

INTERACTIONS BETWEEN HLAWKA TYPE-1 AND TYPE-2 QUANTITIES

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Abstract. The classical Hlawka inequality possesses deep connections with zonotopes and zonoids in convex geometry, and has been related to Minkowski space. We introduce Hlawka Type-1 and Type-2 quantities, and establish a Hlawka-type relation between them, which connects a vast number of strikingly different variants of the Hlawka inequalities, such as Serre's reverse Hlawka inequality in the future cone of the Minkowski space, the Hlawka inequality for subadditive functions on abelian groups by Ressel, and the integral analogs by Takahasi et al. Besides, we announce several enhanced results, such as the Hlawka inequality for the power of measure functions. Particularly, we give a complete study of the Hlawka inequality for quadratic forms which relates to a work of Serre.

1. Introduction

Hlawka's inequality saying for any x, y, z in an inner product space

$$\|x\| + \|y\| + \|z\| + \|x + y + z\| \geq \|x + y\| + \|y + z\| + \|z + x\|, \quad (1)$$

was proved firstly by Hlawka and originally appeared in 1942 in a paper of Hornich [2]. It has a long series of investigations and extensions, such as the Hlawka inequality in integral form [6, 7] and abelian group [3]. The readers can also find an excellent summary of related works in [1], and the beautiful relations to discrete and convex geometry like zonotopes as well as zonoids by Witsenhausen [8, 9, 5].

Recently, Serre consider the pseudo-norm for the future cone of the Minkowski space [4]. There he presented the reverse Hlawka-type inequality.

According to these beautiful works, the classical Hlawka inequality has deep connections with zonotopes and zonoids in convex geometry, and relates to the geometry on the timelike cones of Minkowski spaces.

Note that the proof of (1) depends on the identity

$$\|x\|^2 + \|y\|^2 + \|z\|^2 + \|x + y + z\|^2 = \|x + y\|^2 + \|y + z\|^2 + \|z + x\|^2, \quad (2)$$

which is an equality on quadratic forms. To some extend, the one-homogeneous inequality (1) essentially relates to the two-homogeneous equality (2).

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In this work, we introduce Hlawka Type-1 and Type-2 quantities, and establish a Hlawka-type relation which encodes the signatures of them (see Theorem 1).

This helps us to give the Hlawka inequality for a class of functions on semigroups. By this result, we can connect a vast number of Hlawka inequalities in the literature, even though they come from various perspectives and are very different from each other. Furthermore, we announce several exciting results, such as the Hlawka inequality on the power of measure function. Particularly, we investigate the Hlawka inequality on quadratic forms thoroughly.

For a glimpse of these results, we give some remarkable notes here:

- For a reversed version of the Hlawka inequality, Serre gave a demonstration in the future cone of the Minkowski space [4]. In that paper, he shows: if q is a quadratic form on \mathbb{R}^n with signature $(1, n-1)$, then the length $l = \sqrt{q}$ satisfies

$$l(x) + l(y) + l(z) + l(x+y+z) \leq l(x+y) + l(y+z) + l(z+x) \quad (3)$$

for every vectors x, y, z in the future cone with respect to q . In the present paper, we show a simple proof for (3), and give a systematic study for the Hlawka inequality on quadratic forms (see Section 3.1).

- Ressel [3] shows a generalization of the Hlawka inequality for subadditive functions on abelian groups. In this work, we extend his result to the setting of sub/super-additive functions on semigroups (see Section 3.2). For convenience, Ressel's result is provided in Example 3 as an application.
- Takahasi et al [6, 7] study the integral analogs of the Hlawka inequality. In Section 3.3, we generalize this integral inequality to the form of positive linear operator, and their main theorem is rewritten in Example 5.

This paper provides a theorem combining the above different progresses together in a unified form (see Theorem 1), which also produces several other promotive results.

2. The Hlawka-type relation between Hlawka Type-1 and Type-2 quantities

Basic settings:

- Given a nonempty set Ω , let \mathbb{R}^Ω be the ring of all real valued functions on Ω equipped with the standard addition operator '+' and the standard product operator '·'.
- Let $\mathcal{S} \subset \mathbb{R}^\Omega$ be a sub-ring of real functions in \mathbb{R}^Ω , i.e., \mathcal{S} is closed under both summation and multiplication, as well as $1 \in \mathcal{S}$ and \mathcal{S} is also a real linear space. Here, we use 1 to denote the constant function in \mathcal{S} satisfying $1(\omega) = 1, \forall \omega \in \Omega$.
- Let $T : \mathcal{S} \rightarrow \mathbb{R}$ be a linear and signature-preserving function (i.e., for $\zeta \in \mathcal{S}$ satisfying $\forall \omega \in \Omega, \zeta(\omega) \geq 0$, there holds $T(\zeta) \geq 0$).

From the basic settings, we can see that \mathcal{S} is a ring of functions and it is a real linear space with the base consisted of some functions on Ω , and T is indeed a non-negative linear functional on \mathcal{S} . For convenience, T can be regarded as a summation operator or an integral operator.

THEOREM 1. *Given Ω, \mathcal{S}, T in the basic settings, $a, b \in \mathbb{R}$ with $a \neq 0$ and $a + b > 0$, for a nonempty set X , let $\eta, \xi : \Omega \rightarrow X$ and $f : X \rightarrow \mathbb{R}$ satisfy $f \circ \xi, f \circ \eta \in \mathcal{S}$ and $(f \circ \xi)(\omega) - (f \circ \eta)(\omega) \leq b, \forall \omega \in \Omega$, where ‘ \circ ’ represents the composition operator, we have the following:*

(I) *If $(f \circ \xi)(\omega) + (f \circ \eta)(\omega) \leq a, \forall \omega \in \Omega$, then $H_2 \geq 0$ implies $H_1 \geq 0$, and $H_1 \leq 0$ implies $H_2 \leq 0$;*

(II) *If $(f \circ \xi)(\omega) + (f \circ \eta)(\omega) \geq a, \forall \omega \in \Omega$, then $H_1 \geq 0$ implies $H_2 \geq 0$, and $H_2 \leq 0$ implies $H_1 \leq 0$.*

Here

$$H_1 = (T(1) - c)b + T(f \circ \eta) - T(f \circ \xi)$$

is called the Hlawka Type-1 quantity, and

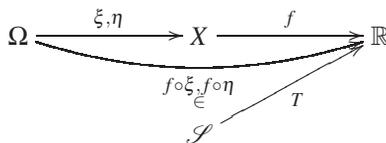
$$H_2 = (T(1) - c)b^2 + T(f^2 \circ \eta) - T(f^2 \circ \xi)$$

is called the Hlawka Type-2 quantity, in which $c := \frac{2}{a}T(f \circ \eta)$.

REMARK 1. In many examples and applications (see Section 3), ξ is some function of η , and f can be thought of as a norm. So, to some extent, H_1 can be seen as a 1-homogeneous function of η .

In summary, Theorem 1 says that under suitable ‘summation control’ and ‘difference control’, the signatures of Hlawka Type-1 quantity H_1 and Hlawka Type-2 quantity H_2 are essentially depended on each other in some way.

Proof. First, the relation among the quantities in Theorem 1 can be shown in the following diagram:



We note the following identities:

$$\begin{aligned}
 & T((a - f \circ \eta - f \circ \xi)(f \circ \eta + b - f \circ \xi)) \\
 &= aT(f \circ \eta) - T(f^2 \circ \eta) + T(f^2 \circ \xi) + T((f \circ \eta) \cdot (f \circ \xi)) - T((f \circ \xi) \cdot (f \circ \eta)) \\
 &\quad - aT(f \circ \xi) - bT(f \circ \eta) - bT(f \circ \xi) + abT(1) \\
 &= aT(f \circ \eta) + (T(1) - c)b^2 + T(f \circ \eta)b + (T(f^2 \circ \xi) - T(f^2 \circ \eta) - (T(1) - c)b^2) \\
 &\quad - T(f \circ \xi)(a + b) + (T(1)a - 2T(f \circ \eta))b + T((f \circ \eta) \cdot (f \circ \xi)) - T((f \circ \xi) \cdot (f \circ \eta)) \\
 &= T(f \circ \eta)(a + b) + (T(1) - c)b^2 + (T(f^2 \circ \xi) - T(f^2 \circ \eta) - (T(1) - c)b^2) \\
 &\quad - T(f \circ \xi)(a + b) + (T(1) - c)ab \\
 &= ((T(1) - c)b + T(f \circ \eta) - T(f \circ \xi))(a + b) + (T(f^2 \circ \xi) - T(f^2 \circ \eta) - (T(1) - c)b^2),
 \end{aligned}$$

where the notation $f^2 \circ \eta := (f \circ \eta) \cdot (f \circ \eta)$ is used. Therefore, we obtain

$$\begin{aligned}
 & T((a - f \circ \eta - f \circ \xi)(f \circ \eta + b - f \circ \xi)) \\
 &= ((T(1) - c)b + T(f \circ \eta) - T(f \circ \xi))(a + b) - ((T(1) - c)b^2 + T(f^2 \circ \eta) - T(f^2 \circ \xi)).
 \end{aligned} \tag{4}$$

(I). For any $\omega \in \Omega$, $f \circ \eta(\omega) + b \geq f \circ \xi(\omega)$ and $a \geq f \circ \eta(\omega) + f \circ \xi(\omega)$. By the assumption, $\forall \omega \in \Omega$,

$$(a - f \circ \eta(\omega) - f \circ \xi(\omega))(f \circ \eta(\omega) + b - f \circ \xi(\omega)) \geq 0.$$

This deduces that

$$T((a - f \circ \eta - f \circ \xi)(f \circ \eta + b - f \circ \xi)) \geq 0.$$

Accordingly, Eq. (4) gives

$$((T(1) - c)b + T(f \circ \eta) - T(f \circ \xi))(a + b) \geq (T(1) - c)b^2 + T(f^2 \circ \eta) - T(f^2 \circ \xi)$$

which arrives the final result.

(II). Since only the assumption $a \leq f \circ \eta(\omega) + f \circ \xi(\omega)$ is reversed, similar process gives

$$((T(1) - c)b + T(f \circ \eta) - T(f \circ \xi))(a + b) \leq (T(1) - c)b^2 + T(f^2 \circ \eta) - T(f^2 \circ \xi)$$

and then the reversed case could be verified immediately. \square

REMARK 2. From the proof of Theorem 1, it is obvious that the conditions could be weakened as follows:

Given $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ with $a + b > 0$, for $\eta, \xi : \Omega \rightarrow X$ and $f : X \rightarrow \mathbb{R}$, let $\tilde{\Omega} = \{\omega \in \Omega | b + f \circ \eta(\omega) - f \circ \xi(\omega) \neq 0\}$.

If $a \geq f \circ \eta(\omega) + f \circ \xi(\omega)$, $b \geq f \circ \xi(\omega) - f \circ \eta(\omega)$, $\forall \omega \in \tilde{\Omega}$, then $H_2 \geq 0 \Rightarrow H_1 \geq 0$, and $H_1 \leq 0 \Rightarrow H_2 \leq 0$.

If $a \leq f \circ \eta(\omega) + f \circ \xi(\omega)$, $b \geq f \circ \xi(\omega) - f \circ \eta(\omega)$, $\forall \omega \in \tilde{\Omega}$, then $H_1 \geq 0 \Rightarrow H_2 \geq 0$, and $H_2 \leq 0 \Rightarrow H_1 \leq 0$.

REMARK 3. To some extent, the two key controls for $f \circ \eta + f \circ \xi$ and $f \circ \xi - f \circ \eta$ via constants a and b are indeed ‘summation control’ and ‘difference control’. The identities used in the proof of Theorem 1 is inspired by product-to-sum formulas.

3. Applications to variant Hlawka inequalities

3.1. Applications to quadratic forms

Given a nondegenerate quadratic form q , i.e., $q(x) = x^\top Qx$, where x^\top is the transpose of the vector x and Q is a matrix of dimension n . Henceforth a pair $(k, n - k)$ is said to be the signature of Q , if Q has k positive eigenvalues and $(n - k)$ negative eigenvalues. Consider $l = \sqrt{q}$, then we have the following:

PROPOSITION 1. (P1) If Q is of the signature $(1, n - 1)$, then l satisfies the reversed Hlawka inequality in the closure of the future cone; (P2) If Q is of the signature $(n, 0)$, then l satisfies the Hlawka inequality in \mathbb{R}^n .

Proof. We will apply Theorem 1 to this setting, where the symbols appearing in Theorem 1 can be concretely chosen (see Table 1).

Firstly, according to the definition of q , there is

$$q(x + y + z) + q(x) + q(y) + q(z) = q(x + y) + q(x + z) + q(y + z). \tag{5}$$

(P1) Since Q is of the signature $(1, n - 1)$, we may assume without loss of generality that $Q = \text{diag}(1, -1, \dots, -1)$ and let $X = \{x = (x_1, \dots, x_n) | q(x) > 0, x_1 > 0\}$, i.e., the future cone in Minkowski space. Indeed, there is no subtraction ‘-’ in X and it is

Table 1: The concrete quantities of Theorem 1 used in the proof of Proposition 1 (1). While, for Proposition 1 (2), we only let X be replaced by \mathbb{R}^n .

Terminologies	Concrete choices in Proposition 1 (1) for fixed $x, y, z \in X$
$\Omega =$	$\{1, 2, 3\}$
$X =$	$\{x = (x_1, \dots, x_n) q(x) > 0, x_1 > 0\}$
$\mathcal{S} =$	$\mathbb{R}^{\{1, 2, 3\}}$
$T =$	$\sum_{\omega \in \{1, 2, 3\}} \eta(\omega)$, i.e., $T(g) = g(1) + g(2) + g(3), \forall g \in \mathcal{S}$
$\eta =$	x, y, z for $\omega = 1, 2, 3$ respectively
$\xi =$	$\sum_{\omega=1}^3 \eta(\omega) - \eta$, i.e., $y + z, z + x, x + y$ for $\omega = 1, 2, 3$ respectively
$f =$	\sqrt{q}
$a =$	$\sqrt{q(x)} + \sqrt{q(y)} + \sqrt{q(z)}$
$b =$	$\sqrt{q(x + y + z)}$

closed under addition. Since $q(x) = x_1^2 - x_2^2 - \dots - x_n^2 > 0$, $q(y) = y_1^2 - y_2^2 - \dots - y_n^2 > 0$, i.e.,

$$x_1^2 > x_2^2 + \dots + x_n^2, \quad y_1^2 > y_2^2 + \dots + y_n^2,$$

by Cauchy inequality, the following inequality holds:

$$x_1^2 y_1^2 > (x_2^2 + \dots + x_n^2)(y_2^2 + \dots + y_n^2) \geq (x_2 y_2 + \dots + x_n y_n)^2.$$

Due to $x, y \in X$, there is $x_1 y_1 > 0$, so $x_1 y_1 > x_2 y_2 + \dots + x_n y_n$, i.e., $x^\top Q y = y^\top Q x = x_1 y_1 - x_2 y_2 - \dots - x_n y_n > 0$. Hence $q(x+y) = q(x) + q(y) + x^\top Q y + y^\top Q x > 0$, which implies $x+y \in X$.

According to the Azteca inequality (i.e., a reversed version of Cauchy inequality), for any $x, y \in X$, there is

$$(x_1 y_1 - \sum_{j \geq 2} x_j y_j)^2 \geq (x_1^2 - \sum_{j \geq 2} x_j^2)(y_1^2 - \sum_{j \geq 2} y_j^2),$$

i.e.,

$$(x^\top Q y)^2 \geq x^\top Q x \cdot y^\top Q y. \tag{6}$$

By further elementary computation, (6) is equivalent to

$$\sqrt{q(x+y)} \geq \sqrt{q(x)} + \sqrt{q(y)} \tag{7}$$

whenever $x, y \in X$. By (7), for any $x, y, z \in X$, there is

$$\sqrt{q(x)} + \sqrt{q(y+z)} \geq a, \quad \sqrt{q(y+z)} - \sqrt{q(x)} \leq b.$$

By the parameters shown in Table 1, we further have $c = 2$ in Theorem 1, $H_2 = q(x+y+z) + q(x) + q(y) + q(z) - q(x+y) - q(x+z) - q(y+z) = 0$ (by Eq. (5)) and

$$H_1 = l(x) + l(y) + l(z) + l(x+y+z) - l(x+y) - l(y+z) - l(z+x).$$

According to Theorem 1 (II), $H_1 \leq 0$, thus

$$l(x) + l(y) + l(z) + l(x+y+z) \leq l(x+y) + l(y+z) + l(z+x),$$

whenever $x, y, z \in X$. By taking limits, one can find that the reversed Hlawka inequality also holds on the boundary of the future cone.

(P2) If Q is $(n, 0)$, we may assume without loss of generality that $Q = \text{diag}(1, 1, \dots, 1)$ and let $X = \mathbb{R}^n$. In this case, the inner product $\langle x, y \rangle := x^\top Q y$ satisfies Cauchy inequality, i.e., $(x^\top Q y)^2 \leq x^\top Q x \cdot y^\top Q y$. By elementary computation, there is $\sqrt{q(x+y)} \leq \sqrt{q(x)} + \sqrt{q(y)}$. From this, we have

$$\sqrt{q(x)} + \sqrt{q(y+z)} \leq a \quad \text{and} \quad \sqrt{q(y+z)} - \sqrt{q(x)} \leq b.$$

In the case of $a + b > 0$, similar to (P1), according to Theorem 1 (I) and Eq. (5), we have $l(x) + l(y) + l(z) + l(x+y+z) \geq l(x+y) + l(y+z) + l(z+x)$. In the case of $a = 0$ or $a + b = 0$, i.e., $x = y = z = 0$, it is obvious that $l(x) + l(y) + l(z) + l(x+y+z) = l(x+y) + l(y+z) + l(z+x) = 0$. Consequently, l satisfies the Hlawka inequality. \square

Proposition 1 contains Hlawka-type inequalities in the settings of both Euclidean case and Minkowski case. Moreover, by using Theorem 1, here we indeed provide an alternative and much easier proof of the reverse Hlawka inequality in Minkowski space (Theorem 1.1 in [4]).

However, there is no similar conclusion on other cases that Q is of the signature $(k, n - k)$ for $2 \leq k \leq n - 1$, and we will give an example to show this.

EXAMPLE 1. If Q is of the signature $(k, n - k)$ for $2 \leq k \leq n - 1$, we may assume without loss of generality that $Q = \text{diag}(\underbrace{1, \dots, 1}_k, \underbrace{-1, \dots, -1}_{n-k})$. By finding suitable cone $X \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid q(x) > 0\}$, one may obtain that both the Hlawka inequality and the reversed Hlawka inequality fail for l . Indeed, take $0 < \varepsilon \ll 1$, let

$$\vec{v}_1 = \vec{v}_2 = (\underbrace{1, 1, \varepsilon, \dots, \varepsilon}_k, \underbrace{1, \varepsilon, \dots, \varepsilon}_{n-k}), \quad \vec{v}_3 = (1, 1, \varepsilon, \dots, \varepsilon)$$

and

$$\vec{v}_4 = (2, 1, \varepsilon, \dots, \varepsilon), \quad \vec{v}_5 = (1, 2, \varepsilon, \dots, \varepsilon).$$

Consider $X = \{t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 + t_4\vec{v}_4 + t_5\vec{v}_5 \mid t_i > 0, 1 \leq i \leq 5\}$. It is clear that $X \subset \{x \mid q(x) > 0\}$. By computation, $l(\vec{v}_1) + l(\vec{v}_2) + l(\vec{v}_3) + l(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) < l(\vec{v}_1 + \vec{v}_2) + l(\vec{v}_2 + \vec{v}_3) + l(\vec{v}_3 + \vec{v}_1)$. While, $l(\vec{v}_5) + l(\vec{v}_4) + l(\vec{v}_3) + l(\vec{v}_3 + \vec{v}_4 + \vec{v}_5) > l(\vec{v}_4 + \vec{v}_5) + l(\vec{v}_3 + \vec{v}_5) + l(\vec{v}_3 + \vec{v}_4)$. Thus, in X , both the Hlawka inequality and the reversed Hlawka inequality do not hold for l .

3.2. Applications to sub/super -additive functions on semigroups

Let X in Theorem 1 be an abelian semigroup $(G, +)$, and let $F : G \rightarrow \mathbb{R}$ be a non-negative real-valued function. We will consider the Hlawka inequality of the form

$$F(x+y)^{\frac{1}{2k}} + F(y+z)^{\frac{1}{2k}} + F(z+x)^{\frac{1}{2k}} \leq F(x)^{\frac{1}{2k}} + F(y)^{\frac{1}{2k}} + F(z)^{\frac{1}{2k}} + F(x+y+z)^{\frac{1}{2k}}, \tag{8}$$

$\forall x, y, z \in G$, where k is an integer.

PROPOSITION 2. Let G be an abelian semigroup, and let $x \mapsto F(x)$ be a non-negative real-valued function on G .

If F is strong subadditive (i.e., $F(x) + F(y) \geq F(x+y)$ and $F(x) + F(x+y) \geq F(y)$, $\forall x, y \in G$), and (8) holds for some $k_0 \geq -1$, then (8) holds for all $k \geq k_0$.

If F is assumed to be superadditive (i.e., $F(x) + F(y) \leq F(x+y)$, $\forall x, y \in G$), and (8) holds for some $k_0 \leq 0$, then (8) holds for all $k \leq k_0$.

Proof. Given $a, b > 0$, the function $(a^t + b^t)^{1/t}$ is decreasing on $(0, \infty)$.

Case (1). F is non-negative and strong subadditive. For any $0 < \alpha \leq 1$,

$$F(x+y)^\alpha \leq (F(x) + F(y))^\alpha \leq F(x)^\alpha + F(y)^\alpha.$$

Table 2: The concrete quantities of Theorem 1 used in the proof of Proposition 2.

Terminologies	Concrete choices in Proposition 2 for fixed $x, y, z \in X$
$\Omega =$	$\{1, 2, 3\}$
$X =$	abelian semigroup G
$\mathcal{S} =$	$\mathbb{R}^{\{1,2,3\}}$
$T =$	$\sum_{\omega \in \{1,2,3\}} \eta(\omega)$, i.e., $T(g) = g(1) + g(2) + g(3), \forall g \in \mathcal{S}$
$\eta =$	x, y, z for $\omega = 1, 2, 3$ respectively
$\xi =$	$\sum_{\omega=1}^3 \eta(\omega) - \eta$, i.e., $y + z, z + x, x + y$ for $\omega = 1, 2, 3$ respectively
$f =$	$F^{\frac{1}{2^k}}$
$a =$	$F(x)^{\frac{1}{2^k}} + F(y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}}$
$b =$	$F(x + y + z)^{\frac{1}{2^k}}$

Suppose (8) holds for some $k_0 \geq -1$. Then for any $k > k_0$, and any $x, y, z \in G$,

$$F(x + y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}} \leq F(x)^{\frac{1}{2^k}} + F(y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}}$$

and

$$F(x + y)^{\frac{1}{2^k}} - F(z)^{\frac{1}{2^k}} \leq F(x + y + z)^{\frac{1}{2^k}}.$$

Here, let a and b in Theorem 1 be $F(x)^{\frac{1}{2^k}} + F(y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}}$ and $F(x + y + z)^{\frac{1}{2^k}}$, respectively. The detailed parameters are shown in Table 2. If $a \neq 0$ and $a + b > 0$, then the proof is finished by Theorem 1 (I). If $a = 0$, then $F(x) = F(y) = F(z) = F(x + y) = F(x + z) = F(y + z) = F(x + y + z) = 0$ and (8) is obvious.

Case (2). F is non-negative and superadditive, i.e., $F(x) + F(y) \leq F(x + y)$.

Note that for any $\alpha \geq 1$,

$$F(x + y)^\alpha \geq (F(x) + F(y))^\alpha \geq F(x)^\alpha + F(y)^\alpha.$$

In consequence, for any $k \leq 0$ and any $x, y, z \in G$, we have

$$F(x + y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}} \geq F(x)^{\frac{1}{2^k}} + F(y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}}$$

and

$$F(x + y + z)^{\frac{1}{2^k}} \geq F(x + y)^{\frac{1}{2^k}} + F(z)^{\frac{1}{2^k}} \geq F(x + y)^{\frac{1}{2^k}} - F(z)^{\frac{1}{2^k}}.$$

If $a \neq 0$ and $a + b > 0$, by Theorem 1, the result is proved. If $a + b = 0$, then $F(x) = F(y) = F(z) = F(x + y) = F(x + z) = F(y + z) = F(x + y + z) = 0$, the result is obvious. If $a = 0$, then $F(x) = F(y) = F(z) = 0$. According to the condition, we have

$$F(x + y)^{1/2^k} + F(y + z)^{1/2^k} + F(z + x)^{1/2^k} \leq F(x + y + z)^{1/2^k}$$

for some $k \leq 0$. Taking the square of the above inequality, there is

$$F(x + y)^{1/2^{k-1}} + F(y + z)^{1/2^{k-1}} + F(z + x)^{1/2^{k-1}} \leq F(x + y + z)^{1/2^{k-1}}.$$

Hereto, the prove is completed. \square

Now we show an interesting example even though this result seems to be elementary.

EXAMPLE 2. Taking $G = L^p$ and $F = \|\cdot\|_p$, together with Corollary 2.1 in [9] and Proposition 2, we have

$$\|a + b\|_p^{\frac{1}{2^k}} + \|b + c\|_p^{\frac{1}{2^k}} + \|c + a\|_p^{\frac{1}{2^k}} \leq \|a\|_p^{\frac{1}{2^k}} + \|b\|_p^{\frac{1}{2^k}} + \|c\|_p^{\frac{1}{2^k}} + \|a + b + c\|_p^{\frac{1}{2^k}}$$

for any $a, b, c \in L^p$, and $k \in \mathbb{N}$, where $1 \leq p \leq 2$.

Replacing a, b, c by $a^{2^k}, b^{2^k}, c^{2^k}$, respectively, one gets

$$\begin{aligned} & \| (a^{2^k} + b^{2^k})^{\frac{1}{2^k}} \|_{2^k p} + \| (b^{2^k} + c^{2^k})^{\frac{1}{2^k}} \|_{2^k p} + \| (c^{2^k} + a^{2^k})^{\frac{1}{2^k}} \|_{2^k p} \\ & \leq \|a\|_{2^k p} + \|b\|_{2^k p} + \|c\|_{2^k p} + \| (a^{2^k} + b^{2^k} + c^{2^k})^{\frac{1}{2^k}} \|_{2^k p}. \end{aligned}$$

For convenience, we define an operation \diamond_k by $a \diamond_k b = (a^{2^k} + b^{2^k})^{\frac{1}{2^k}}$ for $1 \leq k < +\infty$, $a \diamond_0 b := a + b$ and $a \diamond_\infty b := |a| \vee |b| := \max\{|a|, |b|\}$. Then using this notation, we obtain

$$\|a \diamond_k b\|_{2^k p} + \|b \diamond_k c\|_{2^k p} + \|c \diamond_k a\|_{2^k p} \leq \|a\|_{2^k p} + \|b\|_{2^k p} + \|c\|_{2^k p} + \|a \diamond_k b \diamond_k c\|_{2^k p}$$

for any $p \in [1, 2]$ and any $k \in \mathbb{N} \cup \{+\infty\}$. Thus

$$\|a \diamond_k b\|_p + \|b \diamond_k c\|_p + \|c \diamond_k a\|_p \leq \|a\|_p + \|b\|_p + \|c\|_p + \|a \diamond_k b \diamond_k c\|_p$$

holds for $p \in [2^k, 2^{k+1}]$, and by taking $k \rightarrow +\infty$, we have

$$\begin{aligned} & \| \max\{|a|, |b|\} \|_\infty + \| \max\{|b|, |c|\} \|_\infty + \| \max\{|c|, |a|\} \|_\infty \\ & \leq \|a\|_\infty + \|b\|_\infty + \|c\|_\infty + \| \max\{|a|, |b|, |c|\} \|_\infty. \end{aligned}$$

A direct application of Proposition 2 is the following Hlawka inequality on abelian groups.

EXAMPLE 3. (Theorem 2 in [3]) Let G be an abelian group, $x \mapsto |x|$ a non-negative symmetric and subadditive function on G (i.e., $|-x| = |x|$ and $|x| + |y| \geq |x + y|$, $\forall x, y \in G$), and let $S : [0, \infty) \rightarrow [0, \infty)$ be concave. Then, if $\forall x, y, z \in G$,

$$S^2(|x + y|) + S^2(|y + z|) + S^2(|z + x|) \leq S^2(|x|) + S^2(|y|) + S^2(|z|) + S^2(|x + y + z|),$$

so does S .

In fact, the function $F(\cdot) := S(|\cdot|)$ must be non-negative and strong subadditive. So, Proposition 2 is applicable here.

The following measure-type Hlawka inequality is non-trivial and it cannot be deduced from Theorem 2 in [3] (i.e., Example 3 above), because a measure space equipped with any set operation is not a group. But it can be obtained straightforwardly by Proposition 2 since a measure space with any set operation becomes a semigroup.

EXAMPLE 4. Let G be a measure space and $F = \mu$ be the measure. For the case of $k = 0$, note that $\mu(A) + \mu(B) + \mu(C) - \mu(A \cup B) - \mu(B \cup C) - \mu(C \cup A) + \mu(A \cup B \cup C) = \mu(A \cap B \cap C) \geq 0$ and for symmetric difference Δ , $\mu(A) + \mu(B) + \mu(C) - \mu(A \Delta B) - \mu(B \Delta C) - \mu(C \Delta A) + \mu(A \Delta B \Delta C) = 3\mu(A \cap B \cap C) \geq 0$.

According to Proposition 2, we have for any $k \geq 0$,

$$\mu(A)^{\frac{1}{2^k}} + \mu(B)^{\frac{1}{2^k}} + \mu(C)^{\frac{1}{2^k}} + \mu(A \cup B \cup C)^{\frac{1}{2^k}} \geq \mu(A \cup B)^{\frac{1}{2^k}} + \mu(B \cup C)^{\frac{1}{2^k}} + \mu(C \cup A)^{\frac{1}{2^k}}$$

and

$$\mu(A)^{\frac{1}{2^k}} + \mu(B)^{\frac{1}{2^k}} + \mu(C)^{\frac{1}{2^k}} + \mu(A \Delta B \Delta C)^{\frac{1}{2^k}} \geq \mu(A \Delta B)^{\frac{1}{2^k}} + \mu(B \Delta C)^{\frac{1}{2^k}} + \mu(C \Delta A)^{\frac{1}{2^k}}$$

because $\mu(A) + \mu(B) \geq \mu(A \cup B) \geq \mu(A \Delta B)$ and $\mu(A) + \mu(A \cup B) \geq \mu(A) + \mu(A \Delta B) \geq \mu(B)$.

3.3. Applications to integral form

Next, we would pay our attention to the following setting. Let Ω be a nonempty set and let G be an abelian group, and let $x \mapsto |x|$ be a non-negative symmetric and subadditive function on G (i.e., $|-x| = |x|$ and $|x| + |y| \geq |x + y|$, $\forall x, y \in G$). The function spaces G^Ω and \mathbb{R}^Ω are also abelian groups under the natural operation ‘+’. Take an abelian subgroup $\mathcal{F} \subset G^\Omega$ and a linear subspace $\widehat{\mathcal{F}} \subset \mathbb{R}^\Omega$ equipped with $T : \widehat{\mathcal{F}} \rightarrow \mathbb{R}$ satisfying the **basic settings** in the beginning of Section 2. Moreover,¹ $\forall f \in \mathcal{F}, |f| \in \widehat{\mathcal{F}}, 1 \in \widehat{\mathcal{F}}$.

Table 3: The concrete quantities of Theorem 1 used in the proof of Proposition 3.

Terminologies in Theorem 1	Concrete choices in Proposition 3
$X =$	abelian group G
$\mathcal{S} =$	$\widehat{\mathcal{F}}$
$\eta =$	\hat{g}
$\xi =$	$\mathcal{T}g - \hat{g}$
$f =$	$S \cdot $
$a =$	A
$b =$	$T(S \hat{g})$

Applying Theorem 1 to the above restricted situations, we have:

PROPOSITION 3. Given $A \neq 0 \in \mathbb{R}$, an operator $\mathcal{T} : \mathcal{F} \rightarrow G$, and two maps $g, \hat{g} \in \mathcal{F}$, let $S : [0, +\infty) \rightarrow [0, +\infty)$ be a concave function such that $A > 0$ or $S(|\mathcal{T}g|) > 0$ and when x satisfies $S|\hat{g}(x)| + S|\mathcal{T}g| \neq S|\hat{g}(x) - \mathcal{T}g|$, there is $A \geq S|\hat{g}(x)| + S|\mathcal{T}g - \hat{g}(x)|$. Then,

$$(T(1) - C)S^2|\mathcal{T}g| + T(S^2|\hat{g}|) \geq T(S^2|\hat{g} - \mathcal{T}g|) \tag{9}$$

¹Here, for $f \in \mathcal{F}$, $|f|$ is a function mapping Ω to $[0, \infty)$.

implies

$$(T(1) - C)S|\mathcal{T}g| + T(S|\hat{g}|) \geq T(S|\hat{g} - \mathcal{T}g|), \tag{10}$$

where $C = 2T(S|\hat{g}|)/A$.

Proof. Taking $\xi = \mathcal{T}g - \hat{g}$, $\eta = \hat{g}$ and $f(\cdot) = S|\cdot|$ in Theorem 1 (see Table 3 for details), we immediately complete the proof. \square

The main theorems in [6, 3] can be seen as direct conclusions of Proposition 3.

Proof. [A proof of Example 3 (i.e., the main theorem in [3]) via Proposition 3] Take $\Omega = \{1, 2, 3\}$, and for given $x, y, z \in G$, let $\hat{g} = g$ be defined as $g(1) = x$, $g(2) = y$ and $g(3) = z$. Let $T(S|g|) = S|g(1)| + S|g(2)| + S|g(3)|$ and $\mathcal{T}g = g(1) + g(2) + g(3)$ in Proposition 3. Then $T(1) = 3$, $\mathcal{T}g - g(i) = g(j) + g(k)$, where $\{i, j, k\} = \{1, 2, 3\}$. Hence, the result is easy to check. \square

Given an inner product space $(H, \langle \cdot, \cdot \rangle)$, suppose that $\mathcal{F} \subset H^\Omega$ and $\widehat{\mathcal{F}} \subset \mathbb{R}^\Omega$ are linear spaces equipped with linear operators $\mathcal{T} : \mathcal{F} \rightarrow H$ and $T : \widehat{\mathcal{F}} \rightarrow \mathbb{R}$. Then we have the following:

COROLLARY 1. *Let \mathcal{T}, T and $f \in \mathcal{F}$ be such that $T(|f|) > 0$ and for any $a \in H$, there is $T\langle f, a \rangle = \langle \mathcal{T}f, a \rangle$, where $|\cdot|$ is the norm induced by the inner product. If $T(|f|) \geq |f(x)| + |\mathcal{T}f - f(x)|$ whenever x satisfies $-f(x) \neq \alpha \mathcal{T}f$ for any $\alpha \geq 0$, then the following holds:*

$$(T(1) - 2)|\mathcal{T}f| + T(|f|) \geq T(|f - \mathcal{T}f|).$$

Proof. By the basic properties on inner products, we have

$$\begin{aligned} T(|f - \mathcal{T}f|^2) &= T(|f|^2 + |\mathcal{T}f|^2 - 2\langle f, \mathcal{T}f \rangle) \\ &= T(|f|^2) + |\mathcal{T}f|^2 T(1) - 2\langle \mathcal{T}f, \mathcal{T}f \rangle \\ &= T(|f|^2) + |\mathcal{T}f|^2(T(1) - 2). \end{aligned}$$

Let S in Proposition 3 be the identity operator. Then, the other conditions in Proposition 3 are easy to be verified. The prove is completed. \square

It is clear that T and \mathcal{T} are uniquely determined by each other according to Riesz's representation theorem.

COROLLARY 2. *Let (Ω, μ) be a finite measurable space and let $(H, \|\cdot\|)$ be an inner product space. Suppose $f, g : \Omega \rightarrow H$ are two nonzero integrable functions satisfying*

$$\frac{\int_{\Omega} f d\mu}{\int_{\Omega} \|f\| d\mu} = \frac{\int_{\Omega} g d\mu}{\int_{\Omega} \|g\| d\mu}.$$

Assume that for x with $-g(x) \neq \alpha \int_{\Omega} f d\mu$ for any $\alpha \geq 0$, there is

$$\int_{\Omega} \|f\| d\mu \geq \|g(x)\| + \left\| g(x) - \int_{\Omega} f d\mu \right\|.$$

Then we have the following Hlawka inequality

$$(\mu(\Omega) - C) \left\| \int_{\Omega} f d\mu \right\| + \int_{\Omega} \|g\| d\mu \geq \int_{\Omega} \left\| g - \int_{\Omega} f d\mu \right\| d\mu,$$

where $C = 2 \int_{\Omega} \|g\| d\mu / \int_{\Omega} \|f\| d\mu$.

Proof. Take $\mathcal{T}f = \int_{\Omega} f(\omega) d\mu$ and $T(\|f\|) = \int_{\Omega} \|f(t)\| d\mu(t)$. Now it is ready to apply Proposition 3 to complete the proof. \square

COROLLARY 3. Suppose that for x with $-f(x) \neq \alpha \int_{\Omega} f d\mu$ for any $\alpha \geq 0$, there is

$$\int_{\Omega} \|f\| d\mu \geq t \|f(x)\| + \left\| t f(x) - \int_{\Omega} f d\mu \right\|$$

for some $t \geq 0$. Then we have the following Hlawka-type inequality

$$(\mu(\Omega) - 2t) \left\| \int_{\Omega} f d\mu \right\| + t \int_{\Omega} \|f\| d\mu \geq \int_{\Omega} \left\| t f - \int_{\Omega} f d\mu \right\| d\mu.$$

COROLLARY 4. If \bar{f} is a rearrangement of f with the same distribution, and for x with $-\bar{f}(x) \neq \alpha \int_{\Omega} f d\mu$ for any $\alpha \geq 0$, there is

$$\int_{\Omega} \|f\| d\mu \geq \|\bar{f}(x)\| + \left\| \bar{f}(x) - \int_{\Omega} f d\mu \right\|.$$

Then we have the following Hlawka inequality

$$(\mu(\Omega) - 2) \left\| \int_{\Omega} f d\mu \right\| + \int_{\Omega} \|\bar{f}\| d\mu \geq \int_{\Omega} \left\| \bar{f} - \int_{\Omega} f d\mu \right\| d\mu.$$

Proof. Clearly, the properties of the rearrangement imply that $\int_{\Omega} f d\mu = \int_{\Omega} \bar{f} d\mu$ and $\int_{\Omega} \|f\| d\mu = \int_{\Omega} \|\bar{f}\| d\mu$. Hence, Corollary 2 is applicable here. \square

Theorem 1 in [6] could be viewed as a consequence of Corollary 4. In fact, taking $\bar{f} = f$ in Corollary 4, it is easy to verify the following.

EXAMPLE 5. (Theorem 1 in [6]) Let H be a Hilbert space, (Ω, μ) be a finite measure space and let f be a Bochner integrable H -valued function on (Ω, μ) . Suppose that

$$\int_{\Omega} \|f(t)\| d\mu(t) \geq \left\| f(\omega) - \int_{\Omega} f(t) d\mu(t) \right\| + \|f(\omega)\| \quad (a.e., \omega \in \Omega_f),$$

where $\Omega_f = \{\omega \in \Omega : -f(\omega) \neq \alpha \int_{\Omega} f(t) d\mu(t) \text{ for any } \alpha \geq 0\}$. Then

$$(\mu(\Omega) - 2) \left\| \int_{\Omega} f(\omega) d\mu \right\| + \int_{\Omega} \|f(\omega)\| d\mu \geq \int_{\Omega} \left\| f(\omega) - \int_{\Omega} f d\mu \right\| d\mu.$$

The following remark contains some interesting examples as corollaries of Proposition 3.

REMARK 4. Given an inner product space H , we have:

- For any $\lambda \in [0, 1]$, $x, y, z \in H$,

$$(1 - \lambda)(\|x\| + \|y\| + \|z\|) + (1 + 2\lambda)\|x + y + z\| \\ \geq \|\lambda x + y + z\| + \|x + \lambda y + z\| + \|x + \lambda y + z\|.$$

It is deduced by taking $\Omega = \{1, 2, 3\}$ and $t = (1 - \lambda)$ in Corollary 3, which is rather different from Corollary 2 in [6].

- Let $\mu_i, \lambda \geq 0$ be such that $\sum_{i=1}^n \mu_i \|x_i\| \geq \lambda \mu_i \|x_i\| + \|\lambda x_i - \sum_{j=1}^n \mu_j x_j\|$ for any $1 \leq i \leq n$. Then

$$\left(\sum_{i=1}^n \mu_i - 2\lambda \right) \left\| \sum_{i=1}^n \mu_i x_i \right\| + \lambda \sum_{i=1}^n \mu_i \|x_i\| \geq \sum_{i=1}^n \mu_i \left\| \lambda x_i - \sum_{j=1}^n \mu_j x_j \right\|.$$

It is deduced by taking $\Omega = \{1, \dots, n\}$ and $\mu(i) = \mu_i$ in Corollary 3, which is an improved version of Corollary 2 in [6] and Proposition 11 in [7].

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