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NECESSARY AND SUFFICIENT CONDITIONS FOR A DIFFERENCE CONSTITUTED BY FOUR DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION TO BE COMPLETELY MONOTONIC

FENG QI

Abstract. In the paper, by virtue of convolution theorem for the Laplace transforms, Bernstein's theorem for completely monotonic functions, and other techniques, the author finds necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic.

Mathematics subject classification (2020): Primary 33B15; Secondary 26A48, 26A51, 26D07, 44A10.
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