

## IDENTITIES AND INEQUALITIES FOR THE COSINE AND SINE FUNCTIONS

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*Abstract.* Identities and inequalities for the cosine and sine functions are obtained.

### 1. Statements and discussion

The basic result of this note is

**THEOREM 1.1.** *For any real  $x$*

$$\cos \pi x = \sum_{j=1}^{\infty} t_j \pi^{2j} (1/4 - x^2)^j, \quad (1.1)$$

where

$$t_j := \sum_{k=0}^{\infty} a_{j,k}, \quad a_{j,k} := \frac{(-\pi^2/4)^k}{(2j+2k)!} \binom{j+k}{j}. \quad (1.2)$$

Moreover, one has the recurrence

$$t_j = \frac{2(2j-3)}{\pi^2 j} t_{j-1} - \frac{1}{\pi^2 j(j-1)} t_{j-2} \quad \text{for } j = 2, 3, \dots, \quad (1.3)$$

with  $t_0 = 0$  and  $t_1 = 1/\pi$ .

Furthermore, for all natural  $j$

$$0 < t_j < \frac{1}{(2j)!}, \quad (1.4)$$

and

$$t_j \sim \frac{1}{(2j)!} \quad \text{as } j \rightarrow \infty.$$

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The necessary proofs will be given in Section 2.

Recurrence (1.3) allows one to compute the coefficients  $t_j$  in (1.1) quickly and efficiently. In particular, we see that

$$(t_1, \dots, t_5) = \left( \frac{1}{\pi}, \frac{1}{\pi^3}, \frac{12 - \pi^2}{6\pi^5}, \frac{10 - \pi^2}{2\pi^7}, \frac{1680 - 180\pi^2 + \pi^4}{120\pi^9} \right) \\ \approx (0.318, 0.0323, 1.16 \times 10^{-3}, 2.16 \times 10^{-5}, 2.46 \times 10^{-7}).$$

On the other hand, inequalities (1.4) together with identity (1.1) will serve as the source of other inequalities, which begin with the following:

COROLLARY 1.2. *For each natural  $m$ , consider the polynomial*

$$P_m(x) := \sum_{j=1}^m t_j \pi^{2j} (1/4 - x^2)^j, \tag{1.5}$$

which is the  $m$ th partial sum of the series in (1.1). Then for all  $x \in (-1/2, 1/2)$

$$P_m(x) < P_{m+1}(x) < \cos \pi x \tag{1.6}$$

and

$$0 < \delta_m(x) := \cos \pi x - P_m(x) < \frac{\pi^{2m+2} (1/4 - x^2)^{m+1}}{(2m+2)!} \frac{1}{1 - q_m} \\ \underset{m \rightarrow \infty}{\sim} \delta_m^*(x) := \frac{\pi^{2m+2} (1/4 - x^2)^{m+1}}{(2m+2)!}, \tag{1.7}$$

where

$$q_m := \frac{\pi^2/4}{(2m+4)(2m+3)}$$

and the asymptotic relation holds uniformly in  $x \in (-\frac{1}{2}, \frac{1}{2})$ .

REMARK 1.3. Note that the function  $\delta_m$  is analytic. So, in view of [3, Proposition I] (proved e.g. in [1, page 29]), it follows from (1.7) that, for each natural  $m$ ,  $P_m(x)$  is the Hermite interpolating polynomial (HIP) (of degree  $2m$ ) determined by the  $2m+2$  conditions  $\delta_m^{(j)}(\pm\frac{1}{2}) = 0$  for  $j = 0, \dots, m$ ; in fact, by Pólya’s theorem [3, Theorem I], the polynomial  $P_m(x)$  is already determined by any  $2m+1$  of the just mentioned  $2m+2$  conditions.

Explicit expressions of the general HIP were presented, in particular, in [2, 7]. It is unclear, though, how to use those results to show that the polynomial  $P_m(x)$ , as defined in (1.5), is the HIP; nor is it seen how to derive monotonicity property (1.6) or the bound in (1.7) from the mentioned expressions.

Least-squares approximating polynomials of the form

$$\sum_{j=1}^m d_{m,j} (1 - x^2)^j \tag{1.8}$$

for  $\cos(\pi x/2)$  and  $x \in [-1, 1]$  were given in [6, Example 4.1]; see also [5] and [4, §5.3.2]. In contrast with the coefficients  $t_j \pi^{2j}$  in (1.5), the coefficients  $d_{m,j}$  in (1.8) depend on  $m$ .

One also has

PROPOSITION 1.4. *For all natural  $j$*

$$t_j = \frac{\pi^{1-j}}{2^j!} J_{j-1/2}(\pi/2), \tag{1.9}$$

where

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)} \tag{1.10}$$

is an expression defining the Bessel function (of the first kind) – as e.g. is done in [8, page 359].

In view of the identity  $\sin \pi x = \cos \pi(x - 1/2)$ , one immediately obtains the corresponding results for  $\sin \pi x$  instead of  $\cos \pi x$ . More specifically, we have

COROLLARY 1.5. *Take any real  $x$ . Then*

$$\sin \pi x = \sum_{j=1}^{\infty} t_j \pi^{2j} (x(1-x))^j. \tag{1.11}$$

Also, for all natural  $m$  and all  $x \in (0, 1)$

$$Q_m(x) < Q_{m+1}(x) < \sin \pi x \tag{1.12}$$

and

$$0 < \sin \pi x - Q_m(x) < \frac{\pi^{2m+2} (x(1-x))^{m+1}}{(2m+2)!} \frac{1}{1-q_m} \tag{1.13}$$

$$\underset{m \rightarrow \infty}{\sim} \frac{\pi^{2m+2} (x(1-x))^{m+1}}{(2m+2)!},$$

where

$$Q_m(x) := P_m(x - 1/2) = \sum_{j=1}^m t_j \pi^{2j} (x(1-x))^j. \tag{1.14}$$

REMARK 1.6. One may compare expansion (1.11) with the Maclaurin expansion

$$\sin \pi x = - \sum_{j=1}^{\infty} \frac{(-\pi x)^{2j-1}}{(2j-1)!}. \tag{1.15}$$

For any natural  $m$ , the approximation of  $\sin \pi x$  by the corresponding Maclaurin polynomial

$$S_m(x) := - \sum_{j=1}^m \frac{(-\pi x)^{2j-1}}{(2j-1)!}$$

is exact to order  $2m$  at  $x = 0$ , but it is not exact to any order at  $x = 1$ . In contrast, in view of (1.12), the approximation of  $\sin \pi x$  by  $Q_m(x)$  is exact to order  $m$  at both  $x = 0$  and  $x = 1$ . Also, in view of (1.12), the approximation of  $\sin \pi x$  by  $Q_m(x)$  is monotonic in  $m$ , whereas the approximation of  $\sin \pi x$  by  $S_m(x)$  is alternating:

$$S_{2j}(x) < S_{2j+2}(x) < \sin \pi x < S_{2j-1}(x) < S_{2j+1}(x)$$

for all natural  $j$  and all real  $x > 0$ . These observations are illustrated in Fig. 1.

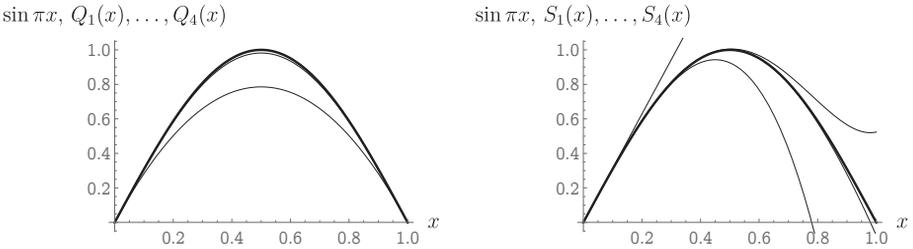


Figure 1: Left panel: Graphs  $\{(x, \sin \pi x) : x \in [0, 1]\}$  (thick) and  $\{(x, Q_m(x)) : x \in [0, 1]\}$  (thin) for  $m = 1, 2, 3, 4$ . Right panel: Graphs  $\{(x, \sin \pi x) : x \in [0, 1]\}$  (thick) and  $\{(x, S_m(x)) : x \in [0, 1]\}$  (thin) for  $m = 1, 2, 3, 4$ .

We see that  $Q_3(x)$  and  $Q_4(x)$  are visually indistinguishable from  $\sin \pi x$  for  $x \in [0, 1]$ ; in contrast,  $S_1(x), S_2(x), S_3(x), S_4(x)$  are all visually distinguishable from  $\sin \pi x$  for  $x \in [0, 1]$ .

REMARK 1.7. Inequalities (1.6) and (1.12) can of course be used to prove other inequalities, which may have exactness or near-exactness properties. For example, we can quickly prove that

$$f(x) := \frac{4}{9} + 15x^2 - 8x + \frac{4(2\sin^2(\pi x) + \sin^2(2\pi x))}{\pi^2} > 0$$

for  $x \in [0, 1/2]$ . Indeed, by (1.12), we have  $f \geq f_4$  on  $[0, 1/2]$ , where  $f_4$  is the polynomial function obtained from  $f$  by replacing the function  $u \mapsto \sin \pi u$  in the above expression for  $f$  by the polynomial function  $Q_4$ . The positivity of any polynomial on any interval can be verified purely algorithmically, which in this case gives  $f_4 > 0$  on  $(0, 1/2]$ , and hence  $f > 0$  on  $[0, 1/2]$ . The graphs of the functions  $f$  and  $f - f_4$  are shown in Fig. 2.

## 2. Proofs

*Proof of Theorem 1.1.* Take any real  $x$  and let

$$y := 1/4 - x^2, \tag{2.1}$$

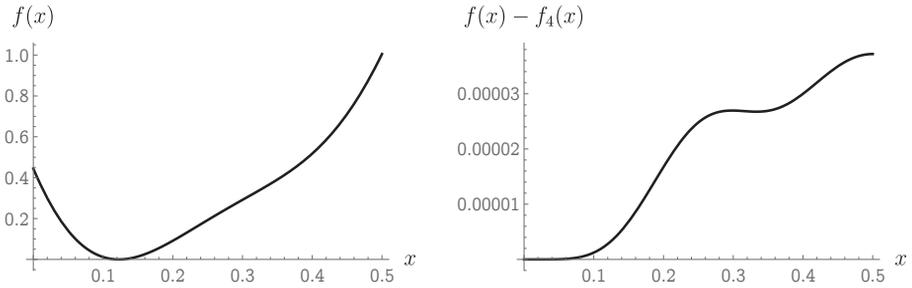


Figure 2: Graphs of the functions  $f$  (left panel) and  $f - f_4$  (right panel).

so that  $y \leq 1/4$  and

$$\begin{aligned}
 \cos \pi x = f(y) &:= \cos\left(\frac{\pi}{2} \sqrt{1-4y}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{2} \sqrt{1-4y}\right)^{2n} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi^2}{4}\right)^n (1-4y)^n \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi^2}{4}\right)^n \sum_{j=0}^n \binom{n}{j} (-4y)^j \\
 &= \sum_{j=0}^{\infty} (-4y)^j \sum_{n=j}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi^2}{4}\right)^n \binom{n}{j} \tag{2.2} \\
 &= \sum_{j=0}^{\infty} (\pi^2 y)^j t_j = \sum_{j=1}^{\infty} (\pi^2 y)^j t_j; \tag{2.3}
 \end{aligned}$$

the equality in (2.2) follows by the Fubini theorem, the first equality in (2.3) follows by the definition of  $t_j$  in (1.2), and the second equality in (2.3) follows because  $t_0 = f(0) = 0$ . Thus, identity (1.1) is proved.

We have already noticed that  $t_0 = f(0) = 0$ . Similarly,  $t_1 = f'(0)/\pi^2 = 1/\pi$ . As for (1.3), it is the special case, with  $z = \pi^2/4$ , of the recurrence

$$T_j(z) = \frac{2j-3}{2jz} T_{j-1}(z) - \frac{1}{4j(j-1)z} T_{j-2}(z) \quad \text{for } j = 2, 3, \dots, \tag{2.4}$$

where

$$T_j(z) := \sum_{k=0}^{\infty} \frac{(-z)^k}{(2j+2k)!} \binom{j+k}{j}, \tag{2.5}$$

so that  $t_j = T_j(\pi^2/4)$ . In turn, identity (2.4) can be verified by a straightforward comparison of the coefficients of the powers of  $z$  on both sides of the identity.

Next, by (1.2), for  $j = 1, 2, \dots$  and  $k = 0, 1, \dots$  the ratio

$$\frac{a_{j,k+1}}{-a_{j,k}} = \frac{\pi^2}{8(k+1)(2j+2k+1)}$$

is positive and less than 1, and this ratio tends to 0 uniformly in  $k = 0, 1, \dots$  as  $j \rightarrow \infty$ . Therefore,  $0 < t_j < a_{j,0} = \frac{1}{(2j)!}$  for all  $j = 1, 2, \dots$ , and  $t_j \sim a_{j,0} = \frac{1}{(2j)!}$  as  $j \rightarrow \infty$ , which verifies the last sentence of Theorem 1.1.

*Proof of Corollary 1.2.* The inequalities in (1.6) follow immediately from (1.5), (1.1), and the first inequality in (1.4).

Recalling the definition of  $\delta_m(x)$  in (1.7), identity (1.1), the definition (2.1) of  $y$ , and the second inequality in (1.4), we see that

$$\delta_m(x) = \sum_{j=m+1}^{\infty} t_j \pi^{2j} y^j < \sum_{j=m+1}^{\infty} b_j(y)$$

for all  $x \in (-1/2, 1/2)$ , where

$$b_j(y) := \frac{(\pi^2 y)^j}{(2j)!}.$$

Moreover, for any natural  $m$ , any natural  $j \geq m + 1$ , and any  $y \in (0, 1/4]$ ,

$$\frac{b_{j+1}(y)}{b_j(y)} = \frac{\pi^2 y}{(2j + 2)(2j + 1)} \leq \frac{\pi^2/4}{(2m + 4)(2m + 3)} = q_m < 1,$$

and  $q_m \rightarrow 0$  as  $m \rightarrow \infty$ . Thus, we have verified (1.7), which completes the proof of Corollary 1.2.

*Proof of Proposition 1.4.* Identity (1.9) is a special case, with  $z = \pi^2/4$ , of the identity

$$T_j(z) = \frac{\sqrt{\pi}}{j! 2^{j+1/2}} z^{1/4-j/2} J_{j-1/2}(\sqrt{z}) \tag{2.6}$$

for real  $z > 0$ , with  $T_j(z)$  as defined in (2.5). In turn, to verify identity (2.6), it is enough to compare the coefficients of the corresponding powers of  $z$  in both sides of (2.6), which is done with the help of the identity

$$\Gamma(n + 1/2) = \frac{\sqrt{\pi} (2n)!}{4^n n!}$$

for  $n = 0, 1, \dots$ , which in turn is easy to check by induction.

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