

## MATRIX REARRANGEMENT INEQUALITIES REVISITED

VICTORIA M. CHAYES

**Abstract.** Let  $\|X\|_p = \text{Tr}[(X^*X)^{p/2}]^{1/p}$  denote the  $p$ -Schatten norm of a matrix  $X \in M_{n \times n}(\mathbb{C})$ , and  $\sigma(X)$  the singular values with  $\uparrow \downarrow$  indicating its increasing or decreasing rearrangements. We wish to examine inequalities between  $\|A+B\|_p^p + \|A-B\|_p^p$ ,  $\|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p$ , and  $\|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p$  for various values of  $1 \leq p < \infty$ . It was conjectured in [6] that a universal inequality  $\|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p \leq \|A+B\|_p^p + \|A-B\|_p^p \leq \|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p$  might hold for  $1 \leq p \leq 2$  and reverse at  $p \geq 2$ , potentially providing a stronger inequality to the generalization of Hanner's Inequality to complex matrices  $\|A+B\|_p^p + \|A-B\|_p^p \geq (\|A\|_p + \|B\|_p)^p + \||A\|_p - \|B\|_p\|^p$ . We extend some of the cases in which the inequalities of [6] hold, but offer counterexamples to any general rearrangement inequality holding. We simplify the original proofs of [6] with the technique of majorization. This also allows us to characterize the equality cases of all of the inequalities considered. We also address the commuting, unitary, and  $\{A, B\} = 0$  cases directly, and expand on the role of the anticommutator. In doing so, we extend Hanner's Inequality for self-adjoint matrices to the  $\{A, B\} = 0$  case for all ranges of  $p$ .

*Mathematics subject classification* (2010): 15A42.

*Keywords and phrases:* Matrix inequality, Hanner's inequality, singular value inequalities,  $p$ -Schatten norm, majorization.

### REFERENCES

- [1] F. J. ALMGREN JR, E. H. LIEB, *Symmetric decreasing rearrangement is sometimes continuous* Journal of the American Mathematical Society, pp. 683–773 (1989).
- [2] T. ANDO, *Majorization, doubly stochastic matrices, and comparison of eigenvalues*, Linear Algebra and its Applications **118**, 163–248 (1989), [https://doi.org/10.1016/0024-3795\(89\)90580-6](https://doi.org/10.1016/0024-3795(89)90580-6), <http://www.sciencedirect.com/science/article/pii/0024379589905806>.
- [3] K. BALL, E. A. CARLEN, E. H. LIEB, *Sharp uniform convexity and smoothness inequalities for trace norms*, Inventiones mathematicae **115** (1), 463–482 (1994), <https://doi.org/10.1007/BF01231769>.
- [4] J. C. BOURIN, E. Y. LEE, *Clarkson-McCarthy inequalities with unitary and isometry orbits*, Linear Algebra and its Applications **601**, 170–179 (2020), <https://doi.org/10.1016/j.laa.2020.04.019>, <http://www.sciencedirect.com/science/article/pii/S0024379520302135>.
- [5] A. BURCHARD, *Cases of equality in the riesz rearrangement inequality*, Annals of Mathematics **143** (3), 499–527 (1996). <http://www.jstor.org/stable/2118534>.
- [6] E. CARLEN, E. H. LIEB, *Some matrix rearrangement inequalities*, Annali di Matematica Pura ed Applicata **185** (5), S315–S324 (2006), <https://doi.org/10.1007/s10231-004-0147-z>.
- [7] K. FAN, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proceedings of the National Academy of Sciences of the United States of America **37** (11), 760–766 (1951), 10.1073/pnas.37.11.760, <https://www.ncbi.nlm.nih.gov/pubmed/16578416>.
- [8] I. M. GEL'FAND, M. A. NAIMARK, *The relation between the unitary representations of the complex unimodular group and its unitary subgroup*, Izv. Akad. Nauk SSSR Ser. Mat. **14** (3), 239–260 (1950).
- [9] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Some simple inequalities satisfied by convex functions*, Messenger Math. **58**, 145–152 (1929), <https://ci.nii.ac.jp/naid/10009422169/en/>.

- [10] G. H. HARDY, G. PÓLYA, *Inequalities*, Cambridge: Cambridge University Press (1934), Bibliography: p. 300–314.
- [11] F. HIAI, *Equality cases in matrix norm inequalities of golden-thompson type*, *Linear and Multilinear Algebra* **36** (4), 239–249 (1994), doi:10.1080/03081089408818297, <https://doi.org/10.1080/03081089408818297>.
- [12] F. HIAI, D. PETZ, *Introduction To Matrix Analysis And Applications*, 1 edn., chap. 6, pp. 227–271, Springer International Publishing, Cham (2014).
- [13] A. HORN, *On the singular values of a product of completely continuous operators*, *Proceedings of the National Academy of Sciences of the United States of America* **36** (7), 374 (1950).
- [14] A. W. MARSHALL, I. OLKIN, B. C. ARNOLD, *Inequalities: Theory of Majorization and Its Applications*, 2 edn., Springer, New York (2011).
- [15] C. MCCARTHY, *c-p cp*, *Isr. J. Math.* **5**, 249–271 (1967).
- [16] N. TOMCZAK-JAEGERMANN, *The moduli of smoothness and convexity and the Rademacher averages of the trace classes  $S_p(1 \leq p \leq \infty)^*$* , *Studia Mathematica* **50** (2), 163–182 (1974), <http://eudml.org/doc/217886>.
- [17] M. TOMIĆ, *Théoreme de gauss relatif au centre de gravité et son application*, *Bull. Soc. Math. Phys. Serbie* **1**, 31–40 (1949).
- [18] H. WEYL, *Inequalities between two types of eigenvalues of a linear transformation*, *Proceedings of the National Academy of Sciences of the United States of America* **35** (7), 408–411 (1949).