \in [0, \frac{1}{6}], \\ \max\left\{24x_0, \frac{3}{2}(1 \pm x_0), \frac{1}{1 + 3x_0}\right\} & \text{if } x_0 \in [\frac{1}{6}, \frac{1}{3}], \\ \max\left\{24x_0, \frac{3}{2}(1 \pm x_0)\right\} & \text{if } x_0 \geq \frac{1}{3}. \end{cases} \\ &= \begin{cases} \frac{2}{1-3x_0} & \text{if } x_0 \in [0, \frac{1}{6}], \\ 24x_0 & \text{if } x_0 \geq \frac{1}{6}. \end{cases} \end{aligned} \tag{17}$$

There remains to optimize the second derivative of the polynomials of type (vi) in Theorem 6. The second derivative of those polynomials, up to the sign, are given by

$$h_{x_0}^+(r, s) = \frac{12(r + s - 2x_0)}{(r - s)^3},$$

$$h_{x_0}^-(r, s) = \frac{12(r + s + 2x_0)}{(r - s)^3}.$$

It is straightforward that

$$\begin{aligned} \nabla h_{x_0}^+(r, s) &= \left( -\frac{24(r + 2s - 3x_0)}{(r - s)^4}, \frac{24(2r + s - 3x_0)}{(r - s)^4} \right), \\ \nabla h_{x_0}^-(r, s) &= \left( \frac{24(2r + s + 3x_0)}{(r - s)^4}, -\frac{24(r + 2s + 3x_0)}{(r - s)^4} \right), \end{aligned}$$

from which the unique critical points of  $h_{x_0}^+(r, s)$  and  $h_{x_0}^-(r, s)$  are  $\pm(x_0, x_0)$ , none of which lies in the region  $\mathcal{R}$  of the  $(r, s)$  plane (see Figure 7). The latter shows that both  $|h_{x_0}^+(r, s)|$  and  $|h_{x_0}^-(r, s)|$  are maximized in  $\partial\mathcal{R}$ . We just need to study  $|h_{x_0}^+(r, s)|$  since the calculations for  $|h_{x_0}^-(r, s)|$  are almost identical. Also, due to the symmetry of  $|h_{x_0}^+(r, s)|$  and  $\mathcal{R}$  with respect to the line  $s = -r$ , we only need to optimize  $|h_{x_0}^+(r, s)|$  on the segments  $L_1$  and  $L_4$  in Figure 7. Let

$$\begin{aligned} h_1(r) &= h_{x_0}^+(r, (r + 2)/3) = \frac{27(4r - 6x_0 + 2)}{2(r - 1)^3}, \\ h_2(r) &= h_{x_0}^+(-1, s) = \frac{12(2x_0 - s + 1)}{(s + 1)^3}, \end{aligned}$$

with  $r \in [-1, -1/2]$  and  $s \in [1/3, 1]$ , be the restrictions of  $h_{x_0}^+$  to  $L_1$  and  $L_4$ , respectively. It is elementary to check that

$$\begin{aligned} h_1'(r) &= -\frac{27(4r - 9x_0 + 5)}{(r - 1)^4}, \\ h_2'(r) &= -\frac{24(3x_0 - s + 2)}{(s + 1)^4}, \end{aligned}$$

from which  $h_1$  and  $h_2$  have a unique critical point at  $r_0 = \frac{9x_0 - 5}{4}$  and  $s_0 = 3x_0 + 2$ , respectively. Actually  $r_0$  happens to be in  $[-1, 1/2]$  if, and only if,  $x_0 \in [1/9, 1/3]$ , whereas  $s_0$  never lies in  $[1/3, 1]$ . Also,  $h_1(-1) = h_2(1/3)$ . Then

$$\sup_{(r,s) \in \partial\mathcal{R}} |h_{x_0}^+(r, s)| = \begin{cases} \max\{|h_1(-1)|, |h_1(\frac{-1}{2})|, |h_2(1)|, |h_1(r_0)|\} & \text{if } x_0 \in [\frac{1}{9}, \frac{1}{3}], \\ \max\{|h_1(-1)|, |h_1(\frac{-1}{2})|, |h_2(1)|\} & \text{otherwise.} \end{cases}$$

Observe that  $|h_1(-1)| = \frac{81}{8}x_0 + \frac{27}{8}$ ,  $h_1(-1/2) = 24x_0$ ,  $h_2(1) = 3x_0$  and  $|h_1(r_0)| = \frac{32}{9(1-x_0)^2}$ , and that  $\frac{32}{9(1-x_0)^2} \geq \max\{\frac{81}{8}x_0 + \frac{27}{8}, 24x_0, 3x_0\}$  whenever  $x_0 \in [1/9, 1/3]$ . To sum it up,

$$\sup_{(r,s) \in \partial\mathcal{R}} |h_{x_0}^+(r, s)| = \begin{cases} \frac{81}{8}x_0 + \frac{27}{8} & \text{if } x_0 \in [0, \frac{1}{9}], \\ \frac{32}{9(1-x_0)^2} & \text{if } x_0 \in [\frac{1}{9}, \frac{1}{3}], \\ 24x_0 & \text{otherwise.} \end{cases} \tag{18}$$

Finally, the result is obtained straightforwardly by comparing the functions (16), (17) and (18).

To obtain the extremal polynomials for every  $x_0 \in \mathbb{R}$ , we may assume here too that  $x_0 \geq 0$ . Then, if  $x_0 \in [0, 1/9]$ , according to the above proof,  $\mathcal{B}(x_0)$  is attained for the polynomials  $P(x) = \pm \left[ 1 + \frac{2}{(1-q^2)^2} (x+q)^2 (2qx-1-q^2) \right]$  with  $q = -3x_0$ . Also, if  $x_0 \in [1/9, 1/3]$ , we have seen in this proof that  $\mathcal{B}_{3,2}(x_0)$  is attained for the polynomials  $P(x) = \pm \left[ 1 + \frac{4}{(s-r)^3} (x-r)^2 \left( x - \frac{3s-r}{2} \right) \right]$  with  $r = r_0 = \frac{9x_0-5}{4}$  and  $s = \frac{r_0+2}{3} = \frac{9x_0+3}{4}$ . To complete the study of the extremal polynomials, it is straightforward that if  $x_0 \geq 1/3$ , then  $\mathcal{B}_{3,2}(x_0)$  is achieved for  $P(x) = \pm(4x^3 - 3x)$ .

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(Received July 15, 2019)

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