

SPLITTING OF OPERATORS FOR FRAME INEQUALITIES

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Abstract. In this paper, we obtain some new inequalities for frames, which are parametrized by a parameter $\lambda \in \mathcal{R}$. By suitable choices of λ , one obtains the known results as special cases. Our new results also make the underlying mathematical structure that gives rise to these inequalities more transparent than previous approaches: our results show that the main point is the splitting $S = S_1 + S_2$ of a positive definite frame operator S into the two positive semidefinite operators S_1 and S_2 .

1. Introduction

Frames in Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [6] to study some deep problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [5], and today frames play important roles in many applications in several areas of mathematics, physics, and engineering, such as coding theory [12, 15], sampling theory [20], quantum measurements [7], filter bank theory [10] and image processing [4].

Let \mathcal{H} be a separable Hilbert space and I a countable index set. A sequence of vectors $\{f_i\}_{i \in I}$ of \mathcal{H} is a frame for \mathcal{H} if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The numbers A, B are called lower and upper frame bounds, respectively. If $A = B$, then this frame is called an A -tight frame, and if $A = B = 1$, then it is called a Parseval frame.

Suppose $\{f_i\}_{i \in I}$ is a frame for \mathcal{H} , then the frame operator is a positive self-adjoint invertible operator, which is given by

$$S: \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

The following reconstruction formula holds:

$$f = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i,$$

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where the family $\{\tilde{f}_i\}_{i \in I} = \{S^{-1}f_i\}_{i \in I}$ is also a frame for \mathcal{H} , which is called the canonical dual frame of $\{f_i\}_{i \in I}$. The frame $\{g_i\}_{i \in I}$ for \mathcal{H} is called an alternate dual frame of $\{f_i\}_{i \in I}$ if the following formula holds:

$$f = \sum_{i \in I} \langle f, f_i \rangle g_i = \sum_{i \in I} \langle f, g_i \rangle f_i$$

for all $f \in \mathcal{H}$ (see [9]).

Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} , for every $J \subset I$, we define the operator

$$S_J = \sum_{i \in J} \langle f, f_i \rangle f_i,$$

and denote $J^c = I \setminus J$.

In [1], the authors solved a long-standing conjecture of the signal processing community. They showed that for suitable frames $\{f_i\}_{i \in I}$, a signal f can (up to a global phase) be recovered from the phase-less measurements $\{|\langle f, f_i \rangle|\}_{i \in I}$. Note, that this only shows that reconstruction of f is in principle possible, but there is not an effective constructive algorithm. While searching for such an algorithm, the authors of [2] discovered a new identity for Parseval frames [3]. The authors of [8, 19] generalized these identities to alternate dual frames and got some general results. The study of inequalities has interested many mathematicians. Some authors have extended the equalities and inequalities for frames in Hilbert spaces to generalized frames [13, 16, 17]. The following form was given in [3] (see [2] for a discussion of the origins of this fundamental identity).

THEOREM 1. *Let $\{f_i\}_{i \in I}$ be a Parseval frame for \mathcal{H} . For every $J \subset I$ and every $f \in \mathcal{H}$, we have*

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 + \left\| \sum_{i \in J^c} \langle f, f_i \rangle f_i \right\|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + \left\| \sum_{i \in J} \langle f, f_i \rangle f_i \right\|^2 \geq \frac{3}{4} \|f\|^2. \quad (1)$$

Later on, the author in [17] generalized Theorem 1 to general frames.

THEOREM 2. *Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with canonical dual frame $\{\tilde{f}_i\}_{i \in I}$. Then for every $J \subset I$ and every $f \in \mathcal{H}$, we have*

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 + \sum_{i \in I} \left| \langle S_{J^c} f, \tilde{f}_i \rangle \right|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + \sum_{i \in I} \left| \langle S_J f, \tilde{f}_i \rangle \right|^2 \geq \frac{3}{4} \sum_{i \in I} |\langle f, f_i \rangle|^2. \quad (2)$$

THEOREM 3. *Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} and $\{g_i\}_{i \in I}$ be an alternate dual frame of $\{f_i\}_{i \in I}$. Then for every $J \subset I$ and every $f \in \mathcal{H}$, we have*

$$\begin{aligned} \operatorname{Re} \left(\sum_{i \in J} \langle f, g_i \rangle \overline{\langle f, f_i \rangle} \right) + \left\| \sum_{i \in J^c} \langle f, g_i \rangle f_i \right\|^2 &= \operatorname{Re} \left(\sum_{i \in J^c} \langle f, g_i \rangle \overline{\langle f, f_i \rangle} \right) + \left\| \sum_{i \in J} \langle f, g_i \rangle f_i \right\|^2 \\ &\geq \frac{3}{4} \|f\|^2. \end{aligned} \quad (3)$$

2. Results and new proofs

First, we give some results for a operator which has same properties of frame operator.

THEOREM 4. *Let $S : \mathcal{H} \rightarrow \mathcal{H}$ be a positive definite operator. Furthermore, let $S_1, S_2 : \mathcal{H} \rightarrow \mathcal{H}$ be positive semidefinite with $S = S_1 + S_2$. Then the following are true:*

1. For $i \in \{1, 2\}$, we have $0 \leq S_i S^{-1} S_i \leq S_i$.
2. We have $S_2 + S_1 S^{-1} S_1 \leq S$.
3. We have $S_1 S^{-1} S_1 + S_2 S^{-1} S_2 \leq S$.
4. $S_2 + S_1 S^{-1} S_1 = S_1 + S_2 S^{-1} S_2$.
5. If $p, q \in \mathcal{R}$ are chosen such that $\rho(a) := a^2 + a \cdot (q - p - 1) + 1 - q \geq 0$ for all $a \in [0, 1]$, then we have

$$p \cdot S_1 + q \cdot S_2 \leq S_2 + S_1 S^{-1} S_1.$$

6. If $p, q \in \mathcal{R}$ are chosen such that $\eta(a) := a^2 - a(1 + p) + q + p \geq 0$ for all $a \in [0, 1]$, then we have

$$S_1 - S_1 S^{-1} S_1 \leq p \cdot S_2 + q \cdot S_2.$$

7. If $p, q \in \mathcal{R}$ are chosen such that $\tau(a) := a^2 + a \cdot (\frac{q-p}{2} - 1) + \frac{1-q}{2} \geq 0$ for all $a \in [0, 1]$, then we have

$$p \cdot S_1 + q \cdot S_2 \leq S_1 S^{-1} S_1 + S_2 S^{-1} S_2.$$

In all of these statements, we write $U \leq V$ for all operators $U, V : \mathcal{H} \rightarrow \mathcal{H}$ if U, V are self-adjoint, and if furthermore $V - U$ is positive semidefinite.

Proof. We first prove the following elementary fact: if $P : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded positive definite operator, then a self-adjoint, bounded operator $X : \mathcal{H} \rightarrow \mathcal{H}$ is positive semidefinite if and only if PXP is positive semidefinite. Indeed, if X is positive semidefinite, then $\langle PXPf, f \rangle = \langle XPf, Pf \rangle$ for all $f \in \mathcal{H}$, so that PXP is positive semidefinite. Conversely, if $PXP \geq 0$, we can apply what we just showed with P^{-1} instead of P to see $X = P^{-1}(PXP)P^{-1} \geq 0$. Overall, this means $\forall U, V : \mathcal{H} \rightarrow \mathcal{H}$ self-adjoint and $P : \mathcal{H} \rightarrow \mathcal{H}$ positive definite, we have

$$U \leq V \Leftrightarrow PUP \leq PVP. \tag{4}$$

Note that $S^{-1/2}$ is positive definite and bounded, so that the operators $U := S^{-1/2} S_1 S^{-1/2}$ and $V := S^{-1/2} S_2 S^{-1/2}$ are positive semidefinite and bounded. Furthermore,

$$U + V = S^{-1/2} (S_1 + S_2) S^{-1/2} = S^{-1/2} S S^{-1/2} = I_{\mathcal{H}}. \tag{5}$$

Now, we properly start the proof:

(1). Since U, V are positive semidefinite, we have $0 \leq U \leq U + V = I_{\mathcal{H}}$, and thus $I_{\mathcal{H}} - U \geq 0$. Since $I_{\mathcal{H}} - U$ and U commute, this implies $U - U^2 = U \cdot (I_{\mathcal{H}} - U) \geq 0$, i.e., $0 \leq U^2 \leq U$. In view of (4), this implies $0 \leq S^{1/2}U^2S^{1/2} \leq S^{1/2}US^{1/2}$. Since $S^{1/2}US^{1/2} = S_1$ and $S^{1/2}U^2S^{1/2} = S_1S^{-1}S_1$, this implies the claim of the first part for $i = 1$. The proof for $i = 2$ is similar.

(2). In view of (5) (with $P = S^{-1/2}$), in view of the definition of U, V and because of $V = I_{\mathcal{H}} - U$ (see (5)), we have the following equivalence:

$$S_2 + S_1S^{-1}S_1 \leq S \Leftrightarrow V + UU \leq I_{\mathcal{H}}.$$

But we saw in the previous part that $U^2 \leq U$, so that $V + UU \leq V + U = I_{\mathcal{H}}$ does hold.

(3). Part (1) shows $S_iS^{-1}S_i \leq S_i$ for $i \in 1, 2$. Hence, $S_1S^{-1}S_1 + S_2S^{-1}S_2 \leq S_1 + S_2 = S$.

(4). By multiplying from the left and from the right by $S^{-1/2}$, we see that the claimed identity is equivalent to $V + UU = U + VV$. Because of $V = I_{\mathcal{H}} - U$, this is in turn equivalent to

$$I_{\mathcal{H}} - U + UU = U + (I_{\mathcal{H}} - U)(I_{\mathcal{H}} - U),$$

which is easy seen to be true by expanding the right-hand side.

(5). In view of (4) (with $P = S^{-1/2}$), from the definition of U, V , and because of $V = I_{\mathcal{H}} - U$ (see (5)), we have the following equivalence:

$$\begin{aligned} p \cdot S_1 + q \cdot S_2 \leq S_2 + S_1S^{-1}S_1 &\Leftrightarrow p \cdot U + q \cdot V \leq V + UU \\ &\Leftrightarrow p \cdot U + q \cdot I_{\mathcal{H}} - q \cdot U \leq I_{\mathcal{H}} - U + UU \\ &\Leftrightarrow U^2 + U \cdot (q - p - 1) + (1 - q) \cdot I_{\mathcal{H}} \geq 0 \\ &\Leftrightarrow \rho(U) \geq 0. \end{aligned}$$

But in the proof of part (1), we saw $0 \leq U \leq I_{\mathcal{H}}$. Since we have $\rho \geq 0$ on $[0, 1]$ by assumption, elementary properties of the spectral calculus (see e.g. [11, Theorem 4.2]) imply that $\rho(U)$ is positive semidefinite, as desired.

(6). Just as in the proof of the previous part, we get the following equivalence:

$$\begin{aligned} S_1 - S_1S^{-1}S_1 \leq p \cdot S_2 + q \cdot S &\Leftrightarrow U - UU \leq p \cdot V + q \cdot I_{\mathcal{H}} \\ &\Leftrightarrow U - U^2 \leq (p + q) \cdot I_{\mathcal{H}} - p \cdot U \\ &\Leftrightarrow U^2 - (1 + p) \cdot U + (p + q) \cdot I_{\mathcal{H}} \geq 0 \\ &\Leftrightarrow \eta(U) \geq 0. \end{aligned}$$

Again, just as in the proof of the previous part, we see that $\eta(U)$ is indeed positive semidefinite, since $0 \leq U \leq I_{\mathcal{H}}$ and since $\eta \geq 0$ on $[0, 1]$ by assumption.

(7). Just as in part (5), we get the following equivalence:

$$\begin{aligned}
 p \cdot S_1 + q \cdot S_2 \leq S_1 S^{-1} S_1 + S_2 S^{-1} S_2 &\Leftrightarrow p \cdot U + q \cdot V \leq UU + VV \\
 &\Leftrightarrow p \cdot U + q \cdot (I_{\mathcal{H}} - U) \leq UU + (I_{\mathcal{H}} - U)(I_{\mathcal{H}} - U) \\
 &\Leftrightarrow 2U^2 + U \cdot (q - p - 2) + I_{\mathcal{H}} \cdot (1 - q) \geq 0 \\
 &\Leftrightarrow \tau(U) \geq 0.
 \end{aligned}$$

Again, just as in the proof of the previous part, we see that $\tau(U)$ is indeed positive semidefinite, since $0 \leq U \leq I_{\mathcal{H}}$ and since $\tau \geq 0$ on $[0, 1]$ by assumption. \square

THEOREM 5. *Let $S : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded, self-adjoint positive definite operator. Furthermore, let $S_1, S_2 : \mathcal{H} \rightarrow \mathcal{H}$ be bounded, self-adjoint, and positive semidefinite with $S = S_1 + S_2$. Then for any $\lambda \in \mathcal{R}$, we have*

$$\left(\lambda - \frac{\lambda^2}{4}\right) \cdot S_1 + \left(1 - \frac{\lambda^2}{4}\right) \cdot S_2 \leq S_2 + S_1 S^{-1} S_1 = S_1 + S_2 S^{-1} S_2 \leq S. \tag{6}$$

Proof. The middle identity is a direct consequence of part (4) of Theorem 4. Likewise, the final estimate follows directly from part (2) of Theorem 4.

To prove the first part of the equation (6), we want to apply part (5) of Theorem 4 with the choices $p = \lambda - \frac{\lambda^2}{4}$ and $q = 1 - \frac{\lambda^2}{4}$. With these choices, the polynomial ρ from Theorem 4 takes the form

$$\begin{aligned}
 \rho(a) &= a^2 + a \cdot (q - p - 1) + 1 - q \\
 &= a^2 - \lambda a + \frac{\lambda^2}{4} = \left(a - \frac{\lambda}{2}\right)^2,
 \end{aligned}$$

so that $\rho(a) \geq 0$ for all $a \in [0, 1]$, as required by part (5) of Theorem 4. An application of that part of the Theorem 4 completes the proof. \square

By choosing S to be the frame operator, and by choosing $S_1 := S_J$ and $S_2 := S_{J^c}$, we see that S , S_1 and S_2 are positive semi-definite, that S is positive definite, and that $S = S_1 + S_2$. Furthermore, directly from the definitions, we see

$$\begin{aligned}
 \langle Sf, f \rangle &= \sum_{i \in I} \|\langle f, f_i \rangle\|^2, \\
 \langle S_1 f, f \rangle &= \sum_{i \in J} \|\langle f, f_i \rangle\|^2, \quad \langle S_2 f, f \rangle = \sum_{i \in J^c} \|\langle f, f_i \rangle\|^2, \\
 \langle S_j S^{-1} S_j f, f_i \rangle &= \langle S(S^{-1} S_j f), S^{-1} S_j f \rangle = \sum_{i \in I} \|\langle S^{-1} S_j f, f_i \rangle\|^2, \quad j = 1, 2.
 \end{aligned} \tag{7}$$

COROLLARY 1. Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with frame operator S . Then for any $\lambda \in \mathcal{R}$, for all $J \subset I$, and any $f \in \mathcal{H}$, we have

$$\begin{aligned} & \left(\lambda - \frac{\lambda^2}{4}\right) \cdot \sum_{i \in J} \|\langle f, f_i \rangle\|^2 + \left(1 - \frac{\lambda^2}{4}\right) \cdot \sum_{i \in J^c} \|\langle f, f_i \rangle\|^2 \\ & \leq \sum_{i \in J^c} \|\langle f, f_i \rangle\|^2 + \sum_{i \in I} \|\langle S^{-1} S_J f, f_i \rangle\|^2 \\ & = \sum_{i \in J} \|\langle f, f_i \rangle\|^2 + \sum_{i \in I} \|\langle S^{-1} S_{J^c} f, f_i \rangle\|^2 \\ & \leq \sum_{i \in I} \|\langle f, f_i \rangle\|^2. \end{aligned} \tag{8}$$

Proof. We choose S_1, S_2 as outlined before equation (7). In view of the “translation table” in equation (7), and by the definition of the relation “ $U \leq V$ ” for self-adjoint operator U, V , the equation (8) is equivalent to (6). By Theorem 5, the result holds. \square

REMARK 1. If we take $\lambda = 1$ in (8), Corollary 1 is equivalent to Theorem 2. If we consider S as a fusion frame operator in Theorem 5, we can easily get the [14, Theorem 3]. If we consider S as a HS-frame operator in Theorem 5, we can easily get the [17, Theorem 3.5].

THEOREM 6. Let $S: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded, self-adjoint positive definite operator. Furthermore, let $S_1, S_2: \mathcal{H} \rightarrow \mathcal{H}$ be bounded, self-adjoint, and positive semidefinite with $S = S_1 + S_2$. Then for any $\lambda \in \mathcal{R}$, we have

$$0 \leq S_1 - S_1 S^{-1} S_1 \leq (\lambda - 1) \cdot S_2 + \left(1 - \frac{\lambda}{2}\right)^2 \cdot S. \tag{9}$$

Proof. The first estimate of equation (9) is a direct consequence of part (1) of Theorem 4. To prove the second estimate, we want to apply part (6) of Theorem 4, with $p = \lambda - 1$ and $q = \left(1 - \frac{\lambda}{2}\right)^2 = 1 - \lambda + \frac{\lambda^2}{4}$. With these choices, the polynomial η from the Theorem 4 takes the form

$$\begin{aligned} \eta(a) &= a^2 - a \cdot (1 + p) + q + p \\ &= a^2 - \lambda a + \frac{\lambda^2}{4} = \left(a - \frac{\lambda}{2}\right)^2, \end{aligned}$$

so that $\eta(a) \geq 0$ for all $a \in [0, 1]$, as required by part (6) of Theorem 4. An application of that theorem thus finishes the proof. \square

COROLLARY 2. Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with frame operator S . Then for any $\lambda \in \mathcal{R}$, for all $J \subset I$, and any $f \in \mathcal{H}$, we have

$$\begin{aligned} 0 &\leq \sum_{i \in J} \|\langle f, f_i \rangle\|^2 - \sum_{i \in I} \|\langle S^{-1}S_J f, f_i \rangle\|^2 \\ &\leq (\lambda - 1) \cdot \sum_{i \in J^c} \|\langle f, f_i \rangle\|^2 + \left(1 - \frac{\lambda}{2}\right)^2 \cdot \sum_{i \in I} \|\langle f, f_i \rangle\|^2. \end{aligned}$$

Proof. By choosing $S_1 = S_J$ and $S_2 = S_{J^c}$, and by using the “translation table” given in equation (7), we see that the claim is equivalent to (9), and result holds by Theorem 6. \square

REMARK 2. If we consider S as a fusion frame operator in Theorem 6, we can easily get the [14, Theorem 5]. If we consider S as a g-frame operator in Theorem 6 for Hilbert C^* -modules, we can easily get the [18, Theorem 2.4].

THEOREM 7. Let $S: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded, self-adjoint positive definite operator. Furthermore, let $S_1, S_2: \mathcal{H} \rightarrow \mathcal{H}$ be bounded, self-adjoint, and positive semidefinite with $S = S_1 + S_2$. Then for any $\lambda \in \mathcal{R}$, we have

$$\left(2\lambda - \frac{\lambda^2}{2} - 1\right) \cdot S_1 + \left(1 - \frac{\lambda^2}{2}\right) \cdot S_2 \leq S_1 S^{-1} S_1 + S_2 S^{-1} S_2 \leq S. \tag{10}$$

Proof. The second of these inequalities is a direct consequence of part (3) of Theorem 4. To prove the first estimate, we want to involve part (7) of Theorem 4 with $p = 2\lambda - \frac{\lambda^2}{2} - 1$ and $q = 1 - \frac{\lambda^2}{2}$. With these choices, the polynomial τ from Theorem 4 takes the form

$$\begin{aligned} \tau(a) &= a^2 + a \cdot \left(\frac{q-p}{2} - 1\right) + \frac{1-q}{2} \\ &= a^2 - \lambda a + \frac{\lambda^2}{4} = \left(a - \frac{\lambda}{2}\right)^2, \end{aligned}$$

so that $\tau(a) \geq 0$ for all $a \in [0, 1]$, as required in part (7) of Theorem 4. An application of that theorem thus finishes the proof. \square

COROLLARY 3. Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with frame operator S . Then for any $\lambda \in \mathcal{R}$, for all $J \subset I$, and any $f \in \mathcal{H}$, we have

$$\begin{aligned} &\left(2\lambda - \frac{\lambda^2}{2} - 1\right) \cdot \sum_{i \in J} \|\langle f, f_i \rangle\|^2 + \left(1 - \frac{\lambda^2}{2}\right) \cdot \sum_{i \in J^c} \|\langle f, f_i \rangle\|^2 \\ &\leq \sum_{i \in I} \|\langle S^{-1}S_J f, f_i \rangle\|^2 + \sum_{i \in I} \|\langle S^{-1}S_{J^c} f, f_i \rangle\|^2 \\ &\leq \sum_{i \in I} \|\langle f, f_i \rangle\|^2. \end{aligned}$$

Proof. By choosing $S_1 = S_J$ and $S_2 = S_{J^c}$, and by using the “translation table” given in equation (7), we see that the claim is equivalent to (10). Then the result holds by Theorem 7. \square

REMARK 3. If we consider S as a continuous fusion frame operator in Theorem 6, we can easily get the [14, Theorem 2.13]. If we consider S as a g -frame operator in Theorem 6 for Hilbert C^* -modules, we can easily get the [18, Theorem 2.4].

Next, we give a new type of inequality of frames of Theorem 3. We first need the following lemma.

LEMMA 1. *Let U, V be two bounded linear operators in \mathcal{H} and $U + V = I_{\mathcal{H}}$, then for any $\lambda \in \mathcal{R}$, we have*

$$U^*U + \lambda \cdot (V^* + V) \geq \lambda(2 - \lambda) \cdot I_{\mathcal{H}}.$$

Proof. Since $U + V = I_{\mathcal{H}}$, we have

$$\begin{aligned} U^*U + \lambda(V^* + V) &= U^*U - \lambda(U^* + U) + 2\lambda \cdot I_{\mathcal{H}} \\ &= U^*U - \lambda \cdot (U^* + U) + 2\lambda \cdot I_{\mathcal{H}} + \lambda^2 \cdot I_{\mathcal{H}} - \lambda^2 \cdot I_{\mathcal{H}} \\ &= (U - \lambda \cdot I_{\mathcal{H}})^*(U - \lambda \cdot I_{\mathcal{H}}) + \lambda(2 - \lambda) \cdot I_{\mathcal{H}} \\ &\geq \lambda(2 - \lambda) \cdot I_{\mathcal{H}}. \quad \square \end{aligned}$$

THEOREM 8. *Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} and $\{g_i\}_{i \in I}$ be an alternate dual frame of $\{f_i\}_{i \in I}$. Then for any $\lambda \in \mathcal{R}$, for all $J \subset I$, and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} \operatorname{Re} \left(\sum_{i \in J} \langle f, g_i \rangle \overline{\langle f, f_i \rangle} \right) + \left\| \sum_{i \in J^c} \langle f, g_i \rangle f_i \right\|^2 &\geq (2\lambda - \lambda^2) \cdot \operatorname{Re} \left(\sum_{i \in J} \langle f, g_i \rangle \overline{\langle f, f_i \rangle} \right) \\ &\quad + (1 - \lambda^2) \cdot \operatorname{Re} \left(\sum_{i \in J^c} \langle f, g_i \rangle \overline{\langle f, f_i \rangle} \right). \end{aligned}$$

Proof. For any $J \subset I$ and $f \in \mathcal{H}$, we define operators U, V as

$$Uf = \sum_{i \in J^c} \langle f, g_i \rangle f_i, \quad Vf = \sum_{i \in J} \langle f, g_i \rangle f_i.$$

Clearly, U, V are bounded linear operators and $U + V = I_{\mathcal{H}}$. From Lemma 1, for any $f \in \mathcal{H}$, we have

$$\langle U^*Uf, f \rangle + \lambda \overline{\langle Vf, f \rangle} + \lambda \langle Vf, f \rangle \geq (2\lambda - \lambda^2) \langle I_{\mathcal{H}}f, f \rangle,$$

and then,

$$\|Uf\|^2 + 2\lambda \operatorname{Re} \langle Vf, f \rangle \geq (2\lambda - \lambda^2) \operatorname{Re} \langle I_{\mathcal{H}}f, f \rangle,$$

which implies

$$\begin{aligned} \|Uf\|^2 &\geq (2\lambda - \lambda^2)\operatorname{Re}\langle I_{\mathcal{H}}f, f \rangle - 2\lambda\operatorname{Re}\langle Vf, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle (U + V)f, f \rangle - 2\lambda\operatorname{Re}\langle Vf, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle Uf, f \rangle - \lambda^2\operatorname{Re}\langle Vf, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle Uf, f \rangle + (1 - \lambda^2)\operatorname{Re}\langle Vf, f \rangle - \operatorname{Re}\langle Vf, f \rangle. \end{aligned}$$

Hence

$$\|Uf\|^2 + \operatorname{Re}\langle Vf, f \rangle \geq (2\lambda - \lambda^2)\operatorname{Re}\langle Uf, f \rangle + (1 - \lambda^2)\operatorname{Re}\langle Vf, f \rangle,$$

thus

$$\begin{aligned} \operatorname{Re}\left(\sum_{i \in J} \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right) + \left\| \sum_{i \in J^c} \langle f, g_i \rangle f_i \right\|^2 &\geq (2\lambda - \lambda^2) \cdot \operatorname{Re}\left(\sum_{i \in J} \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right) \\ &\quad + (1 - \lambda^2) \cdot \operatorname{Re}\left(\sum_{i \in J^c} \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right). \quad \square \end{aligned}$$

In the sequel we give a more general result. Consider a bounded sequence of complex numbers $\{a_i\}_{i \in I}$. In Theorem 8 we take

$$Uf = \sum_{i \in J^c} a_i \langle f, g_i \rangle f_i, \quad Vf = \sum_{i \in J} (1 - a_i) \langle f, g_i \rangle f_i.$$

We can get the following result.

COROLLARY 4. *Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} and $\{g_i\}_{i \in I}$ be an alternate dual frame of $\{f_i\}_{i \in I}$. Then for all bounded sequences $\{a_i\}_{i \in I}$ we have*

$$\begin{aligned} \operatorname{Re}\left(\sum_{i \in J} (1 - a_i) \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right) + \left\| \sum_{i \in J^c} a_i \langle f, g_i \rangle f_i \right\|^2 \\ \geq (2\lambda - \lambda^2) \cdot \operatorname{Re}\left(\sum_{i \in J} (1 - a_i) \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right) + (1 - \lambda^2) \cdot \operatorname{Re}\left(\sum_{i \in J^c} a_i \langle f, g_i \rangle \overline{\langle f, f_i \rangle}\right) \end{aligned}$$

REMARK 4. If we take $\lambda = \frac{1}{2}$ Theorem 8, we can obtain the inequality in Theorem 3.

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REFERENCES

- [1] R. BALAN AND P. CASAZZA AND D. EDIDIN, *On signal reconstruction without phase*, Applied and Computational Harmonic Analysis, **20**, 3 (2006), 345–356.
- [2] R. BALAN AND P. CASAZZA AND D. EDIDIN AND G. KUTYNIOK, *Decompositions of frames and a new frame identity*, Proceedings of SPIE, **5914**, (2005), 1–10.
- [3] R. BALAN AND P. CASAZZA AND D. EDIDIN AND G. KUTYNIOK, *A new identity for Parseval frames*, Proceedings of the American Mathematical Society, **135**, 4(2007), 1007–1015.
- [4] D. DONOHO AND E. CANDÉS, *Continuous curvelet transform: II. Discretization and frames*, Applied and Computational Harmonic Analysis, **19**, 2 (2005), 198–222.
- [5] I. DAUBECHIES AND A. GROSSMANN AND Y. MEYER, *Painless nonorthogonal expansions*, Journal of Mathematical Physics, **27**, 5 (1986), 1271–1283.
- [6] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Transactions of the American Mathematical Society, **72**, 2 (1952), 341–366.
- [7] Y. C. ELДАР AND G. D. FORNEY, *Optimal tight frames and quantum measurement*, IEEE Transactions on Information Theory, **48**, 3 (2002), 599–610.
- [8] P. GĂVRUȚA, *On some identities and inequalities for frames in Hilbert spaces*, Journal of mathematical analysis and applications, **321**, 1 (2006), 469–478.
- [9] D. HAN AND D. R. LARSON, *Frames, bases and group representations*, American Mathematical Society, **697**, 2006.
- [10] J. KOVACEVIC AND P. L. DRAGOTTI AND V. K. GOYAL, *Filter bank frame expansions with erasures*, IEEE Transactions on Information Theory, **48**, 6 (2002), 1439–1450.
- [11] S. LANG, *Real and functional analysis*, Springer Science and Business Media, 2012.
- [12] J. LENG AND D. HAN AND T. HUANG, *Optimal dual frames for communication coding with probabilistic erasures*, IEEE transactions on signal processing, **59**, 11 (2011), 5380–5389.
- [13] D. LI AND J. LENG, *On Some New Inequalities For Continuous Fusion Frames in Hilbert Spaces*, Mediterranean Journal of Mathematics, **15**, 4 (2018), 173.
- [14] D. LI AND J. LENG, *On some new inequalities for fusion frames in Hilbert spaces*, Mathematical Inequalities & Applications, **20**, 3 (2017), 889–900.
- [15] D. LI AND J. LENG AND T. HUANG AND Q. GAO, *Frame expansions with probabilistic erasures*, Digital Signal Processing, **72**, (2018), 75–82.
- [16] J. LI AND Y. ZHU, *Some equalities and inequalities for g -Bessel sequences in Hilbert spaces*, Applied Mathematics Letters, **25**, (2012), 1601–1607.
- [17] A. PORIA, *Some identities and inequalities for Hilbert-Schmidt frames*, Mediterranean Journal of Mathematics, **14**, 2 (2017), 59.
- [18] Z. XIANG, *New inequalities for g -frames in Hilbert C^* -modules*, Journal of Mathematical Inequalities, **10**, 3 (2016), 889–897.
- [19] X. ZHU AND G. WU, *A note on some equalities for frames in Hilbert spaces*, Applied Mathematics Letters, **23**, 7 (2010), 788–790.
- [20] P. ZHAO AND C. ZHAO AND P. CASAZZA, *Perturbation of regular sampling in shift-invariant spaces for frames*, IEEE Transactions on Information Theory, **52**, 10 (2006), 4643–4648.

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