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HERMITE–HADAMARD TYPE INEQUALITIES FOR MULTIDIMENSIONAL STRONGLY h–CONVEX FUNCTIONS

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Abstract. We establish some Hermite-Hadamard type inequalities for strongly h-convex function on balls and ellipsoids, which extend some known results. Some mappings connected with these inequalities and related applications are also obtained.

1. Introduction

In 2007, Varošanec [26] introduced the concept of h-convexity, which has received extensive attentions in recent years, see e.g. [3, 10, 15, 19].

DEFINITION 1. Let $h:[0,1]\to[0,\infty)$ be a given function. We say that $f:\mathscr{D}\to\mathbb{R}$, where $\mathscr{D}$ is a convex subset of $\mathbb{R}^n$, is h-convex if for any $X,Y\in\mathscr{D}$ and $\alpha\in[0,1]$,

$$f(\alpha X+(1-\alpha)Y)\leqslant h(\alpha)f(X)+h(1-\alpha)f(Y). \tag{1}$$

This notion unifies several other classes of convex functions, s-convex functions (in the second sense) [4], P-functions [22] and Godunova-Levin functions [9], which are obtained by putting in (1) $h(\alpha)=\alpha$, $h(\alpha)=\alpha^s(s\in(0,1))$, $h(\alpha)=1$ and $h(\alpha)=1/\alpha\ \ (0<\alpha\leqslant 1)$, respectively.

Strongly convex functions were introduced by Polyak [20] in 1966, and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see e.g. [14, 16, 17, 18, 20, 23, 25, 27, 28]). In 2011, Angulo, Gimenez, Moros and Nikodem [2] generalized the classes of strongly convex functions and h-convex functions as follows:

DEFINITION 2. Let $h:[0,1]\to[0,\infty)$ be a given function and $\lambda>0$ be a constant. We say that $f:\mathscr{D}\to\mathbb{R}$, where $\mathscr{D}$ is a convex subset of $\mathbb{R}^n$, is strongly h-convex with modulus λ if for any $X=(x_1,x_2,\dots,x_n),Y=(y_1,y_2,\dots,y_n)\in\mathscr{D}$ and $\alpha\in[0,1]$,

$$f(\alpha X+(1-\alpha)Y)\leqslant h(\alpha)f(X)+h(1-\alpha)f(Y)-\lambda\alpha(1-\alpha)|X-Y|^2, \tag{2}$$

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where
$$|X-Y|^2=(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2.$$

In particular, if f satisfies (2) with $h(\alpha)=\alpha$, $h(\alpha)=\alpha^s (s\in(0,1))$, $h(\alpha)=1$ and $h(\alpha)=1/\alpha$ $(0<\alpha\leqslant 1)$, then f is said to be a *strongly convex function, strongly s-convex function (in the second sense), strongly P-function* and a *strongly Godunova-Levin function*, respectively.

Convexity and its generalizations have a very important position in pure mathematics and in applications. A famous application of convex functions is the following Hermite-Hadamard inequality.

THEOREM A. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}.$$

In 1999, Dragomir and Fitzpatrick obtained the variant of Hermite-Hadamard's inequality for s-convex functions in the second sense.

THEOREM B. [8] *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a nonnegative s-convex function in the second sense with $0<s<1$. Then*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{s+1}.$$

In 2008, Sarikaya, Saglam and Yildririm proved the following analogue inequalities for h-convex functions.

THEOREM C. [24] *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an h-convex function on $[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant[f(a)+f(b)]\int_0^1 h(x)dx.$$

In 2011, the authors established the following inequality for strongly h-convex functions.

THEOREM D. [2] *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a strongly h-convex function with modulus λ on $[a,b]$. Then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right)+\frac{\lambda}{12}(b-a)^2\right]&\leqslant\frac{1}{b-a}\int_a^b f(x)dx\\&\leqslant[f(a)+f(b)]\int_0^1 h(t)dt-\frac{\lambda}{6}(b-a)^2.\end{aligned}$$

Meanwhile, there are large number of works dedicated to study Hermite-Hadamard's type inequalities in multidimensional spaces. For instance, some inequalities for convex type functions on rectangles can be referred to [1, 7, 12], and on disks can be referred to [5, 6, 13, 29]. The motivation of this paper is to deal with analogue inequalities for strongly h-convex functions on balls and ellipsoids. Compared with the methods taken on rectangles, which used on balls and ellipsoids are much more complicated.

In the sequel, unless otherwise specified, $\mathbb{R}^n$ denotes the Euclidean space of dimension n and $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, $d\sigma(X)$ is the arc length $(n=2)$ or the usual surface measure $(n \geqslant 3)$ in general. For any points $X=(x_1,x_2,\dots,x_n)$, $Y=(y_1,y_2,\dots,y_n) \in \mathbb{R}^n$ and $a,b \in \mathbb{R}$, define the product of vectors by

$$X \circ Y = (x_1y_1, x_2y_2, \dots, x_ny_n),$$

the linear combination of vectors by

$$aX+bY = (ax_1+by_1, ax_2+by_2, \dots, ax_n+by_n),$$

and the norm of X by

$$|X| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}.$$

$B_n(C,r)$ and $\delta_n(C,r)$ are the n-dimensional ball and its sphere respectively centered at the point $C=(c_1,c_2,\dots,c_n) \in \mathbb{R}^n$ with radius $r>0$. $E_n(C,R)$ denotes the n-dimensional ellipsoid centered at the point $C=(c_1,c_2,\dots,c_n) \in \mathbb{R}^n$ with semiaxises $R=(r_1,r_2,\cdots,r_n) \in \mathbb{R}^n$, that is

$$\frac{(x_1-c_1)^2}{r_1^2}+\frac{(x_2-c_2)^2}{r_2^2}+\cdots+\frac{(x_n-c_n)^2}{r_n^2} \leqslant 1, \;\; 0<r_1,r_2,\dots,r_n<\infty,$$

and $S_n(C,R)$ is the sphere of $E_n(C,R)$. It is well known that

$$|B_n(C,r)| = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}, \;\; |\delta_n(C,r)| = \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(\frac{n}{2}+1)}, \tag{3}$$

$$|E_n(C,R)| = \frac{\pi^{\frac{n}{2}} r_1\cdots r_n}{\Gamma(\frac{n}{2}+1)}, \;\; |S_n(C,tR)| = t^{n-1}|S_n(C,R)|, \;\; t>0, \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Throughout the paper, we also assume that the functions h in Definition 1 and Definition 2 are always Lebesgue integrable on $[0,1]$ and are chosen such that $h\left(\frac{1}{2}\right) > 0$.

Now we recall some known results. In 2000, Dragomir obtained the Hermite-Hadamard type inequality of convex functions on disks in $\mathbb{R}^2$ [5] and on balls in $\mathbb{R}^3$ [6]. In 2014, Matłoka [13] generalized the conclusions for h-convex functions on disks in $\mathbb{R}^2$. In 2019, the authors [29] extended the above results to the more general cases as follows.

THEOREM E. [29] *Let $f: B_n(C,r) \to \mathbb{R}$ be an h-convex function on $B_n(C,r)$. Suppose that h satisfies*

$$1-2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt > 0. \tag{5}$$

Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f(C) \leqslant \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \leqslant \frac{\mathscr{K}(n)}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X),$$

where

$$\mathscr{K}(n) = \frac{n\int_0^1 t^{n-1}h(t)dt}{1-2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt}. \tag{6}$$

THEOREM F. [29] *Let $f : E_n(C,R) \to \mathbb{R}$ be an h-convex function on the ellipsoid $E_n(C,R)$. Suppose that h satisfies (5). Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f(C) \leqslant \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\mathscr{K}(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right),$$

where $\mathscr{K}(n)$ is as in Theorem E and $\widetilde{X} = R\circ X' + C \in S_n(C,R)$.

Furthermore, if $f \geqslant 0$, we have

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\tilde{\mathscr{F}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)d\sigma(X),$$

where

$$\tilde{\mathscr{F}}(R) = \frac{|S_n(C,R)|}{r^{n-1}}\frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathscr{K}(n) \text{ and } r = \min\{r_1,\, r_2,\dots,\, r_n\}. \tag{7}$$

With these motivations, one of the purposes of this paper is to establish analogues of the above inequalities for strongly h-convex functions. Now we are in a position to state our results.

THEOREM 1. *Let $f : E_n(C,R) \to \mathbb{R}$ be a strongly h-convex function with modulus λ on the ellipsoid $E_n(C,R)$. Suppose that h satisfies (5). Then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left(f(C)+\frac{\lambda|R|^2}{n+2}\right) &\leqslant \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \\ &\leqslant \frac{\mathscr{K}(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right) - \lambda\widetilde{\mathscr{K}}(n)|R|^2, \end{aligned} \tag{8}$$

where $\widetilde{X} = R\circ X' + C \in S_n(C,R)$, $\mathscr{K}(n)$ is defined by (6) in Theorem E and

$$\widetilde{\mathscr{K}}(n) = \frac{1+n(n+1)\int_0^1 t^{n-1}h(1-t)dt}{(n+1)(n+2)\left[1-2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt\right]}. \tag{9}$$

Especially, if $f \geqslant 0$, we have

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\tilde{\mathscr{F}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)d\sigma(X) - \lambda\widetilde{\mathscr{K}}(n)|R|^2, \tag{10}$$

where $\tilde{\mathscr{F}}(R)$ is defined by (7) in Theorem F.

As a consequence, with the aid of (3), we immediately derive that

THEOREM 2. *Let $f : B_n(C,r) \to \mathbb{R}$ be a strongly h-convex function with modulus λ on $B_n(C,r)$. Suppose that h satisfies (5). Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left(f(C)+\frac{\lambda nr^2}{n+2}\right) \leqslant \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX$$
$$\leqslant \frac{\mathscr{K}(n)}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X) - \lambda n\widetilde{\mathscr{K}}(n)r^2, \quad (11)$$

where $\mathscr{K}(n)$ and $\widetilde{\mathscr{K}}(n)$ are as in Theorem 1.

In particular, letting $\lambda \to 0$, Theorem 1 and Theorem 2 reduce to Theorem F and Theorem E respectively.

It is easy to check that (5) holds for $h(t) = t$. Actually,

$$1-2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt = 1-n\int_0^1 (t^{n-1}-t^n)dt = \frac{n}{n+1} > 0.$$

And, a direct calculation yields that $\mathscr{K}(n) = 1, \widetilde{\mathscr{K}}(n) = \frac{2}{n(n+2)}$. These facts show that

COROLLARY 1. *If $f : E_n(C,R) \to \mathbb{R}$ be a strongly convex function with modulus λ, then*

$$f(C)+\frac{\lambda |R|^2}{n+2} \leqslant \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\int_{\delta_n(0,1)} f(\widetilde{X})d\sigma(X') - \frac{2\lambda |R|^2}{n(n+2)},$$

where $\widetilde{X}$ are as in Theorem 1.

Furthermore, if f is a nonnegative convex function on $E_n(C,R)$ and $r = \min\{r_1, r_2, \ldots, r_n\}$, then

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}r^{n-1}}\int_{S_n(C,R)} f(X)d\sigma(X) - \frac{2\lambda |R|^2}{n(n+2)}.$$

COROLLARY 2. *If $f : B_n(C,r) \to \mathbb{R}$ be a strongly convex function with modulus λ, then*

$$f(C)+\frac{\lambda nr^2}{n+2} \leqslant \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \leqslant \frac{1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X) - \frac{2\lambda r^2}{n+2}.$$

When $h(t) = t^s$, $0 < s < 1$, integration by parts implies that

$$\int_0^1 t^{n-1}h(1-t)dt = \int_0^1 (t-1)^{n-1}t^s dt = \frac{(n-1)!}{(s+1)(s+2)\cdots(s+n)}. \quad (12)$$

By (12), Theorem 1 and Theorem 2, we get the following inequalities for s-convex functions.

COROLLARY 3. *Let $f : E_n(C,R) \to \mathbb{R}$ be a strongly s-convex function (in the second sense) with modulus λ on the ellipsoid $E_n(C,R)$. If $0 < s < 1$ and*

$$2^s(s+1)(s+2)\cdots(s+n) > 2n!, \tag{13}$$

then

$$\begin{aligned}\frac{2^s}{2}\left(f(C)+\frac{\lambda|R|^2}{n+2}\right) &\leqslant \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \\ &\leqslant \mathscr{K}_2\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right)-\lambda\widetilde{\mathscr{K}_1}|R|^2,\end{aligned}$$

where $\widetilde{X}, r$ are as in Theorem 1 and

$$\mathscr{K}_1 = \frac{n2^s(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n)-2n!}, \tag{14}$$

$$\widetilde{\mathscr{K}_1} = \frac{2^s\left[(s+1)(s+2)\cdots(s+n)+(n+1)!\right]}{(n+1)(n+2)\left[2^s(s+1)(s+2)\cdots(s+n)-2n!\right]}, \tag{15}$$

$$\mathscr{K}_2 = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}}\frac{2^sn(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n)-2n!} = \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathscr{K}_1.$$

Furthermore, if $f \geqslant 0$, we have

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leqslant \frac{\tilde{\mathbf{F}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)d\sigma(X)-\lambda\widetilde{\mathscr{K}_1}|R|^2,$$

where

$$\tilde{\mathbf{F}}(R) = \frac{\Gamma(\frac{n}{2}+1)|S_n(C,R)|}{n\pi^{\frac{n}{2}}r^{n-1}}\frac{2^sn(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n)-2n!} = \frac{|S_n(0,R)|}{|\delta_n(0,r)|}\mathscr{K}_1.$$

COROLLARY 4. *Let $f : B_n(C,r) \to \mathbb{R}$ be a strongly s-convex function (in the second sense) with modulus λ on $B_n(C,r)$ and $\mathscr{K}_1, \widetilde{\mathscr{K}_1}$ be the constants defined in Corollary 3. If $0 < s < 1$ and (13) holds, then*

$$\begin{aligned}\frac{2^s}{2}\left(f(C)+\frac{\lambda nr^2}{n+2}\right) &\leqslant \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \\ &\leqslant \frac{\mathscr{K}_1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X)-\lambda n\widetilde{\mathscr{K}_1}r^2.\end{aligned}$$

The second purpose in this paper is to provide some applications of the Hermite-Hadamard inequalities for strongly h-convex functions. In [5] and [6], Dragomir studied some properties of the mappings connected to the Hermite-Hadamard type inequality of convex function on disks in $\mathbb{R}^2$ and on balls in $\mathbb{R}^3$. In [13], Matłoka considered the similar mappings connected to the h-convex function on disks in $\mathbb{R}^2$. Recently, the authors [29] extended the above results to the general high-dimension balls and ellipsoids in $\mathbb{R}^n$.

THEOREM G. [29] *Define the mapping* $\tilde{\mathfrak{H}}:[0,1]\to\mathbb{R}$ *by*

$$\tilde{\mathfrak{H}}(t)=\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(tX+(1-t)C)dX. \tag{16}$$

If f is an h-convex function on the ellipsoid $E_n(C,R)$, then
(i) *the function $\tilde{\mathfrak{H}}$ is an h-convex function on $[0,1]$,*
(ii) *for any $t\in(0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)}\leqslant \tilde{\mathfrak{H}}(t)\leqslant \tilde{\mathfrak{H}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]. \tag{17}$$

THEOREM H. [29] *Define the mapping* $\tilde{\mathbf{H}}:[0,1]\to\mathbb{R}$ *by*

$$\tilde{\mathbf{H}}(t)=\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(tX+(1-t)C)dX. \tag{18}$$

If f is an h-convex function on the ball $B_n(C,r)$, then the mapping $\tilde{\mathbf{H}}$ enjoys the same properties as $\tilde{\mathfrak{H}}$ in Theorem G.

THEOREM I. [29] *Define the mapping* $\tilde{\mathfrak{G}}:[0,1]\to\mathbb{R}$ *by*

$$\tilde{\mathfrak{G}}(t)=\begin{cases}\dfrac{1}{|S_n(C,tR)|}\displaystyle\int_{S_n(C,tR)} f(X)d\sigma(X),\ t\in(0,1],\\ f(C),\ t=0.\end{cases} \tag{19}$$

If f is an h-convex function on the ellipsoid $E_n(C,R)$ and (5) holds, then
(i) *the function $\tilde{\mathfrak{G}}(t)$ is an h-convex function on $[0,1]$,*
(ii) *when $f\geqslant 0$, for any $t\in(0,1]$, $\tilde{\mathfrak{H}}(t)\leqslant \tilde{\mathscr{F}}(R)\tilde{\mathfrak{G}}(t)$,*
(iii) *when $f\geqslant 0$, for any $t\in(0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\tilde{\mathscr{F}}(R)}\leqslant \tilde{\mathfrak{G}}(t)\leqslant \tilde{\mathfrak{G}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\tilde{\mathscr{F}}(R)\right], \tag{20}$$

where $\tilde{\mathscr{F}}(R)$ is defined by (7) in Theorem F.

THEOREM J. [29] *Define the mapping* $\tilde{\mathbf{G}}:[0,1]\to\mathbb{R}$ *by*

$$\tilde{\mathbf{G}}(t)=\begin{cases}\dfrac{1}{|\delta_n(C,tr)|}\displaystyle\int_{\delta_n(C,tr)} f(X)d\sigma(X),\ t\in(0,1],\\ f(C),\ t=0.\end{cases} \tag{21}$$

If f is an h-convex function on the ball $B_n(C,r)$ and (5) holds, then
(i) *the function $\tilde{\mathbf{G}}(t)$ is an h-convex function on $[0,1]$,*
(ii) *for any $t\in(0,1]$, $\tilde{\mathbf{H}}(t)\leqslant \mathscr{K}(n)\tilde{\mathbf{G}}(t)$,*
(iii) *for any $t\in(0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\mathscr{K}(n)}\leqslant \tilde{\mathbf{G}}(t)\leqslant \tilde{\mathbf{G}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\mathscr{K}(n)\right]. \tag{22}$$

Now, we will prove some properties of these four mappings assuming that the function f is strongly h-convex.

THEOREM 3. *If f is a strongly h-convex function with modulus λ on the ellipsoid $E_n(C,R)$ and the mapping $\tilde{\mathfrak{H}}:[0,1]\to\mathbb{R}$ is defined by (16) in Theorem G, then*

(i) *$\tilde{\mathfrak{H}}$ is a strongly h-convex function with modulus $\frac{\lambda}{n+2}|R|^2$ on $[0,1]$,*

(ii) *for any $t\in(0,1]$,*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)}\left(f(C)+\frac{\lambda|R|^2t^2}{n+2}\right)\\ &\leqslant \tilde{\mathfrak{H}}(t)\leqslant \tilde{\mathfrak{H}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]-\frac{\lambda|R|^2}{n+2}\left[h(1-t)+t(1-t)\right].\end{aligned}\tag{23}$$

As a consequence, we have the following conclusion.

COROLLARY 5. *If f is a strongly h-convex function with modulus λ on the ball $B_n(C,r)$ and the mapping $\tilde{\mathbf{H}}:[0,1]\to\mathbb{R}$ is defined by (18) in Theorem H, then $\tilde{\mathbf{H}}$ enjoys the same properties as in Theorem 3 with $|R|^2=nr^2$.*

THEOREM 4. *Let the mapping $\tilde{\mathfrak{G}}:[0,1]\to\mathbb{R}$ be defined by (19) in Theorem I. If f is a strongly h-convex function with modulus λ on the ellipsoid $E_n(C,R)$ and (5) holds, then*

(i) *the function $\tilde{\mathfrak{G}}(t)$ is a strongly h-convex function with modulus λr^2 on $[0,1]$,*

(ii) *when $f\geqslant 0$, for any $t\in(0,1]$, $\tilde{\mathfrak{H}}(t)+\lambda\widetilde{K}(n)|R|^2t^2\leqslant\tilde{\mathscr{F}}(R)\tilde{\mathfrak{G}}(t)$,*

(iii) *when $f\geqslant 0$, for any $t\in(0,1]$,*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)\tilde{\mathscr{F}}(R)}\left[f(C)+\lambda\widehat{\mathscr{K}}(n)|R|^2t^2\right]\\ &\leqslant\tilde{\mathfrak{G}}(t)\leqslant\tilde{\mathfrak{G}}(1)\left[h(t)+2\tilde{\mathscr{F}}(R)h\left(\frac{1}{2}\right)h(1-t)\right]-\lambda\widehat{K}(n)|R|^2h(1-t)-\lambda r^2t(1-t),\end{aligned}\tag{24}$$

where $r=\min\{r_1,r_2,\dots,r_n\}$, $\tilde{\mathscr{F}}(R)$, $\mathscr{K}(n)$, $\widetilde{\mathscr{K}}(n)$ are as in Theorem 1 and

$$\widehat{\mathscr{K}}(n)=\frac{1}{n+2}\left[1+2(n+2)\widetilde{\mathscr{K}}(n)h\left(\frac{1}{2}\right)\right].$$

THEOREM 5. *Let the mapping $\tilde{\mathbf{G}}:[0,1]\to\mathbb{R}$ be defined by (21) in Theorem J. If f is a strongly h-convex function with modulus λ on the ball $B_n(C,r)$ and (5) holds, then*

(i) *the function $\tilde{\mathbf{G}}(t)$ is a strongly h-convex function with modulus λr^2 on $[0,1]$,*

(ii) *for any $t\in(0,1]$, $\tilde{\mathbf{H}}(t)\leqslant\mathscr{K}(n)\tilde{\mathbf{G}}(t)-\lambda n\widetilde{\mathscr{K}}(n)r^2t^2$,*

(iii) *for any $t\in(0,1]$,*

$$\begin{aligned}&\frac{f(C)+\lambda n\widehat{\mathscr{K}}(n)r^2t^2}{2h\left(\frac{1}{2}\right)\mathscr{K}(n)}\leqslant\tilde{\mathbf{G}}(t)\\ &\leqslant\tilde{\mathbf{G}}(1)\left[h(t)+2\mathscr{K}(n)h\left(\frac{1}{2}\right)h(1-t)\right]-\lambda r^2\left[t(1-t)+n\widehat{\mathscr{K}}(n)h(1-t)\right],\end{aligned}\tag{25}$$

where $\mathscr{K}(n),\widetilde{\mathscr{K}}(n)$ and $\widehat{\mathscr{K}}(n)$ are as in Theorem 4.

2. Proof of the Theorems

2.1. Proof of Theorem 1

(i) The facts of $f(C) = f\left(\frac{X}{2} + \frac{2C-X}{2}\right)$ and

$$\int_{E_n(C,R)} f(X)dX = \int_{E_n(C,R)} f(2C-X)dX$$

suggest that

$$\begin{aligned} f(C) &= \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f\left(\frac{X}{2}+\frac{2C-X}{2}\right)dX \\ &\leqslant \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} \left[h\left(\frac{1}{2}\right)f(X)+h\left(\frac{1}{2}\right)f(2C-X)-\frac{\lambda}{4}\left|2(X-C)\right|^2\right]dX \\ &= \frac{2h\left(\frac{1}{2}\right)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX - \frac{\lambda}{|E_n(0,R)|}\int_{E_n(0,R)} |X|^2 dX. \end{aligned} \tag{26}$$

On the other hand,

$$\int_{E_n(0,R)} |X|^2\, dX = \int_{E_n(0,R)} \left(x_1^2+x_2^2+\cdots+x_n^2\right) dx_1dx_2\cdots dx_n, \tag{27}$$

and (4) implies that

$$\begin{aligned} \int_{E_n(0,R)} x_n^2 dx_1dx_2\cdots dx_n &= 2\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}+1\right)} r_1\cdots r_{n-1}\int_0^{r_n} x_n^2\left(1-\frac{x_n^2}{r_n^2}\right)^{\frac{n-1}{2}} dx_n \\ &= 2\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} r_1\cdots r_{n-1}r_n^3\int_0^1 t^2(1-t^2)^{\frac{n-1}{2}}dt \\ &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} r_1\cdots r_{n-1}r_n^3\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{n-1}{2}}dt \\ &= r_1\cdots r_{n-1}r_n^3\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}B\left(\frac{3}{2},\frac{n+1}{2}\right), \end{aligned}$$

where $B(\cdot,\cdot)$ denotes the Beta function. It follows from the basic properties of the Gamma function and the Beta function that

$$\begin{aligned} \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}B\left(\frac{3}{2},\frac{n+1}{2}\right) &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{2\Gamma\left(\frac{n+4}{2}\right)} \\ &= \frac{\pi^{\frac{n}{2}}}{(n+2)\Gamma\left(\frac{n}{2}+1\right)}, \end{aligned}$$

which tells us that

$$\int_{E_n(0,R)} x_n^2 dx_1dx_2\cdots dx_n = \frac{\pi^{\frac{n}{2}}}{(n+2)\Gamma\left(\frac{n}{2}+1\right)} r_1\cdots r_{n-1}r_n^3 = \frac{|E_n(0,R)|}{n+2}r_n^2. \tag{28}$$

With the aid of (27) and similar arguments as in (28), we have

$$\frac{1}{|E_n(0,R)|}\int_{E_n(0,R)}|X|^2\,dX=\frac{r_1^2+\cdots+r_n^2}{n+2}=\frac{|R|^2}{n+2}. \tag{29}$$

Therefore, the first part of (8) is obtained by (26) and (29).

(ii) Now we turn to prove the second part of of (8). It is not difficult to see that

$$\begin{aligned}
&\int_{E_n(C,R)} f(X)dX\\
&= r_1r_2\cdots r_n\int_{B_n(0,1)} f(R\circ X+C)dX\\
&= r_1r_2\cdots r_n\int_0^1\int_{\delta_n(0,1)} f\left(t\left(R\circ X'+C\right)+(1-t)C\right)t^{n-1}d\sigma(X')dt\\
&\leqslant r_1r_2\cdots r_n\left\{\int_0^1 t^{n-1}h(t)dt\int_{\delta_n(0,1)} f\left(R\circ X'+C\right)d\sigma\left(X'\right)\right.\\
&\quad\left.+f(C)|\delta_n(0,1)|\int_0^1 t^{n-1}h(1-t)dt-\frac{\lambda}{(n+1)(n+2)}\int_{\delta_n(0,1)}\left|R\circ X'\right|^2d\sigma\left(X'\right)\right\}.
\end{aligned}$$

On the other hand, it follows from (29) that

$$\begin{aligned}
\frac{|R|^2}{n+2}|E_n(0,R)| &= \int_{E_n(0,R)}|X|^2\,dX\\
&= r_1r_2\cdots r_n\int_0^1 t^{n+1}dt\int_{\delta_n(0,1)}\left|R\circ X'\right|^2d\sigma\left(X'\right)\\
&= \frac{r_1r_2\cdots r_n}{n+2}\int_{\delta_n(0,1)}\left|R\circ X'\right|^2d\sigma\left(X'\right),
\end{aligned}$$

which means that

$$\int_{\delta_n(0,1)}\left|R\circ X'\right|^2d\sigma\left(X'\right)=|B_n(0,1)|\,|R|^2. \tag{30}$$

Due to (30) and the inequality

$$f(C)\leqslant\frac{2h(\frac{1}{2})}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX-\frac{\lambda}{n+2}|R|^2,$$

we finish the proof of the right part of (8).

(iii) Next we will prove inequality (10). Since $f\geqslant 0$ and $r=\min\{r_1,r_2,\ldots,r_n\}$,

$$\begin{aligned}
\int_{S_n(C,R)} f(X)d\sigma(X) &= \int_{S_n(0,R)} f(X+C)d\sigma(X)\\
&\geqslant \int_{\delta_n(0,r)} f\left(\frac{R}{r}\circ X+C\right)d\sigma(X)\\
&= r^{n-1}\int_{\delta_n(0,1)} f\left(R\circ X'+C\right)d\sigma(X').
\end{aligned}$$

That is

$$\int_{\delta_n(0,1)} f\left(\widetilde{X}\right) d\sigma\left(X'\right) \leqslant \frac{1}{r^{n-1}} \int_{S_n(C,R)} f(X) d\sigma(X). \tag{31}$$

By combing (8) and (31) we complete the proof of Theorem 1. □

2.2. Proof of Theorem 3

(i) Let $t_1, t_2 \in [0,1]$, and $\alpha, \beta \geqslant 0, \alpha+\beta=1$. It follows from (29) that

$$\begin{aligned}
&\tilde{\mathfrak{H}}(\alpha t_1+\beta t_2) \\
&= \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f\left(\alpha\left[t_1 X+(1-t_1)C\right]+\beta\left[t_2 X+(1-t_2)C\right]\right) dX \\
&\leqslant \frac{h(\alpha)}{|E_n(C,R)|} \int_{E_n(C,R)} f(t_1 X+(1-t_1)C) dX+\frac{h(\beta)}{|E_n(C,R)|} \int_{E_n(C,R)} f(t_2 X+(1-t_2)C) dX \\
&\quad -\frac{\lambda \alpha \beta (t_1-t_2)^2}{|E_n(C,R)|} \int_{E_n(C,R)} |X-C|^2 dX \\
&= h(\alpha) \tilde{\mathfrak{H}}(t_1)+h(\beta) \tilde{\mathfrak{H}}(t_2)-\frac{\lambda |R|^2}{n+2} \alpha \beta (t_1-t_2)^2,
\end{aligned}$$

which means that $\tilde{\mathfrak{H}}$ is a strongly h-convex function with modulus $\frac{\lambda}{n+2}|R|^2$ on $[0,1]$.

(ii) For any fixed $t \in (0,1]$, taking the substitution $\eta=(\eta_1, \eta_2, \cdots, \eta_n)$, where $\eta_i=t x_i+(1-t) c_i$, we have

$$\begin{aligned}
\tilde{\mathfrak{H}}(t) &= \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(tX+(1-t)C) dX \\
&= \frac{1}{|E_n(C,R)|} \int_{E_n(C,tR)} f(\eta) \left|\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(\eta_1, \eta_2, \ldots, \eta_n}\right| d\eta \\
&= \frac{1}{t^n |E_n(C,R)|} \int_{E_n(C,tR)} f(\eta) d\eta \\
&= \frac{1}{|E_n(C,tR)|} \int_{E_n(C,tR)} f(X) dX.
\end{aligned} \tag{32}$$

Then Theorem 1 gives us that

$$\frac{1}{2h(\frac{1}{2})}\left(f(C)+\frac{\lambda}{n+2}|R|^2 t^2\right) \leqslant \tilde{\mathfrak{H}}(t).$$

In this way the first part of the inequality (23) is proved.

It follows from the first inequality of (8), the definition of $\tilde{\mathfrak{H}}$ and (29) that

$$\begin{aligned}
\tilde{\mathfrak{H}}(t) &\leqslant \frac{h(t)}{|E_n(C,R)|} \int_{E_n(C,R)} f(X) dX+h(1-t) f(C)-\frac{\lambda t(1-t)}{|E_n(C,R)|} \int_{E_n(C,R)} |X-C|^2 dX \\
&\leqslant h(t) \tilde{\mathfrak{H}}(1)+h(1-t)\left[2h\left(\frac{1}{2}\right) \tilde{\mathfrak{H}}(1)-\frac{\lambda}{n+2}|R|^2\right]-\frac{\lambda t(1-t)}{n+2}|R|^2 \\
&\leqslant \left[h(t)+2h\left(\frac{1}{2}\right) h(1-t)\right] \tilde{\mathfrak{H}}(1)-\frac{\lambda |R|^2}{n+2}\left[h(1-t)+t(1-t)\right],
\end{aligned}$$

which completes the proof. $\square$

2.3. Proof of Theorem 4

For any $t \in (0,1]$, it follows from (4) that

$$\begin{aligned}\tilde{\mathfrak{G}}(t) &= \frac{1}{|S_n(0,tR)|}\int_{S_n(0,tR)} f(X+C)d\sigma(X)\\ &= \frac{1}{|S_n(0,R)|}\int_{S_n(0,R)} f(tX+C)d\sigma(X).\end{aligned}$$

(i) Let $t_1,t_2 \in [0,1]$ and $\alpha,\beta \geqslant 0,\ \alpha+\beta = 1$. Then

$$\begin{aligned}&\tilde{\mathfrak{G}}(\alpha t_1+\beta t_2)\\ &= \frac{1}{|S_n(0,R)|}\int_{S_n(0,R)} f\left(\alpha\left(t_1X+C\right)+\beta\left(t_2X+C\right)\right)d\sigma(X)\\ &\leqslant \frac{h(\alpha)}{|S_n(0,R)|}\int_{S_n(0,R)} f\left(t_1X+C\right)d\sigma(X)+\frac{h(\beta)}{|S_n(0,R)|}\int_{S_n(0,R)} f\left(t_2X+C\right)d\sigma(X)\\ &\quad -\lambda\alpha\beta\left(t_1-t_2\right)^2\frac{1}{|S_n(0,R)|}\int_{S_n(0,R)}|X|^2d\sigma(X)\\ &\leqslant h(\alpha)\tilde{\mathfrak{G}}(t_1)+h(\beta)\tilde{\mathfrak{G}}(t_2)-\lambda r^2\alpha\beta\left(t_1-t_2\right)^2.\end{aligned}$$

This concludes the proof of (i).

(ii) For any given $t \in (0,1]$, the identity (32) provides us that

$$\tilde{\mathfrak{H}}(t) = \frac{1}{|E_n(C,tR)|}\int_{E_n(C,tR)} f(X)dX.$$

Since $f \geqslant 0$, by Theorem 1, we claim that

$$\frac{1}{|E_n(C,tR)|}\int_{E_n(C,tR)} f(X)dX \leqslant \frac{\tilde{\mathscr{F}}(tR)}{|S_n(C,tR)|}\int_{S_n(C,tR)} f(X)d\sigma(X)-\lambda\widetilde{K}(n)t^2|R|^2.$$

That is

$$\tilde{\mathfrak{H}}(t) \leqslant \tilde{\mathscr{F}}(tR)\tilde{\mathfrak{G}}(t)-\lambda\widetilde{K}(n)|R|^2t^2,\ \ t \in (0,1],$$

where

$$\tilde{\mathscr{F}}(tR) = \frac{|S_n(0,tR)|}{|\delta_n(0,tr)|}\mathscr{K}(n).$$

On the other hand, it is clear from (3) and (4) that

$$\tilde{\mathscr{F}}(tR) = \tilde{\mathscr{F}}(R).$$

This observation implies that

$$\tilde{\mathfrak{H}}(t) \leqslant \tilde{\mathscr{F}}(R)\tilde{\mathfrak{G}}(t)-\lambda\widetilde{K}(n)|R|^2t^2 \tag{33}$$

holds for all $t \in (0,1]$. We complete the proof of (ii).

(iii) Since the first inequality of (25) is easily reached by Theorem 1, (32) and (33), it remains to prove the second part of (25).

Recalling that

$$\tilde{\mathfrak{G}}(t) = \frac{1}{|S_n(0,R)|} \int_{S_n(0,R)} f(tX+C) d\sigma(X),$$

we have

$$\begin{aligned}
\tilde{\mathfrak{G}}(t) &= \frac{1}{|S_n(0,R)|} \int_{S_n(0,R)} f(t(X+C)+(1-t)C) d\sigma(X) \\
&\leqslant \frac{h(t)}{|S_n(0,R)|} \int_{S_n(0,R)} f(X+C) d\sigma(X) + h(1-t) f(C) \\
&\quad -\lambda t(1-t) \frac{1}{|S_n(0,R)|} \int_{S_n(0,R)} |X|^2 d\sigma(X) \\
&\leqslant h(t)\tilde{\mathfrak{G}}(1) + 2\tilde{\mathscr{F}}(R) h\left(\frac{1}{2}\right) h(1-t)\tilde{\mathfrak{G}}(1) - \lambda \widetilde{K}(n)|R|^2 h(1-t) - \lambda r^2 t(1-t) \\
&= \tilde{\mathfrak{G}}(1)\left[h(t) + 2\tilde{\mathscr{F}}(R) h\left(\frac{1}{2}\right) h(1-t)\right] - \lambda \widetilde{K}(n)|R|^2 h(1-t) - \lambda r^2 t(1-t),
\end{aligned}$$

which completes the proof. $\square$

2.4. Proof of Theorem 5

Since (i) is a special case of Theorem 4 (i), it remains to prove (ii) and (iii). With the aid of (3), we can arrive at

$$\widetilde{\mathbf{G}}(t) = \frac{1}{|\delta_n(0,1)|} \int_{\delta_n(0,1)} f\left(trX' + C\right) d\sigma\left(X'\right). \tag{34}$$

As a special case of (32), we easily to see that

$$\widetilde{\mathbf{H}}(t) = \frac{1}{|B_n(C,tr)|} \int_{B_n(C,tr)} f(X) dX. \tag{35}$$

Thus Theorem 2 means that

$$\begin{aligned}
\widetilde{\mathbf{H}}(t) &\leqslant \frac{\mathscr{K}(n)}{|\delta_n(C,tr)|} \int_{\delta_n(C,tr)} f(X) d\sigma(X) - \lambda n \widetilde{\mathscr{K}}(n) r^2 t^2 \\
&= \mathscr{K}(n)\widetilde{\mathbf{G}}(t) - \lambda n \widetilde{\mathscr{K}}(n) t^2 r^2
\end{aligned} \tag{36}$$

holds for all $t \in (0,1]$, which completes the proof of (ii).

Now we will prove (iii). According to (35), (36) and the first part of (11), we can deduce that

$$\frac{f(C) + \frac{\lambda n}{n+2} t^2 r^2}{2h\left(\frac{1}{2}\right)} \leqslant \widetilde{\mathbf{H}}(t) \leqslant \mathscr{K}(n)\widetilde{\mathbf{G}}(t) - \lambda n \widetilde{\mathscr{K}}(n) t^2 r^2 \tag{37}$$

for all $t \in (0,1]$. Especially,

$$f(C) \leqslant 2h\left(\frac{1}{2}\right)\mathscr{K}(n)\widetilde{\mathbf{G}}(1) - \frac{\lambda n r^2}{n+2}\left[1+2(n+2)\widetilde{\mathscr{K}}(n)h\left(\frac{1}{2}\right)\right]$$
$$= 2h\left(\frac{1}{2}\right)\mathscr{K}(n)\widetilde{\mathbf{G}}(1) - \lambda n\widehat{\mathscr{K}}(n)r^2. \tag{38}$$

On the other hand, (34) and (38) tell us that

$$\begin{aligned}
\widetilde{\mathbf{G}}(t) &= \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(t\left(rX'+C\right)+(1-t)C\right)d\sigma\left(X'\right) \\
&\leqslant \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} \left[h(t)f\left(rX'+C\right)+h(1-t)f(C)\right]d\sigma\left(X'\right) - \lambda t(1-t)r^2 \\
&= h(t)\widetilde{\mathbf{G}}(1)+h(1-t)f(C)-\lambda t(1-t)r^2 \\
&\leqslant \widetilde{\mathbf{G}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\mathscr{K}(n)\right]-\lambda r^2 t(1-t)-\lambda n\widehat{\mathscr{K}}(n)r^2 h(1-t).
\end{aligned}$$

Thus we finish the proof of Theorem 5. □

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