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OPERATOR INEQUALITIES AND GYROLINES OF THE WEIGHTED GEOMETRIC MEANS

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Abstract. We consider in this paper two different types of the weighted geometric means of positive definite operators. We show the component-wise bijection of these geometric means and give a geometric property of the spectral geometric mean as a metric midpoint. Moreover, several interesting inequalities related with the geometric means of positive definite operators will be shown. We also see the meaning of weighted geometric means in the gyrogroup structure with finite dimension and find the formulas of weighted geometric means of 2-by-2 positive definite matrices and density matrices.

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