

FURTHER SUBADDITIVE MATRIX INEQUALITIES

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Abstract. Matrix inequalities that extend certain scalar ones have been in the center of numerous researchers' attention. In this article, we explore the celebrated subadditive inequality for matrices via concave functions and present a reversed version of this result. Our approach will be tackling concave functions properties and some delicate manipulations of matrices and inner products.

1. Introduction

In 1999, Ando and Zhan proved that an operator monotone function $f : [0, \infty) \rightarrow [0, \infty)$ satisfies the subadditive inequality [1]

$$\| \|f(A+B)\| \| \leq \| \|f(A) + f(B)\| \|, \quad (1)$$

for all $n \times n$ positive semidefinite matrices A, B (written $A, B \geq 0$) and any unitarily invariant norm $\| \| \cdot \| \|$ on the algebra \mathcal{M}_n of all complex $n \times n$ matrices, with identity I .

In this context, a function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be operator monotone if it preserves the partial order among Hermitian matrices. That is, if it satisfies $f(A) \leq f(B)$ whenever $A \leq B$ are two positive semidefinite matrices. The partial order " \leq " among Hermitian matrices is defined by

$$A \leq B \Leftrightarrow B - A \geq 0.$$

It is quite interesting that a non-negative function f defined on $[0, \infty)$ is operator monotone if and only if it is operator concave, in the sense that for all $A, B \geq 0$,

$$f((1-t)A + tB) \geq (1-t)f(A) + tf(B), \quad \forall 0 \leq t \leq 1.$$

Later, in 2007, Bourin and Uchiyama proved (1) for concave functions; a condition that is much weaker than operator monotony (or operator concavity), [3].

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The motivation behind (1) is that a concave function $f : [0, \infty) \rightarrow [0, \infty)$ necessarily satisfies

$$f(a+b) \leq f(a) + f(b), \quad a, b \in [0, \infty). \quad (2)$$

Our first goal in this article is to show a new refinement and reverse for (2). More precisely, we show that

$$\begin{aligned} & 2 \left(1 + \frac{|a-b|}{a+b} \right) \left(\frac{f(a+b)}{2} - f \left(\frac{a+b}{2} \right) \right) \\ & \leq f(a+b) - (f(a) + f(b)) \\ & \leq 2 \left(1 - \frac{|a-b|}{a+b} \right) \left(\frac{f(a+b)}{2} - f \left(\frac{a+b}{2} \right) \right). \end{aligned} \quad (3)$$

It is unfortunate that the matrix version of (2) is not true. That is, an operator concave function f does not necessarily satisfy

$$f(A+B) \leq f(A) + f(B) \quad (4)$$

for the positive semidefinite matrices A, B .

In [5], (4) was discussed in details, where additional assumptions were assumed to obtain different forms of such inequalities.

Searching the literature, we find no mention for a reverse of (1). Our second and main goal of this article is to find a positive term Γ such that for a concave function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, one has the following reversed version of (1)

$$|||f(A+B)||| + \Gamma \geq |||f(A) + f(B)|||$$

for all positive semidefinite matrices A, B and any unitarily invariant norm $|||\cdot|||$. This will be done in Lemma 2 and Theorem 2 below.

Our approach to prove Lemma 2 and Theorem 2 will be a delicate treatment of concave functions and inner product properties, where (3) will play a major role.

2. Main results

In this section, we present our results, where we begin with the discussion of concave functions inequalities, then we apply those results to matrices.

Recall that a concave function is distinguished by the fact that it lies above its secants on the interval of concavity. However, if $f : [a, b] \rightarrow \mathbb{R}$ is concave, then one can easily see that the function $g(t) = f((1-t)a + tb) - ((1-t)f(a) + tf(b))$ is concave on $[0, 1]$. Consequently, the graph of $g(t)$ is above its secants on $[0, 1/2]$ and $[1/2, 1]$. This observation leads to the well known inequality [4, 6, 7]

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb) + 2r \left(\frac{f(a) + f(b)}{2} - f \left(\frac{a+b}{2} \right) \right), \quad (5)$$

where $0 \leq t \leq 1$ and $r = \min\{t, 1 - t\}$.

Noting negativity of $\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)$, we see how (5) refines the inequality

$$(1 - t)f(a) + tf(b) \leq f((1 - t)a + tb)$$

for concave functions. Manipulating concave inequalities also lead to a reversed version as follows [4, 8]

$$(1 - t)f(a) + tf(b) + 2R \left(f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2} \right) \geq f((1 - t)a + tb), \tag{6}$$

where $0 \leq t \leq 1$ and $R = \max\{t, 1 - t\}$.

Our first result provides a refinement and a reverse for (2). The proof will use both (5) and (6). As far as we know, this approach has never been tickled in the literature.

THEOREM 1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a concave function with $f(0) = 0$. Then for any $a, b \geq 0$,*

$$\begin{aligned} & 2 \left(1 + \frac{|a - b|}{a + b} \right) \left(\frac{f(a + b)}{2} - f\left(\frac{a + b}{2}\right) \right) \\ & \leq f(a + b) - (f(a) + f(b)) \\ & \leq 2 \left(1 - \frac{|a - b|}{a + b} \right) \left(\frac{f(a + b)}{2} - f\left(\frac{a + b}{2}\right) \right). \end{aligned} \tag{7}$$

Proof. For $a, b \geq 0$ and $0 \leq t \leq 1$, (6) implies

$$(1 - t)f(a) + tf(b) + 2R \left(f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2} \right) \geq f((1 - t)a + tb).$$

where $R = \max\{t, 1 - t\}$.

Replacing a by 0 and b by $x \geq 0$, we have

$$f(tx) = f(tx + (1 - t) \cdot 0) \leq (1 - t)f(0) + tf(x) + 2R \left(f\left(\frac{x}{2}\right) - \frac{f(0) + f(x)}{2} \right).$$

Since $f(0) = 0$, the above inequality implies

$$f(tx) \leq tf(x) + 2R \left(f\left(\frac{x}{2}\right) - \frac{f(x)}{2} \right),$$

where $R = \max\{t, 1 - t\}$ and $0 \leq t \leq 1$.

Applying this inequality twice implies

$$\begin{aligned} f(a + b) &= \frac{a}{a + b} f(a + b) + \frac{b}{a + b} f(a + b) \\ &\geq f\left(\frac{a}{a + b} \cdot (a + b)\right) - 2R \left(f\left(\frac{a + b}{2}\right) - \frac{f(a + b)}{2} \right) \\ &\quad + f\left(\frac{b}{a + b} \cdot (a + b)\right) - 2R \left(f\left(\frac{a + b}{2}\right) - \frac{f(a + b)}{2} \right) \\ &= f(a) + f(b) - 4R \left(f\left(\frac{a + b}{2}\right) - \frac{f(a + b)}{2} \right), \end{aligned}$$

where $R = \max \left\{ \frac{a}{a+b}, \frac{b}{a+b} \right\}$.
Consequently,

$$f(a) + f(b) \leq f(a+b) + 4R \left(f\left(\frac{a+b}{2}\right) - \frac{f(a+b)}{2} \right).$$

Noting that $R = \max \left\{ \frac{a}{a+b}, \frac{b}{a+b} \right\} = \frac{a+b+|a-b|}{2(a+b)}$, we reach

$$f(a) + f(b) \leq 2 \left(1 + \frac{|a-b|}{a+b} \right) \left(f\left(\frac{a+b}{2}\right) - \frac{f(a+b)}{2} \right) + f(a+b),$$

which proves the first inequality in (7).

Now we shall prove the second inequality in (7). From (5), we have

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb) + 2r \left(\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right),$$

where $r = \min \{t, 1-t\}$.

This implies, when $a = 0$,

$$tf(x) \leq f(tx) + 2r \left(\frac{f(x)}{2} - f\left(\frac{x}{2}\right) \right).$$

Consequently,

$$\begin{aligned} f(a+b) &= \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(a+b) \\ &\leq f(a) + 2r \left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right) \right) + f(b) + 2r \left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right) \right) \\ &= f(a) + f(b) + 4r \left(\frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right) \right), \end{aligned}$$

where $r = \min \left\{ \frac{a}{a+b}, \frac{b}{a+b} \right\} = \frac{a+b-|a-b|}{2(a+b)}$. This completes the proof of the theorem.

REMARK 1. Notice that if $f : [0, \infty) \rightarrow \mathbb{R}$ is a concave function with $f(0) = 0$, then for any $a, b \geq 0$,

$$0 \geq \frac{f(a+b)}{2} - f\left(\frac{a+b}{2}\right).$$

Since

$$\frac{f(a+b)}{2} \leq \frac{f(a) + f(b)}{2} \leq f\left(\frac{a+b}{2}\right),$$

where the first inequality follows from the subadditivity of concave function and the second inequality follows directly from the definition of a concave function.

COROLLARY 1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a concave function with $f(0) = 0$. Then for any $a, b \geq 0$,

$$f\left(\frac{a+b}{2}\right) - \left(\frac{f(a)+f(b)}{2}\right) \leq \frac{|a-b|}{a+b} \left(f\left(\frac{a+b}{2}\right) - \frac{f(a+b)}{2}\right).$$

In the sequel, we will present our applications of the above scalar inequalities to matrices. For this purpose, we will need the following well known lemma.

LEMMA 1. ([2, p. 281]) If $f : J \rightarrow \mathbb{R}$ is concave and if $A \in \mathcal{M}_n$ is Hermitian with spectrum in J , then

$$\langle f(A)x, x \rangle \leq f(\langle Ax, x \rangle),$$

for all unit vectors $x \in \mathbb{C}^n$.

As an application of Theorem 1, we have the following reversed version of the celebrated subadditive inequality (1) for concave functions. For the next two main results, we adopt the notations $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ to denote the least and largest eigenvalues of the Hermitian matrix $X \in \mathcal{M}_n$, respectively.

In the following lemma, we present the reversed version of (1) for the usual operator norm. The unitarily invariant norm version is shown then.

LEMMA 2. Let $A, B \in \mathcal{M}_n$ be two positive definite matrices and let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function, with $f(0) = 0$. Then

$$\|f(A) + f(B)\| \leq \frac{\alpha + \beta}{\alpha} \left(2f\left(\frac{\eta}{2}\right) - f(\alpha)\right) + \|f(A+B)\|,$$

where $\alpha = \lambda_{\min}(A+B)$, $\beta = \lambda_{\max}(|A-B|)$, $\eta = \lambda_{\max}(A+B)$ and $\|\cdot\|$ is the usual operator norm.

Proof. If $\|x\| = 1$, we have $\langle (A+B)x, x \rangle \geq \alpha$. Now since $f : [0, \infty) \rightarrow [0, \infty)$ is concave it follows that f is increasing. Consequently, $-f(\langle (A+B)x, x \rangle) \leq -f(\alpha)$. This together with the fact that f is increasing imply

$$\begin{aligned} & 2\left(1 + \frac{|\langle (A-B)x, x \rangle|}{\langle (A+B)x, x \rangle}\right) \left(f\left(\left\langle \left(\frac{A+B}{2}\right)x, x \right\rangle\right) - \frac{f(\langle (A+B)x, x \rangle)}{2}\right) \\ & \leq \left(1 + \frac{\beta}{\alpha}\right) \left(2f\left(\frac{\eta}{2}\right) - f(\alpha)\right). \end{aligned} \tag{8}$$

Consequently, by applying Theorem 1, with $a = \langle Ax, x \rangle$ and $b = \langle Bx, x \rangle$, we have

$$\begin{aligned} & \langle f(A) + f(B)x, x \rangle \\ &= \langle f(A)x, x \rangle + \langle f(B)x, x \rangle \\ &\leq f(\langle Ax, x \rangle) + f(\langle Bx, x \rangle) \quad (\text{by Lemma 1}) \\ &\leq 2 \left(1 + \frac{|\langle (A - B)x, x \rangle|}{\langle (A + B)x, x \rangle} \right) \left(f \left(\left\langle \left(\frac{A + B}{2} \right) x, x \right\rangle \right) - \frac{f(\langle (A + B)x, x \rangle)}{2} \right) \\ &\quad + f(\langle (A + B)x, x \rangle) \quad (\text{by Theorem 1}) \\ &\leq \left(1 + \frac{\beta}{\alpha} \right) \left(2f \left(\frac{\eta}{2} \right) - f(\alpha) \right) + f(\langle (A + B)x, x \rangle) \quad (\text{by (8)}). \end{aligned}$$

This implies

$$\langle f(A) + f(B)x, x \rangle \leq \frac{\alpha + \beta}{\alpha} \left(2f \left(\frac{\eta}{2} \right) - f(\alpha) \right) + f(\langle (A + B)x, x \rangle),$$

for any unit vector $x \in \mathbb{C}^n$. Now, by taking supremum over unit vector x , and recalling that f is increasing, we obtain the desired inequality.

Now we are ready to present the main result in this article, where we show the unitarily invariant norm version of Lemma 2. In the proof, we will need the following basic lemma [2, Problem 1.6. 15].

LEMMA 3. *Let $A \in \mathcal{M}_n$ be Hermitian and let $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ denote all eigenvalues of A , counting multiplicities. Then, for $1 \leq k \leq n$,*

$$\sum_{i=1}^k \lambda_i(A) = \max \sum_{i=1}^k \langle Ax_i, x_i \rangle,$$

where the maximum is taken over all sets of k orthonormal vectors x_1, \dots, x_k in \mathbb{C}^n .

THEOREM 2. *Let $A, B \in \mathcal{M}_n$ be two positive definite matrices and let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function, with $f(0) = 0$. If $\|\cdot\|$ is a unitarily invariant norm on \mathcal{M}_n , normalized so that $\|I\| = 1$, then*

$$\|f(A) + f(B)\| \leq \frac{\alpha + \beta}{\alpha} \left(2f \left(\frac{\eta}{2} \right) - f(\alpha) \right) + \|f(A + B)\|,$$

where α, β and η are as in Lemma 2.

Proof.

Let x_1, x_2, \dots, x_n be orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of $f(A) + f(B)$. For simplicity, let

$$\gamma = \frac{\alpha + \beta}{\alpha} \left(2f \left(\frac{\eta}{2} \right) - f(\alpha) \right).$$

Then, for $1 \leq k \leq n$,

$$\begin{aligned} \sum_{i=1}^k \lambda_i(f(A) + f(B)) &= \sum_{i=1}^k \langle (f(A) + f(B))x_i, x_i \rangle \\ &\leq \sum_{i=1}^k \langle (\gamma I + f(A + B))x_i, x_i \rangle \text{ (by Lemma 2)} \\ &\leq \sum_{i=1}^k \lambda_i(\gamma I + f(A + B)) \text{ (by Lemma 3)}. \end{aligned}$$

Now, since A and B are positive and $f : [0, \infty) \rightarrow [0, \infty)$, we have

$$|||f(A) + f(B)|||_{(k)} \leq |||\gamma I + f(A + B)|||_{(k)},$$

where $|||\cdot|||_{(k)}$ denotes the ky-Fan norms. From this, it follows that (see [2, Theorem IV.2.2, p. 93])

$$|||f(A) + f(B)||| \leq |||\gamma I + f(A + B)|||,$$

for any unitarily invariant norm $|||\cdot|||$. But this latter inequality implies that

$$|||f(A) + f(B)||| \leq \gamma |||I||| + |||f(A + B)|||,$$

which completes the proof.

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