

WEIGHTED NORM INEQUALITIES FOR THE GENERALIZED MULTILINEAR STIELTJES TRANSFORMATION

VÍCTOR GARCÍA GARCÍA AND PEDRO ORTEGA SALVADOR*

(Communicated by P. Tradacete Perez)

Abstract. We characterize some weighted strong and weak-type inequalities for the generalized Stieltjes and Calderón multilinear operators. As applications, we characterize a weighted multilinear Hilbert's inequality and a weighted Hilbert's multiple series theorem.

1. Introduction and results

The generalized Stieltjes transform, also known as generalized Hilbert operator, is defined for non-negative functions f on $(0, \infty)$ by

$$\mathcal{S}_\lambda f(x) = \int_0^\infty \frac{f(t)}{(x+t)^\lambda} dt, \quad x \in (0, \infty),$$

where $\lambda > 0$.

Another classical operator, closely related to \mathcal{S}_λ , is the generalized Calderón operator \mathcal{E}_λ , defined also for non-negative functions f on $(0, \infty)$ by the sum of the Hardy-type operator P_λ and its adjoint Q_λ , i.e.,

$$\mathcal{E}_\lambda f(x) = P_\lambda f(x) + Q_\lambda f(x) = \frac{1}{x^\lambda} \int_0^x f(t) dt + \int_x^\infty \frac{f(t)}{t^\lambda} dt.$$

K. Andersen characterized in [2] the pairs of weights (u, v) for which the weighted Stieltjes inequality

$$\left(\int_0^\infty \mathcal{S}_\lambda f(x)^q u(x) dx \right)^{\frac{1}{q}} \leq K \left(\int_0^\infty f^p v \right)^{\frac{1}{p}} \quad (1)$$

holds for all non-negative f in the case $1 \leq p \leq q \leq \infty$. For $\lambda > 0$, he proved that (1) holds if and only there exists a constant $K > 0$ such that for all $r > 0$, the inequality

$$r^\lambda \left(\int_0^\infty \frac{u(x)}{(x+r)^{\lambda q}} dx \right)^{\frac{1}{q}} \left(\int_0^\infty \frac{\sigma(x)}{(x+r)^{\lambda p'}} dx \right)^{\frac{1}{p'}} \leq K \quad (2)$$

Mathematics subject classification (2020): 26D15.

Keywords and phrases: Calderón operator, Hilbert inequality, Hilbert operator, multilinear Hardy operators, multilinear Stieltjes transform, Stieltjes transform, weighted inequalities, weights.

This research has been supported in part by Ministerio de Ciencia, Innovación y Universidades (Grant PGC2018-096166-B-I00) and Junta de Andalucía (Grants FQM354 and UMA18-FEDERJA-002).

* Corresponding author.

holds, where p' is the conjugate exponent of p and $\sigma = v^{1-p'}$.

The same result holds for \mathcal{C}_λ , since $\frac{1}{2^\lambda} \mathcal{C}_\lambda f(x) \leq \mathcal{S}_\lambda f(x) \leq \mathcal{C}_\lambda f(x)$ for all f and $x \in (0, \infty)$.

The m -linear generalized Stieltjes transform was defined in [6] and [8] for m -tuples (f_1, f_2, \dots, f_m) of non-negative functions on $(0, \infty)$ by

$$\mathcal{S}_\lambda(f_1, f_2, \dots, f_m)(x) = \int_{(0, \infty)^m} \frac{f_1(y_1) f_2(y_2) \cdots f_m(y_m)}{(x + y_1 + y_2 + \cdots + y_m)^{\lambda m}} dy_1 dy_2 \cdots dy_m.$$

It is interesting to observe that multilinear operators of this type have been studied in [4] and [5] in connection with the boundedness of the Bergman projection on tube domains.

We also define the m -linear generalized Calderón operator as

$$\mathcal{C}_\lambda(f_1, f_2, \dots, f_m)(x) = \prod_{i=1}^m P_\lambda f_i(x) + \sum_{i=1}^m Q_\lambda(f_i \prod_{\substack{j=1 \\ j \neq i}}^m P_\lambda f_j)(x),$$

i.e., the sum of the generalized multilinear Hardy operator and its adjoints.

If $\lambda = 1$, we simply write \mathcal{S} and \mathcal{C} instead of \mathcal{S}_1 and \mathcal{C}_1 , respectively.

The operators \mathcal{S}_λ and \mathcal{C}_λ are equivalent, in the sense that there are two positive constants K_1 and K_2 independent of f_1, f_2, \dots, f_m and x such that

$$\mathcal{C}_\lambda(f_1, f_2, \dots, f_m)(x) \leq K_1 \mathcal{S}_\lambda(f_1, f_2, \dots, f_m)(x) \leq K_2 \mathcal{C}_\lambda(f_1, f_2, \dots, f_m)(x). \quad (3)$$

The boundedness of the operator \mathcal{S} from $L^{p_1}(|x|^{\alpha_1}) \times L^{p_2}(|x|^{\alpha_2}) \times \cdots \times L^{p_m}(|x|^{\alpha_m})$ to $L^p(|x|^\alpha)$ in the case $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ was studied in [11]. The authors characterized in [12] the boundedness of \mathcal{S} and \mathcal{C} from $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \cdots \times L^{p_m}(v_m)$ to $L^p(u)$ in the case $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and $u = \prod_{i=1}^m v_i^{\frac{1}{p_i}}$. In this paper, our purpose is to extend the previously cited Andersen's result to the multilinear setting, in the sense of assuming some relationships between the exponents and weights different from the ones considered in [12]. The main theorem is the next one.

THEOREM 1. *Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m > 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Assume that $p_i \leq \min_{i+1 \leq j \leq m} \{p'_j\}$ for each $i \in \{1, 2, \dots, m-1\}$. Let u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$ and $\sigma_j = v_j^{1-p'_j}$. The next statements are equivalent:*

- (i) *The Stieltjes transform \mathcal{S}_λ is bounded from $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \cdots \times L^{p_m}(v_m)$ to $L^p(u)$.*
- (ii) *The Calderón operator \mathcal{C}_λ is bounded from $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \cdots \times L^{p_m}(v_m)$ to $L^p(u)$.*

(iii) There exists $K > 0$ such that for each $i \in \{1, 2, \dots, m\}$ and every $r > 0$ the inequalities

$$\left(\int_0^r u\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{\sigma_i(t)}{t^{\lambda m p_i}} dt\right)^{\frac{1}{p_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\int_0^r \sigma_j\right)^{\frac{1}{p_j}} \leq K \quad (4)$$

and

$$\left(\int_r^\infty \frac{u(t)}{t^{\lambda m p}} dt\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\int_0^r \sigma_j\right)^{\frac{1}{p_j}} \leq K \quad (5)$$

hold.

(iv) There exists $K > 0$ such that for each $r > 0$ the inequality

$$r^{\lambda m^2} \left(\int_0^\infty \frac{u(t)}{(t+r)^{\lambda m p}} dt\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\int_0^\infty \frac{\sigma_j(t)}{(t+r)^{\lambda m p_j}} dt\right)^{\frac{1}{p_j}} \leq K \quad (6)$$

holds.

We can also prove a weak-type result. In this setting, we will work with a more general operator, the modified Stieltjes transform $\mathcal{S}_{g,\lambda}$, defined by

$$\mathcal{S}_{g,\lambda}(f_1, f_2, \dots, f_m)(x) = g(x) \int_0^\infty \frac{f_1(t_1) f_2(t_2) \dots f_m(t_m)}{(x+t_1+t_2+\dots+t_m)^{\lambda m}} dt_1 dt_2 \dots dt_m,$$

where g is a positive function. It is clear that the operator $\mathcal{S}_{g,\lambda}$ is equivalent to the modified multilinear Calderón operator $\mathcal{C}_{g,\lambda}$, defined by

$$\begin{aligned} \mathcal{C}_{g,\lambda}(f_1, \dots, f_m)(x) &= \frac{g(x)}{x^{\lambda m}} \left(\int_0^x f_1\right) \dots \left(\int_0^x f_m\right) + \sum_{i=1}^m g(x) \int_x^\infty \frac{f_i(s)}{s^{\lambda m}} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\int_0^s f_j\right) ds \\ &= \mathcal{P}_{g,\lambda}(f_1, \dots, f_m)(x) + \sum_{i=1}^m \mathcal{D}_{g,\lambda}^i(f_1, \dots, f_m)(x). \end{aligned}$$

Weak-type inequalities for modified linear or sublinear operators are included in the topic of mixed weak-type inequalities. This kind of inequalities goes back to Andersen and Muckenhoupt's paper [3], where they studied weighted mixed weak-type inequalities for Hardy operators, Hilbert transform and the maximal operator. Andersen and Muckenhoupt's results were extended and improved by E. Sawyer in [18]. Later, many papers have dealt with this topic, even in the multilinear setting (see, for instance, [9], [14], [15], [16] and [17]).

The weak-type result reads as follows.

THEOREM 2. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m \geq 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Assume that $p_i \leq \min_{i+1 \leq j \leq m} \{p'_j\}$ for each $i \in \{1, 2, \dots, m-1\}$. Let

u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$ and $\sigma_j = v_j^{1-p'_j}$. The next statements are equivalent:

- (i) The modified Stieltjes transform $\mathcal{S}_{g,\lambda}$ is bounded from $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$ to $L^{p,\infty}(u)$.
- (ii) The modified Calderón operator $\mathcal{C}_{g,\lambda}$ is bounded from $L^{p_1}(v_1) \times L^{p_2}(v_2) \times \dots \times L^{p_m}(v_m)$ to $L^{p,\infty}(u)$.
- (iii) For each $i \in \{1, 2, \dots, m\}$, the conditions

$$\sup_{r>0} \|g\chi_{(0,r)}\|_{p,\infty;u} \left(\int_r^\infty \frac{\sigma_i(t)}{t^{\lambda m p'_i}} dt \right)^{\frac{1}{p'_i}} \prod_{\substack{j=1 \\ j \neq i}}^m \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} < \infty \tag{7}$$

and

$$\sup_{r>0} \left\| \frac{g(x)}{x^{\lambda m}} \chi_{(r,\infty)}(x) \right\|_{p,\infty;u} \prod_{j=1}^m \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} < \infty \tag{8}$$

hold, where $(\int_0^r \sigma_j)^{\frac{1}{p'_j}}$ and $(\int_r^\infty \frac{\sigma_j(t)}{t^{\lambda m p'_j}} dt)^{\frac{1}{p'_j}}$ have to be understood as

$\text{ess sup}_{t \in (0,r)} v_j^{-1}(t)$ and $\text{ess sup}_{t \in (r,\infty)} \frac{v_j^{-1}(t)}{t^{\lambda m}}$, respectively, if $p_j = 1$.

- (iv) There exists $K > 0$ such that for each $r > 0$ the inequality

$$r^{\lambda m^2} \left\| \frac{g(t)}{(t+r)^{\lambda m}} \right\|_{p,\infty;u} \prod_{j=1}^m \left(\int_0^\infty \frac{\sigma_j(t)}{(t+r)^{\lambda m p'_j}} dt \right)^{\frac{1}{p'_j}} \leq K \tag{9}$$

holds, where $(\int_0^\infty \frac{\sigma_j(t)}{(t+r)^{\lambda m p'_j}} dt)^{\frac{1}{p'_j}}$ stands for $\text{ess sup}_{t \in (0,\infty)} \frac{v_j^{-1}(t)}{(t+r)^{\lambda m}}$ when $p_j = 1$.

Observe that the conditions on the exponents p_1, p_2, \dots, p_m hold, for instance, if $p_i \leq 2$ for all $i \in \{1, 2, \dots, m\}$. Observe also that $p_i \leq \min_{i+1 \leq j \leq m} \{p'_j\}$ for each $i \in \{1, 2, \dots, m-1\}$ is equivalent to $\max_{1 \leq j \leq m, j \neq i} \{p_j\} \leq p'_i$ for all $i \in \{1, 2, \dots, m\}$.

The boundedness of the Stieltjes transform \mathcal{S} is closely related to the celebrated Hilbert’s inequality [13], which asserts that if $p > 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p \right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'} \right)^{\frac{1}{p'}}.$$

The next result, which characterizes a weighted multilinear Hilbert’s inequality, can be got as a simple consequence of Theorem 1 with a duality argument.

THEOREM 3. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m > 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Assume that $p_i \leq \min_{i+1 \leq j \leq m} \{p'_j\}$ for each $i \in \{1, 2, \dots, m-1\}$. Let

u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$ and $\sigma = u^{1-p'}$. Then the weighted multilinear Hilbert's inequality

$$\int_{(0, \infty)^{m+1}} \frac{f(y)f_1(y_1)\cdots f_m(y_m)}{(y+y_1+\cdots+y_m)^{\lambda m}} dy dy_1 \cdots dy_m \leq K \|f\|_{p', \sigma} \|f_1\|_{p_1, v_1} \cdots \|f_m\|_{p_m, v_m} \quad (10)$$

holds if and only if (6) does.

It is worth noting that the authors have already characterized in [12] the weights u, v_1, v_2, \dots, v_m for (10) to hold in the case $\frac{1}{p} = \sum_j \frac{1}{p_j}$ and $u = \prod_j v_j^{\frac{p}{p_j}}$.

The discrete version of Theorem 3 yields to the next multiple weighted analogue of the classical Hilbert's double series inequality ([13]).

THEOREM 4. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m > 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Assume that $p_i \leq \min_{i+1 \leq j \leq m} \{p'_j\}$ for each $i \in \{1, 2, \dots, m-1\}$. Let $\{u_n\}, \{v_n^1\}, \{v_n^2\}, \dots, \{v_n^m\}$ be sequences of positive numbers. Let $\sigma_n = u_n^{1-p'}$ and $\sigma_n^j = (v_n^j)^{1-p'_j}$ for $j = 1, 2, \dots, m$. Then the weighted Hilbert's multiple series inequality

$$\sum_{n=1}^{\infty} \sum_{i_1, i_2, \dots, i_m=1}^{\infty} \frac{a_n a_{i_1}^1 a_{i_2}^2 \cdots a_{i_m}^m}{(n+i_1+i_2+\cdots+i_m)^{\lambda m}} \leq K \left(\sum_{n=1}^{\infty} a_n^{p'} \sigma_n \right)^{\frac{1}{p'}} \prod_{j=1}^m \left(\sum_{i_j=1}^{\infty} (a_{i_j}^j)^{p_j} v_{i_j}^j \right)^{\frac{1}{p_j}}$$

holds for all positive sequences $\{a_n\}, \{a_{n_1}^1\}, \{a_{n_2}^2\}, \dots, \{a_{n_m}^m\}$ if and only if

$$\sup_{r>0} r^{\lambda m^2} \left(\sum_{n=1}^{\infty} \frac{u_n}{(n+r)^{\lambda m p}} \right)^{\frac{1}{p'}} \prod_{j=1}^m \left(\sum_{i_j=1}^{\infty} \frac{\sigma_{i_j}^j}{(i_j+r)^{\lambda m p'_j}} \right)^{\frac{1}{p'_j}} < \infty.$$

The next two sections consist of the proofs of Theorems 1 and 2. In order to prove them, we will characterize some weighted weak and strong-type inequalities for the generalized multilinear Hardy operator and its adjoints. It will be done in several lemmas of independent interest.

Observe that, all along the paper, the letter K stands for a positive constant, not necessarily the same at each occurrence.

2. Proof of Theorem 1

(iii) \Rightarrow (ii)

The proof of this implication is based on the following lemmas. The first one characterizes the weighted inequalities for the generalized multilinear Hardy operator. The bilinear case with $\lambda = 1$ was studied in [1].

LEMMA 1. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m > 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Let u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$. Then the multilinear generalized Hardy operator

$$\mathcal{P}_\lambda(f_1, f_2, \dots, f_m)(x) = \prod_{j=1}^m P_\lambda f_j(x)$$

is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$ to $L^p(u)$ if and only if (5) holds.

Proof. We will prove the Lemma by induction on m . The case $m = 1$ is Bradley's Theorem ([7]). Let us suppose that the result is true for $m - 1$ and let us prove it for m . The inequality

$$\left(\int_0^\infty \left(\prod_{j=1}^m P_\lambda f_j(x) \right)^p u(x) dx \right)^{\frac{1}{p}} \leq K \prod_{j=1}^m \|f_j\|_{p_j, v_j}$$

is equivalent to

$$\left(\int_0^\infty \left(\prod_{j=1}^{m-1} P_\lambda f_j(x) \right)^p \left(P_\lambda \left(\frac{f_m}{\|f_m\|_{p_m, v_m}} \right) (x) \right)^p u(x) dx \right)^{\frac{1}{p}} \leq K \prod_{j=1}^{m-1} \|f_j\|_{p_j, v_j}.$$

This is a $(m - 1)$ -linear Hardy inequality with weights $w = u \left(P_\lambda \left(\frac{f_m}{\|f_m\|} \right) \right)^p, v_1, v_2, \dots, v_{m-1}$. By induction hypothesis, that inequality holds if and only if there exists a constant $K > 0$ such that for every $r > 0$ and every $f_m \geq 0$

$$\left(\int_r^\infty \left(P_\lambda \left(\frac{f_m}{\|f_m\|_{p_m, v_m}} \right) (x) \right)^p \frac{u(x)}{x^{\lambda(m-1)p}} dx \right)^{\frac{1}{p}} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} \leq K$$

holds, i.e.,

$$\left(\int_r^\infty \left(\int_0^x f_m \right)^p \frac{u(x)}{x^{\lambda m p}} dx \right)^{\frac{1}{p}} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} \leq K \|f_m\|_{p_m, v_m}. \quad (11)$$

Taking into account that $\int_0^\infty f_m = \int_0^r f_m + \int_r^\infty f_m$ and also that the inequality above remains valid for f_m supported on (r, ∞) , we have that (11) is equivalent to the two following inequalities:

$$\left(\int_r^\infty \frac{u(x)}{x^{\lambda m p}} dx \right)^{\frac{1}{p}} \left(\int_0^r f_m \right)^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} \leq K \|f_m\|_{p_m, v_m} \quad (12)$$

and

$$\left(\int_r^\infty \left(\int_r^x f_m \right)^p \frac{u(x)}{x^{\lambda m p}} dx \right)^{\frac{1}{p}} \leq K \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \left(\int_r^\infty f_m^{p_m} v_m \right)^{\frac{1}{p_m}}. \quad (13)$$

The inequality (12) holds for all f_m if and only if (5) does. It is clear, indeed, that (12) implies (5) by testing (12) with $f_m = \sigma_m \chi_{(0,r)}$, whereas the converse follows from (5) applying Hölder's inequality. On the other hand, by Bradley's Theorem ([7]), (13) is equivalent to the existence of a constant $K > 0$ such that for each $r > 0$ and every $s > r$,

$$\left(\int_s^\infty \frac{u(x)}{x^{\lambda mp}} dx \right)^{\frac{1}{p}} \left(\int_r^s \sigma_m \right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} \leq K. \quad (14)$$

Observe that we have strongly used that the boundedness constant of Bradley's Theorem is equivalent to the one appearing in the equivalent condition on the weights.

Therefore, (11) is equivalent to (5) and (14). Finally, let us observe that (5) implies (14), what shows that (11) is equivalent to (5) and finishes the proof. \square

The next lemma characterizes the weighted inequalities for the adjoints of the multilinear Hardy operator.

LEMMA 2. *Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m > 1$ with $p_j \leq p$ for each $j \in \{1, 2, \dots, m\}$. Let $i_0 \in \{1, 2, \dots, m\}$ such that $\max_{\substack{1 \leq j \leq m \\ j \neq i_0}} \{p_j\} \leq p'_{i_0}$. Let u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$. Then the i_0 -adjoint of the multilinear Hardy operator*

$$\mathcal{D}_\lambda^{i_0}(f_1, f_2, \dots, f_m)(x) = \mathcal{Q}_\lambda(f_{i_0} \prod_{j \neq i_0} P_\lambda f_j)(x)$$

is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$ to $L^p(u)$ if and only if (4) holds for $i = i_0$.

Proof. By duality, the operator $\mathcal{D}_\lambda^{i_0}$ is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$ to $L^p(u)$ if and only if the multilinear Hardy operator \mathcal{P}_λ is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_{i_0-1}}(v_{i_0-1}) \times L^{p'}(\sigma) \times L^{p_{i_0+1}}(v_{i_0+1}) \times \dots \times L^{p_m}(v_m)$ to $L^{p'_{i_0}}(\sigma_{i_0})$, where $\sigma = u^{1-p'}$ and $\sigma_{i_0} = v_{i_0}^{1-p'_{i_0}}$. Since $p' \leq p'_{i_0}$ and $\max_{\substack{1 \leq j \leq m \\ j \neq i_0}} \{p_j\} \leq p'_{i_0}$, we can apply Lemma 1 to get the equivalence with condition (4) for $i = i_0$. \square

Now, the implication (iii) \Rightarrow (ii) is immediate. In fact, we only have to observe that, by Lemma 1, (5) implies the boundedness of the multilinear generalized Hardy operator and that, by Lemma 2, (4) implies that the adjoints \mathcal{D}_λ^i , $i \in \{1, 2, \dots, m\}$ are bounded. Then, the m -linear Calderón operator is bounded.

(iv) \Rightarrow (iii)

Assume that (6) holds. Let $r > 0$. The inequality (6) implies

$$r^{\lambda m^2} \left(\int_r^\infty \frac{u(t)}{(t+r)^{\lambda mp}} dt \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\int_0^r \frac{\sigma_j(t)}{(t+r)^{\lambda m p'_j}} dt \right)^{\frac{1}{p'_j}} \leq K.$$

Since $t + r \leq 2t$ if $t \in (r, \infty)$ and $t + r \leq 2r$ if $t \in (0, r)$, the previous inequality gives

$$r^{\lambda m^2} \left(\int_r^\infty \frac{u(t)}{t^{\lambda m p}} dt \right)^{\frac{1}{p}} \prod_{j=1}^m \frac{1}{r^{\lambda m}} \left(\int_0^r \sigma_j(t) dt \right)^{\frac{1}{p_j}} \leq K,$$

which is (5). In the same way, we deduce the inequalities (4) for $i \in \{1, 2, \dots, m\}$ from (6).

(i) \Rightarrow (iv)

For this implication, we will need the following lemma, which gives a generalization of inequality (1.14) in [2].

LEMMA 3. *Let $m \in \mathbb{N}$ and $\lambda > 0$. Then there is a positive constant $K > 0$ such that for all $r, t, x_1, \dots, x_m \in (0, \infty)$, the inequality*

$$\frac{1}{(t + x_1 + \dots + x_m)^{\lambda m}} \geq K \frac{r^{\lambda m^2}}{(r + t)^{\lambda m} \prod_{j=1}^m (r + x_j)^{\lambda m}} \tag{15}$$

holds.

Proof. Applying successively the inequality (1.14) in [2], we have

$$\begin{aligned} \frac{1}{t + x_1 + \dots + x_m} &\geq \frac{1}{r + t + x_2 + \dots + x_m} \cdot \frac{r}{r + x_1} \\ &\geq \frac{1}{2r + t + x_3 + \dots + x_m} \cdot \frac{r}{r + x_2} \cdot \frac{r}{r + x_1} \\ &\geq \dots \\ &\geq \frac{1}{mr + t} \cdot \frac{r^m}{\prod_{j=1}^m (r + x_j)} \\ &\geq K \frac{r^m}{(r + t) \prod_{j=1}^m (r + x_j)}. \end{aligned}$$

Raising this inequality to λm , we get (15). \square

Now, we can prove the implication (i) \Rightarrow (iv). Assume that (i) holds, i.e., there is $K > 0$ such that

$$\left(\int_{(0, \infty)} \left(\int_{(0, \infty)^m} \frac{f_1(x_1) f_2(x_2) \dots f_m(x_m)}{(t + x_1 + x_2 + \dots + x_m)^{\lambda m}} dx_1 dx_2 \dots dx_m \right)^p u(t) dt \right)^{\frac{1}{p}} \leq K \prod_{j=1}^m \|f_j\|_{p_j, v_j}. \tag{16}$$

Let $r > 0$ and $f_j(x) = \frac{\sigma_j(x)}{(x+r)^{\frac{\lambda m}{p_j-1}}}$. Then, by Lemma 3, the left-hand side of (16) is greater than

$$\begin{aligned} & r^{\lambda m^2} \left(\int_0^\infty \frac{u(t)}{(r+t)^{\lambda m p}} dt \right)^{\frac{1}{p}} \prod_{j=1}^m \int_0^\infty \frac{\sigma_j(x_j)}{(x_j+r)^{\lambda m \left(\frac{1}{p_j-1}+1\right)}} dx_j \\ &= r^{\lambda m^2} \left(\int_0^\infty \frac{u(t)}{(r+t)^{\lambda m p}} dt \right)^{\frac{1}{p}} \prod_{j=1}^m \int_0^\infty \frac{\sigma_j(x_j)}{(x_j+r)^{\lambda m p_j}} dx_j. \end{aligned}$$

Finally, (16) gives

$$r^{\lambda m^2} \left(\int_0^\infty \frac{u(t)}{(r+t)^{\lambda m p}} dt \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\int_0^\infty \frac{\sigma_j(x_j)}{(x_j+r)^{\lambda m p_j}} dx_j \right)^{\frac{1}{p_j}} \leq K.$$

3. Proof of Theorem 2

The proof follows the pattern of the previous one. We only have to show that condition (8) characterizes the boundedness of $\mathcal{P}_{g,\lambda}$ and (7) characterizes the boundedness of $\mathcal{D}_{g,\lambda}^i$ for all $i \in \{1, 2, \dots, m\}$. In fact, we will only prove (iii) \Rightarrow (ii). The remaining implications can be proved as in the proof of Theorem 1 with obvious changes.

(iii) \Rightarrow (ii)

As in the proof of Theorem 1, we will need the following lemmas, of independent interest. The interest resides in the fact that, as far as we know, there are no results about weighted weak-type inequalities for multilinear Hardy type operators in the literature. The first lemma characterizes the weighted weak-type inequalities for the modified multilinear generalized Hardy operator $\mathcal{P}_{g,\lambda}$.

LEMMA 4. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m \geq 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Let u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$. Then, for any positive function g , the modified multilinear generalized Hardy operator

$$\mathcal{P}_{g,\lambda}(f_1, f_2, \dots, f_m)(x) = g(x) \prod_{j=1}^m P_\lambda f_j(x)$$

is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$ to $L^{p,\infty}(u)$ if and only if (8) holds.

Proof. We will prove the lemma by induction on m . The result for $m = 1$ was proved in [10] (see also [17]). Let us suppose that the result is true for $m - 1$ and let us prove it for m . The weak-type inequality

$$\left\| \frac{g(x)}{x^{\lambda m}} \prod_{j=1}^m \left(\int_0^x f_j \right) \right\|_{p,\infty;u} \leq K \prod_{j=1}^m \|f_j\|_{p_j,v_j}$$

is equivalent to

$$\left\| \frac{g(x)}{x^{\lambda m}} \left(\int_0^x \frac{f_m}{\|f_m\|_{p_m, v_m}} \prod_{j=1}^{m-1} \left(\int_0^x f_j \right) \right) \right\|_{p, \infty; u} \leq K \prod_{j=1}^{m-1} \|f_j\|_{p_j, v_j}.$$

Let $h_{f_m}(x) = \frac{g(x)}{x^\lambda} \int_0^x \frac{f_m}{\|f_m\|_{p_m, v_m}}$. Then, the inequality above is equivalent to the next one:

$$\left\| \frac{h_{f_m}(x)}{x^{\lambda(m-1)}} \prod_{j=1}^{m-1} \left(\int_0^x f_j \right) \right\|_{p, \infty; u} \leq K \prod_{j=1}^{m-1} \|f_j\|_{p_j, v_j}.$$

This is a $(m - 1)$ -linear weak-type inequality for the operator $P_{h_{f_m}, \lambda}$. By induction hypothesis, it is equivalent to the existence of a constant $K > 0$ such that for each $r > 0$ and each positive function f_m

$$\left\| \frac{h_{f_m}(x)}{x^{\lambda(m-1)}} \chi_{(r, \infty)}(x) \right\|_{p, \infty; u} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} \leq K.$$

It can also be written as

$$\left\| \frac{g(x)}{x^{\lambda m}} \chi_{(r, \infty)}(x) \left(\int_0^x f_m \right) \right\|_{p, \infty; u} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} \leq K \|f_m\|_{p_m, v_m}. \tag{17}$$

Taking into account that $\int_0^x f_m = \int_0^r f_m + \int_r^x f_m$ and that the last inequality remains valid for f_m supported on (r, ∞) , we have that the inequality above is equivalent to the two following ones:

$$\left\| \frac{g(x)}{x^{\lambda m}} \chi_{(r, \infty)}(x) \left(\int_r^x f_m \right) \right\|_{p, \infty; u} \leq K \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{-\frac{1}{p_j}} \left(\int_r^\infty f_m^{p_m} v_m \right)^{\frac{1}{p_m}} \tag{18}$$

and

$$\left\| \frac{g(x)}{x^{\lambda m}} \chi_{(r, \infty)}(x) \right\|_{p, \infty; u} \left(\int_0^r f_m \right) \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} \leq K \|f_m\|_{p_m, v_m}. \tag{19}$$

On one hand, (19) is equivalent to (8). We simply test with the function $f_m = \sigma_m \chi_{(0, r)}$ for one implication and we apply Hölder’s inequality for the converse. On the other hand, we can apply the result in [10] (see also [17]) and (18) is equivalent to

$$\sup_{s > r > 0} \left\| \frac{g(x)}{x^{\lambda m}} \chi_{(s, \infty)}(x) \right\|_{p, \infty; u} \left(\int_r^s \sigma_m \right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} < \infty. \tag{20}$$

As in the proof of Lemma 1, we have used the equivalence between the boundedness constant of the Hardy operator and the constant appearing in the condition on the weights. This fact will be also used in the proof of Lemma 5. Therefore, (17) is equivalent to (8) and (20). To complete the proof, we only have to observe that (8) implies (20), what shows that (17) is equivalent to (8). \square

Now, we will characterize the weighted inequalities for the modified adjoints $\mathcal{D}_{g,\lambda}^{i_0}, i_0 \in \{1, 2, \dots, m\}$, of the multilinear generalized Hardy operator.

LEMMA 5. Let $\lambda > 0$ and $p, p_1, p_2, \dots, p_m \geq 1$ with $p_i \leq p$ for each $i \in \{1, 2, \dots, m\}$. Let $i_0 \in \{1, 2, \dots, m\}$ such that $\max_{\substack{1 \leq j \leq m \\ j \neq i_0}} \{p_j\} \leq p'_{i_0}$. Let u, v_1, v_2, \dots, v_m be positive measurable functions on $(0, \infty)$. Then, for any positive function g , the modified i_0 -adjoint of the multilinear generalized Hardy operator

$$\mathcal{D}_{g,\lambda}^{i_0}(f_1, f_2, \dots, f_m)(x) = g(x)Q_\lambda(f_{i_0} \prod_{\substack{j=1 \\ j \neq i_0}}^m P_\lambda f_j)(x)$$

is bounded from $L^{p_1}(v_1) \times \dots \times L^{p_m}(v_m)$ to $L^{p,\infty}(u)$ if and only if (7) holds for $i = i_0$.

Proof. We will work by induction on m . The result for $m = 1$ was proved in [10] (see also [17]). Let us suppose that $i_0 \in \{1, 2, \dots, m-1\}$ and that if $p_1, p_2, \dots, p_{m-1} > 1$ verify $p_i \leq p$ for all i and $\max_{\substack{1 \leq j \leq m \\ j \neq i_0}} \{p_j\} \leq p'_{i_0}$, then the $(m-1)$ -linear weak-type inequality

$$\left\| \left\| g(x) \int_x^\infty \frac{f_{i_0}(s)}{s^{\lambda(m-1)}} \prod_{\substack{j=1 \\ j \neq i_0}}^{m-1} \left(\int_0^s f_j \right) ds \right\|_{p,\infty;u} \right\| \leq K \prod_{j=1}^{m-1} \|f_j\|_{p_j,v_j}$$

holds, with a constant K independent of f_1, f_2, \dots, f_{m-1} if and only if

$$\sup_{r>0} \|g\chi_{(0,r)}\|_{p,\infty;u} \left(\int_r^\infty \frac{\sigma_i(t)}{t^{\lambda(m-1)p'_{i_0}}} dt \right)^{\frac{1}{p'_{i_0}}} \prod_{\substack{j=1 \\ j \neq i_0}}^{m-1} \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} < \infty.$$

Let us prove the result for m . Let v_1, v_2, \dots, v_m, u be the weights. Let $j_0 \in \{1, 2, \dots, m\}, j_0 \neq i_0$. We define the function h as

$$h(s) = \frac{f_{i_0}(s)}{s^\lambda} \int_0^s \frac{f_{j_0}(t)}{\|f_{j_0}\|_{p_{j_0},v_{j_0}}} dt.$$

Then, the weak-type inequality

$$\left\| \left\| g(x) \int_x^\infty \frac{f_{i_0}(s)}{s^{\lambda m}} \prod_{\substack{j=1 \\ j \neq i_0}}^m \left(\int_0^s f_j \right) ds \right\|_{p,\infty;u} \right\| \leq K \prod_{j=1}^m \|f_j\|_{p_j,v_j}$$

is equivalent to the following one:

$$\left\| g(x) \int_x^\infty \frac{h(s)}{s^{\lambda(m-1)}} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^s f_j \right) ds \right\|_{p, \infty; u} \leq K \left\| \frac{h(s)s^\lambda}{\int_0^s \frac{f_{j_0}(t)}{\|f_{j_0}\|_{p_{j_0}, v_{j_0}}} dt} \right\|_{p_{i_0}, v_{i_0}} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \|f_j\|_{p_j, v_j}. \tag{21}$$

Then, the $(m - 1)$ -linear operator $\mathcal{D}_{g, \lambda}^{i_0}$ is bounded from $L^{p_{i_0}}(w_{i_0}) \times \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m L^{p_j}(v_j)$ to $L^{p, \infty}(u)$, where

$$w_{i_0}(s) = v_{i_0}(s) \frac{s^{\lambda p_{i_0}}}{\left(\int_0^s \frac{f_{j_0}(t)}{\|f_{j_0}\|_{p_{j_0}, v_{j_0}}} dt \right)^{p_{i_0}}},$$

$p_j \leq p$ for all j and $\max_{\substack{1 \leq j \leq m \\ j \neq i_0}} \{p_j\} \leq p'_{i_0}$. Now, we are going to apply the induction

hypothesis. We will distinguish the case $p_{i_0} = 1$ from $p_{i_0} > 1$. Assume first that $p_{i_0} > 1$. By induction hypothesis, (21) is equivalent to the existence of a constant $K > 0$ such that for all $r > 0$ and for all f_{j_0} ,

$$\|g\mathcal{X}_{(0,r)}\|_{p, \infty; u} \left(\int_r^\infty \frac{\sigma_{i_0}(t)}{t^{\lambda m p'_{i_0}}} \left(\int_0^t \frac{f_{j_0}(s)}{\|f_{j_0}\|_{p_{j_0}, v_{j_0}}} ds \right)^{p'_{i_0}} dt \right)^{\frac{1}{p'_{i_0}}} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} \leq K.$$

We can also write the inequality above as follows:

$$\left(\int_r^\infty \left(\int_0^t f_{j_0} \right)^{p'_{i_0}} \frac{\sigma_{i_0}(t)}{t^{\lambda m p'_{i_0}}} dt \right)^{\frac{1}{p'_{i_0}}} \leq K \|g\mathcal{X}_{(0,r)}\|_{p, \infty; u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0}\|_{p_{j_0}, v_{j_0}}. \tag{22}$$

Taking into account that $\int_0^t f_{j_0} = \int_0^r f_{j_0} + \int_r^t f_{j_0}$ and also that the inequality (22) remains valid for f_{j_0} supported on (r, ∞) , we have that (22) is equivalent to the two following inequalities:

$$\left(\int_0^r f_{j_0} \right) \left(\int_r^\infty \frac{\sigma_{i_0}(t)}{t^{\lambda m p'_{i_0}}} dt \right)^{\frac{1}{p'_{i_0}}} \leq K \|g\mathcal{X}_{(0,r)}\|_{p, \infty; u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0}\|_{p_{j_0}, v_{j_0}} \tag{23}$$

and

$$\left(\int_r^\infty \left(\int_r^t f_{j_0} \right)^{p'_{i_0}} \frac{\sigma_{i_0}(t)}{t^{\lambda m p'_{i_0}}} dt \right)^{\frac{1}{p'_{i_0}}} \leq K \|g\mathcal{X}_{(0,r)}\|_{p, \infty; u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0}\mathcal{X}_{(r, \infty)}\|_{p_{j_0}, v_{j_0}}. \tag{24}$$

On one hand, the inequality (23) holds for all f_{j_0} if and only if (7) does. It is clear, indeed, that (23) implies (7) by testing (23) with $f_{j_0} = \sigma_{j_0} \mathcal{X}_{(0,r)}$, whereas the converse follows from (7) applying Hölder's inequality. On the other hand, taking into account that $p_{j_0} \leq p'_{j_0}$, we can apply Bradley's Theorem ([7]) and we get that (24) is equivalent to

$$\sup_{s>r>0} \|g\mathcal{X}_{(0,r)}\|_{p,\infty;u} \left(\int_s^\infty \frac{\sigma_{i_0}(t)}{t^{\lambda m p'_{i_0}}} dt \right)^{\frac{1}{p'_{i_0}}} \left(\int_r^s \sigma_{j_0} \right)^{\frac{1}{p'_{j_0}}} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{\frac{1}{p'_j}} < \infty. \quad (25)$$

In summary, we have seen that (22) is equivalent to (7) and (25). Finally, let us observe that (7) implies (25), what shows that (22) is equivalent to (7) in the case $p_{i_0} > 1$.

Assume now that $p_{i_0} = 1$. In this case, by the induction hypothesis, (21) is equivalent to the existence of a constant $K > 0$ such that for all $r > 0$ and all f_{j_0} ,

$$\operatorname{ess\,sup}_{t \in (r,\infty)} \left(\frac{v_{i_0}^{-1}(t)}{t^{\lambda m}} \int_0^t f_{j_0} \right) \leq K \|g\mathcal{X}_{(0,r)}\|_{p,\infty;u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0}\|_{p_{j_0}, v_{j_0}}.$$

Splitting the integral as in the previous case, we see that the inequality above is equivalent to the two following ones:

$$\left(\int_0^r f_{j_0} \right) \left(\operatorname{ess\,sup}_{t \in (r,\infty)} \frac{v_{i_0}^{-1}(t)}{t^{\lambda m}} \right) \leq K \|g\mathcal{X}_{(0,r)}\|_{p,\infty;u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0}\|_{p_{j_0}, v_{j_0}} \quad (26)$$

and

$$\operatorname{ess\,sup}_{t \in (r,\infty)} \left(\frac{v_{i_0}^{-1}(t)}{t^{\lambda m}} \int_r^t f_{j_0} \right) \leq K \|g\mathcal{X}_{(0,r)}\|_{p,\infty;u}^{-1} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{-\frac{1}{p'_j}} \|f_{j_0} \mathcal{X}_{(r,\infty)}\|_{p_{j_0}, v_{j_0}}. \quad (27)$$

As in the case $p_{i_0} > 1$, inequality (26) holds for all f_{j_0} if and only if (7) does. In order to characterize inequality (27) we will need the next lemma.

LEMMA 6. *Let $r > 0$, $p \geq 1$ and w, v two positive functions on (r, ∞) . Then, there exists $K > 0$ such that for all positive function f supported on (r, ∞) the inequality*

$$\operatorname{ess\,sup}_{t \in (r,\infty)} \left(w(t) \int_r^t f \right) \leq K \|f\|_{p,v} \quad (28)$$

holds if and only if

$$\operatorname{ess\,sup}_{t \in (r,\infty)} \left(w(t) \left(\int_r^t \sigma \right)^{\frac{1}{p'}} \right) < \infty, \quad (29)$$

where $\sigma = v^{1-p'}$.

Proof. Assume that (29) holds. Then, if $t \in (r, \infty)$, by Hölder’s inequality

$$w(t) \int_r^t f \leq w(t) \left(\int_r^t \sigma \right)^{\frac{1}{p'}} \left(\int_r^t f^{p_V} \right)^{\frac{1}{p}}.$$

Since the inequality above holds for all $t \in (r, \infty)$, we have

$$\operatorname{ess\,sup}_{t \in (r, \infty)} \left(w(t) \int_r^t f \right) \leq K \|f\|_{p, v},$$

where $K = \operatorname{ess\,sup}_{t \in (r, \infty)} \left(w(t) \left(\int_r^t \sigma \right)^{\frac{1}{p'}} \right)$, which is finite by (29).

Conversely, let $s \in (r, \infty)$ such that

$$w(s) \int_r^s \sigma \leq \operatorname{ess\,sup}_{t \in (r, \infty)} \left(w(t) \int_r^t \sigma \right).$$

Testing inequality (28) with $f = \sigma \chi_{(r, s)}$, we get

$$w(s) \int_r^s \sigma \leq K \left(\int_r^s \sigma \right)^{\frac{1}{p}},$$

i.e.,

$$w(s) \left(\int_r^s \sigma \right)^{\frac{1}{p'}} \leq K.$$

Since the inequality above holds for almost all $s \in (r, \infty)$, we get (29). \square

By Lemma 6, (27) is equivalent to the existence of a constant $K > 0$ such that

$$\operatorname{ess\,sup}_{t \in (r, \infty)} \left(\frac{v_{i_0}^{-1}(t)}{t^{\lambda m}} \left(\int_r^t \sigma_{j_0} \right)^{\frac{1}{p_{j_0}}} \right) \|g \chi_{(0, r)}\|_{p, \infty; u} \prod_{\substack{j=1 \\ j \neq i_0, j_0}}^m \left(\int_0^r \sigma_j \right)^{\frac{1}{p_j}} \leq K. \tag{30}$$

Therefore, we have shown that when $p_{i_0} = 1$, (21) is equivalent to (7) and (30). Finally, since (7) implies (30), we have that (21) is equivalent to (7), as we wished to prove. \square

Now, the implication (iii) \Rightarrow (ii) is straightforward. As in the proof of Theorem 1, applying the previous lemmas we get the boundedness of the operators $\mathcal{P}_{g, \lambda}$ and $\mathcal{D}_{g, \lambda}^i$, $i \in \{1, \dots, m\}$. Then, the modified Calderón operator is bounded.

Acknowledgement. The authors would like to thank the referee for some comments and suggestions that have improved the paper.

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(Received November 6, 2020)

Víctor García García
 Análisis Matemático, Facultad de Ciencias
 Universidad de Málaga
 29071 Málaga, Spain
 e-mail: victorgarcia2@uma.es

Pedro Ortega Salvador
 Análisis Matemático, Facultad de Ciencias
 Universidad de Málaga
 29071 Málaga, Spain
 e-mail: portega@uma.es