

## REMARK ON THE CHAIN RULE OF FRACTIONAL DERIVATIVE IN THE SOBOLEV FRAMEWORK

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*Abstract.* A chain rule for power product is studied with fractional differential operators in the framework of Sobolev spaces. The fractional differential operators are defined by the Fourier multipliers. The chain rule is considered newly in the case where the order of differential operators is between one and two. The study is based on the analogy of the classical chain rule or Leibniz rule.

### 1. Introduction

The chain rule or Leibniz rule is an essential tool to study nonlinear differential equations. In the study of nonlinear partial differential equations (PDEs), fractional differential operators are also known as powerful tools. So, in order to analyze nonlinear PDEs, chain rules for fractional differential operators are naturally required. Even though fractional differential operators may be non-local unlike classical operators, estimates for fractional derivative have been studied on the analogy of classical chain rules. The history of this study can go back at least to the work of Kato and Ponce [9].

In this paper, we consider a chain rule corresponding to the identity

$$F(u)' = F'(u)u'$$

in the framework of Riesz potential space  $\dot{H}_p^s = D^{-s}L^p$ , where  $s \in \mathbb{R}$ ,  $n \geq 1$ , and  $L^p = L^p(\mathbb{R}^n)$  is the usual Lebesgue space.  $\dot{H}_p^s$  is also called as homogeneous Sobolev space. The fractional differential operator  $D^s = (-\Delta)^{s/2}$  is recognized as a Fourier multiplier by  $D^s = \mathfrak{F}^{-1}|\cdot|^s\mathfrak{F}$ , where  $\mathfrak{F}$  is the standard Fourier transform. Especially, we study the case where  $F$  behaves like power product, that is,  $F(z) \sim |z|^{p-1}z$ .

In [2], Christ and Weinstein showed the following estimate:

**PROPOSITION 1.** ([2, Proposition:3.1]) *Let  $n \geq 1$  and  $s \in (0, 1)$ . Let  $F \in C(\mathbb{C})$  and  $G \in C(\mathbb{C} : [0, \infty))$  satisfy*

$$|F(u) - F(v)| \leq (G(u) + G(v))|u - v|.$$

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Let  $1 \leq p < \infty$ ,  $1 < r, q < \infty$  satisfy

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \tag{1}$$

The estimate

$$\|F(u)\|_{\dot{H}_p^s} \lesssim \|G(u)\|_{L^q} \|u\|_{\dot{H}_r^s}.$$

holds for any  $u \in \dot{H}_r^s$  with  $G(u) \in L^q$ .

Here  $a \lesssim b$  stands for  $a \leq Cb$  with some positive constant  $C$ . We also denote  $a \sim b$  when  $a \lesssim b$  and  $b \lesssim a$ . We use these notation through this paper. We note that if  $F(z) = |z|^{p-1}z$ , then  $G(z) = C|z|^p$  with some positive  $C$ . Roughly speaking, Proposition 1 asserts that  $D^s F(u)$  behaves like  $F'(u)D^s u$ . Since the Riesz operator  $R = D^{-1}\nabla$  is bounded on  $L^p$  when  $p \in (1, \infty)$ ,  $\|F(u)\|_{\dot{H}_p^s}$  may be estimated even when  $s \geq 1$  by combining the classical chain rule, Proposition 1, and the Hölder inequality with more regular  $F$ .

On the other hand, when  $\rho \in (1, 2)$  and  $s \in (1, \rho)$ ,  $D^s(|u|^{\rho-1}u)$  cannot be controlled directly by Proposition 1. Indeed, one may regard

$$D^s(|u|^{\rho-1}u) \sim D^{s-1}(|u|^{\rho-1}\nabla u)$$

and distribute  $D^{s-1}$  to  $\nabla u$  and  $|u|^{\rho-1}$  by using the fractional Leibniz rule (for example, see [7, Theorem 1], [10], [4], and references therein). However, since  $f(z) = |z|^{\rho-1}$  is not Lipschitz continuous for  $\rho < 2$ , it is impossible to control  $D^{s-1}|u|^{\rho-1}$  by applying Proposition 1 directly.

Here we note that  $D^s(|u|^{\rho-1}u)$  is controlled by Proposition 1 for any  $s \in (0, \rho)$  when  $\rho$  is a positive integer. Therefore, one may expect that  $D^s(|u|^{\rho-1}u)$  is controlled for any  $\rho \geq 1$  and  $s \in (0, \rho)$ . This is what interests us in this paper; we consider a generalization of Proposition 1 for  $s \in (1, 2)$ . We remark that the case where  $s > 2$  is reduced to the case where  $s \in (0, 2)$ , so it is sufficient to show the estimate with  $s \in (1, 2)$ .

We generalize the form of  $F$  slightly. With  $\rho > 1$ , we put  $F_\rho \in C^1(\mathbb{C})$  satisfying  $F_\rho(0) = F'_\rho(0) = 0$  and

$$\begin{aligned} |F_\rho(u) - F_\rho(v)| &\lesssim \max\{|u|, |v|\}^{\rho-1} |u - v|, \\ |F'_\rho(u) - F'_\rho(v)| &\lesssim \begin{cases} \max\{|u|, |v|\}^{\rho-2} |u - v| & \text{if } \rho \geq 2, \\ |u - v|^{\rho-1} & \text{if } \rho < 2. \end{cases} \end{aligned} \tag{2}$$

In [5], Ginibre, Ozawa, and Velo showed the expectation above in the framework of homogeneous Besov spaces  $\dot{B}_{p,q}^s$ .

PROPOSITION 2. ([5, Lemma:3.4]) *Let  $n \geq 1$ ,  $\rho \geq 1$ , and  $s \in (0, \min\{2, \rho\})$ . Let  $1 \leq p, r \leq \infty$  and  $(\rho - 1)^{-1} \leq q \leq \infty$  satisfy (1). If  $u \in \dot{B}_{r,2}^s \cap L^{(\rho-1)q}$ , then*

$$\|F_\rho(u)\|_{\dot{B}_{p,2}^s} \lesssim \|u\|_{L^{(\rho-1)q}}^{\rho-1} \|u\|_{\dot{B}_{r,2}^s}. \tag{3}$$

We note that Proposition 2 has been used to study  $H_p^s$ -valued solutions to some nonlinear PDEs because Besov spaces may play a role in useful auxiliary spaces. The advantage to consider the fractional chain rule in the framework of Besov space is the following representation of homogeneous Besov norms:

$$\|u\|_{\dot{B}_{p,q}^s} \sim \left( \int_0^\infty \lambda^{-sq-1} \sup_{|y|<\lambda} \|(\tau_y - 2 + \tau_{-y})u\|_{L^p}^q d\lambda \right)^{1/q}, \tag{4}$$

where  $\tau_y u = u(\cdot + y)$ ,  $s \in (0, 2)$ , and  $1 \leq p, q \leq \infty$  (See [1, 6.2.5. Theorem], for example.). With this representation, the term  $(\tau_y - 2 + \tau_{-y})u$  gives a clear explanation of the connection between the classical and fractional chain rules. We remark that even though the fractional differential operators are defined by Fourier multipliers, it is nontrivial why  $F'(u) \sim |u|^{\rho-1}$  appears as an upper bound of (3) from the viewpoint of Fourier transform. However, we further remark that when  $\rho = 2$ , the chain rule may be shown by the argument of Fourier multipliers. For the detail, we refer the reader to [4].

Proposition 2 is an effective estimate but it seems more handy to close the argument only with Sobolev spaces. So one of the main purpose of this paper is to show a similar estimate to Proposition 2 in the framework of  $\dot{H}_p^s$ . This is one of the main statement of this paper.

**PROPOSITION 3.** *Let  $n \geq 1$ . Let  $\rho > 1$  and  $s \in (1, \min\{2, \rho\})$ . Let  $1 < p, r < \infty$  and  $(\rho - 1)^{-1} < q < \infty$  satisfy (1). The estimate*

$$\|F_\rho(u)\|_{\dot{H}_p^s} \lesssim \|u\|_{L^{(\rho-1)q}}^{\rho-1} \|u\|_{\dot{H}_p^s}$$

holds for any  $u \in \dot{H}_r^s \cap L^{(\rho-1)q}$ .

We remark that the case when  $p$  or  $r = 1, \infty$  is included in the assumption of Proposition 2 but not in that of Proposition 3. It is because our proof is based on the boundedness of Hardy-Littlewood maximal operators. See the next section.

The main idea for the proof of Proposition 3 is to express  $\|F(u)\|_{\dot{H}_p^s}$  with  $(\tau_y - 2 + \tau_{-y})F(u)$  like the proof of Proposition 2. Since  $\dot{H}_p^s$  norms seem not to be expressed like (4), we rewrite each dyadic components of  $F(u)$  by  $(\tau_y - 2 + \tau_{-y})F(u)$ . Namely, the identity

$$Q_j F(u)(x) = \frac{1}{2} \int (\tau_y - 2 + \tau_{-y})F(u)(x) \psi_j(y) dy$$

plays a critical role in this paper, where  $Q_j$  is the standard  $j$ -th Littlewood-Paley dyadic operator and  $\psi_j$  is the corresponding kernel. The details of  $Q_j$  and  $\psi_j$  are stated in the next section. We note that the identity

$$Q_j F(u)(x) = \int (\tau_y - 1)F(u)(x) \psi_j(y) dy$$

plays an essential role in [2] as well. In this paper, we deploy a similar but more careful approach with the identities above.

Our second purpose is to extend Proposition 1 for the difference between two functions. Here we put  $a_+ = \max\{a, 0\}$  and  $a_- = \min\{a, 0\}$  for  $a \in \mathbb{R}$ .

**COROLLARY 1.** *Let  $n \geq 1$ . Let  $s \in (0, 1)$  and  $\rho \geq 1$ . Let  $1 \leq p < \infty$  and  $1 < q, r < \infty$  satisfy (1). Then the estimate*

$$\begin{aligned} \|F_\rho(u) - F_\rho(v)\|_{\dot{H}_p^s} &\lesssim (\|u\|_{L^q(\rho-1)} + \|v\|_{L^q(\rho-1)})^{\rho-1} \|u - v\|_{\dot{H}_p^s} \\ &\quad + (\|u\|_{\dot{H}_p^s} + \|v\|_{\dot{H}_p^s})(\|u\|_{L^q(\rho-1)} + \|v\|_{L^q(\rho-1)})^{(\rho-2)_+} \|u - v\|_{L^q(\rho-1)}^{\min\{\rho-1, 1\}} \end{aligned}$$

holds for any  $u, v \in \dot{H}_r^s \cap L^q(\rho-1)$ .

Corollary 1 corresponds to the identity

$$\nabla(u^\rho - v^\rho) = \rho u^{\rho-1} \nabla(u - v) + \rho(u^{\rho-1} - v^{\rho-1}) \nabla v.$$

Corollary 1 is as worthy as Proposition 1 to show the locally-in-time well-posedness of semilinear PDEs. We remark that Corollary 1 is not necessary to construct solutions with a contraction argument because one may deploy a contraction argument in the ball of  $H_p^s$  with the distance of  $L^p$ , for example. However, the proof of the continuous dependence of solution maps on initial data may require Corollary 1.

We also extend Corollary 1 to the case where  $s \in (1, 2)$ :

**PROPOSITION 4.** *Let  $n \geq 1$ . Let  $\rho > 1$  and  $s \in (1, \rho)$ . Let  $1 \leq p < \infty$ ,  $1 < r < \infty$ ,  $(\rho - 1)^{-1} < q < \infty$  satisfy (1). Let  $d_{\rho,s} \in (0, \min\{\rho - s, 1\})$ . Then*

$$\begin{aligned} \|F_\rho(u) - F_\rho(v)\|_{\dot{H}_p^s} &\lesssim (\|u\|_{L^q(\rho-1)} + \|v\|_{L^q(\rho-1)})^{\rho-1} \|u - v\|_{\dot{H}_p^s} \\ &\quad + (\|u\|_{\dot{H}_p^s} + \|v\|_{\dot{H}_p^s})(\|u\|_{L^q(\rho-1)} + \|v\|_{L^q(\rho-1)})^{\rho-1-d_{\rho,s}} \|u - v\|_{L^q(\rho-1)}^{d_{\rho,s}}. \end{aligned} \tag{5}$$

We note that if  $\rho \geq \max\{s' \in \mathbb{Z} \mid s' \leq s\} + 2$ , then one may obtain (5) with  $d_{\rho,s} = 1$  by combining Corollary 1, boundedness Riesz operator, and classical chain rule. We remark that an extension of Corollary 1 in the framework of Besov spaces was given in [11, Proposition 2.1] and [3, Lemma 6.2]. We remark that on the analogy of classical chain rules, it is expected that Proposition 4 may hold with  $d_{\rho,s} = \rho - s$  but a similar technical difficulty arises in both Sobolev and Besov frameworks. The case where  $\rho = s$  is similar.

In the next section, we collect some notation and basic estimates. In Section 3, we revisit the proofs of Proposition 1 and Corollary 1 for the completeness. In Section 4, we give the proofs of Propositions 3 and 4

## 2. Preliminary

### 2.1. Notation

Here we collect some notation.

Let  $\psi$  be a radial Schwartz function and satisfy

$$\text{supp } \mathfrak{F}\psi \subset \{\xi \mid 1/2 \leq |\xi| \leq 2\}, \quad \text{range } \mathfrak{F}\psi \subset [0, 1]$$

and

$$\sum_j \tilde{\mathfrak{F}}\psi(2^{-j}\xi) = 1$$

for any  $\xi \neq 0$ . For  $j \in \mathbb{Z}$ , let  $\psi_j = 2^{jn}\psi(2^j\cdot)$  satisfy that  $\|\psi_j\|_{L^1} = \|\psi\|_{L^1}$  and let  $Q_j = \psi_j^*$ . We also put  $\tilde{Q}_j = Q_{j-1} + Q_j + Q_{j+1}$  and  $\tilde{\psi}_j = \psi_{j-1} + \psi_j + \psi_{j+1}$ . We remark that  $\tilde{Q}_j Q_j = Q_j$  holds for any  $j$ . It is known that for  $s \in \mathbb{R}$  and  $1 < p, q < \infty$ , the homogeneous Sobolev and Triebel-Lizorkin norms are equivalent:

$$\|f\|_{\dot{H}_p^s} \sim \|2^{js} Q_j f\|_{L^p(\ell^2)}.$$

For the details of this equivalence, we refer the reader to [6, Theorem 5.1.2], for example. For  $f \in L^1_{\text{loc}}$ , we define the Hardy-Littlewood maximal operator by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(r)|} \int_{B(r)} |f(x+y)| dy$$

and set

$$M^{(P)}f(x) = M(|f|^P)^{1/P}$$

for  $P > 0$ , where  $B(r) \subset \mathbb{R}^n$  is the ball with radius  $r$  centered at the origin. It is known that  $M$  is bounded operator on  $L^p$  and  $L^p(\ell^q)$  for  $1 < p, q < \infty$  (See [6, Theorems 2.1.6 and 4.6.6] and references therein). Moreover, for  $1 \leq p < \infty$ , we define weighted  $L^p$  norm with weight function  $w$  by

$$\|f\|_{L^p_w} = \left( \int |f(x)|^p w(x) dx \right)^{1/p}.$$

### 2.2. Basic Estimates

Here we collect some estimates.

LEMMA 1. ([6, Theorem :2.1.10]) *Let  $w \in L^1$  be a positive radially decreasing function. Then the estimate*

$$|w * g(x)| \leq \|w\|_{L^1} M g(x)$$

holds for any  $x \in \mathbb{R}^n$ .

LEMMA 2. *The estimate*

$$|\tilde{Q}_k g(x)| \lesssim M g(x)$$

holds for any  $k \in \mathbb{Z}$  and  $x \in \mathbb{R}^n$ . Moreover, if  $|y| < 2^{-k}$ , then the estimates

$$\begin{aligned} |\tilde{Q}_k g(x+y) - \tilde{Q}_k g(x)| &\lesssim 2^k |y| M g(x), \\ |\tilde{Q}_k g(x+y) - 2\tilde{Q}_k g(x) + \tilde{Q}_k g(x-y)| &\lesssim 2^{2k} |y|^2 M g(x) \end{aligned}$$

hold for any  $x \in \mathbb{R}^n$ .

*Proof.* The first estimate follows directly from Lemma 1.

The estimate for  $|\tilde{Q}_k g(x+y) - \tilde{Q}_k g(x)|$  is given in [2] but for completeness, we give the proof.

By the fundamental theorem of calculus implies that the identities

$$\psi_k(x+y) - \psi_k(x) = \int_0^1 (\nabla \psi)_k(x + \theta y) d\theta \cdot 2^k y$$

and

$$\begin{aligned} & \psi_k(x+y) - 2\psi_k(x) + \psi_k(x-y) \\ &= \int_0^1 ((\nabla \psi)_k(x + \theta y) - (\nabla \psi)_k(x - \theta y)) d\theta \cdot 2^k y \\ &= \sum_{|\alpha|=2} \int_0^1 \int_0^1 (\partial^\alpha \psi)_k(x + (2\theta' - 1)\theta y) d\theta' d\theta \cdot 2^{2k} y^2. \end{aligned}$$

hold for any  $x, y$ . Since for  $|\alpha| \leq 2$ , the estimates

$$(\partial^\alpha \psi)_k(x+y) \lesssim 2^{kn} (2 + 2^k |x+y|)^{-n-1} \leq 2^{kn} (1 + 2^k |x|)^{-n-1},$$

hold for  $|y| < 2^{-k}$ . Therefore, the last two estimates are obtained by combining this and Lemma 1.  $\square$

LEMMA 3. For  $0 < S < S', Q \geq 1$ , and  $a \in \ell^Q$ , the following estimate holds:

$$\|2^{jS} \sum_k 2^{(k-j)-S'} a_k\|_{\ell_j^Q} \leq \left( \frac{2^S}{2^S - 1} + \frac{1}{2^{S'-S} - 1} \right) \|2^{kS} a_k\|_{\ell_k^Q}.$$

*Proof.* By the Minkowski inequality, we have

$$\begin{aligned} \|2^{jS} \sum_k 2^{(k-j)-S'} a_k\|_{\ell_j^Q} &= \|2^{jS} \sum_{\ell < 0} 2^{\ell S'} a_{\ell+j}\|_{\ell_j^Q} + \|2^{jS} \sum_{\ell \geq 0} a_{\ell+j}\|_{\ell_j^Q} \\ &\leq \sum_{\ell < 0} 2^{\ell(S'-S)} \|2^{S(\ell+j)} a_{\ell+j}\|_{\ell_j^Q} + \sum_{\ell \geq 0} 2^{-\ell S} \|2^{S(\ell+j)} a_{\ell+j}\|_{\ell_j^Q} \\ &\leq \frac{1}{2^{S'-S} - 1} \|2^{Sk} a_k\|_{\ell_k^Q} + \frac{2^S}{2^S - 1} \|2^{Sk} a_k\|_{\ell_k^Q}. \quad \square \end{aligned}$$

LEMMA 4. Let  $0 < S < 1$  and  $1 \leq P, Q < \infty$ . The following estimate holds:

$$\begin{aligned} & \|2^{jS} \|v(y+x)(u(y+x) - u(x))\|_{L^P_{w_{j,y}}}\|_{\ell_j^Q} \\ & \lesssim M^{(P)} v(x) \|2^{ks} M Q_k u\|_{\ell_k^Q} + \|2^{ks} M^{(P)}(v M Q_k u)(x)\|_{\ell_k^Q}. \end{aligned}$$

*Proof.* Lemmas 1 and 2 imply that we have

$$\begin{aligned}
 & \|v(y+x)(u(y+x) - u(x))\|_{L^P_{\psi_j,y}} \\
 &= \sum_k \|v(y+x)(\tilde{Q}_k Q_k u(y+x) - \tilde{Q}_k Q_k u(x))\|_{L^P_{\psi_j,y}} \\
 &\lesssim \sum_{k < j} M Q_k u(x) 2^{k-j} \left( \int_{|y| < 2^{-k}} |v(y+x)|^P |2^j y|^P |\psi_j(y)| dy \right)^{1/P} \\
 &\quad + \sum_{k < j} \left( \int_{|y| > 2^{-k}} |(M Q_k u(x) + M Q_k u(x+y))v(y+x)|^P |\psi_j(y)| dy \right)^{1/P} \\
 &\quad + \sum_{k \geq j} \left( \|v(y+x)M Q_k u(x+y)\|_{L^P_{\psi_j,y}} + M Q_k u(x) \|v(y+x)\|_{L^P_{\psi_j,y}} \right) \\
 &\lesssim \sum_k 2^{(k-j)-} \left( M^{(P)}(vM Q_k u)(x) + M^{(P)}v(x)M Q_k u(x) \right),
 \end{aligned}$$

where we have used the estimates

$$\int_{|y| > 2^{-k}} |\psi_j(y)| dy = \int_{|y| > 2^{-k+j}} |\psi(y)| dy \lesssim \int_{|y| > 2^{-k+j}} |y|^{-n-P} dy \lesssim 2^{P(k-j)}$$

with Lemma 1 for the last estimate. Therefore, Lemma 4 follows from these estimates above and Lemma 3.  $\square$

**COROLLARY 2.** *Let  $0 < S < 1$  and  $1 \leq P_0 < 2$ . Let  $P_0 < P < \infty$  and  $1 < Q, R < \infty$  satisfy*

$$\frac{1}{P} = \frac{1}{Q} + \frac{1}{R}.$$

*The estimates*

$$\|2^{jS} \|(u(y + \cdot) - u(\cdot))\|_{L^{P_0}_{\psi_j,y}} \|_{L^P(\ell^2_j)} \lesssim \|u\|_{\dot{H}^S}$$

*and*

$$\|2^{jS} \|v(y + \cdot)(u(y + \cdot) - u(\cdot))\|_{L^{P_0}_{\psi_j,y}} \|_{L^P(\ell^2_j)} \lesssim \|v\|_{L^Q} \|u\|_{\dot{H}^S}$$

*hold.*

*Proof.* For  $P_1 > P_0$ , the boundedness of  $M$  implies that we have

$$\|M^{(P_0)} f\|_{L^{P_1}} = \|M|f|^{P_0}\|_{L^{P_1/P_0}}^{1/P_0} \lesssim \| |f|^{P_0} \|_{L^{P_1/P_0}}^{1/P_0} = \|f\|_{L^{P_1}}.$$

Since  $P_0 < P < Q$ , Corollary 2 follows from Lemma 4 and the Hölder inequality.  $\square$

**3. Revisit of the Chain rules when  $s \in (0, 1)$**

In this section, we revisit the proof of Proposition 1 and Corollary 1. The proofs are essentially given in [2] but for completeness, we give the proofs.

*Proof of Proposition 1.* Since  $\int \psi(y)dy = 0$ , the identity

$$Q_j F(u)(x) = \int (F(u)(x+y) - F(u)(x)) \psi_j(y) dy$$

holds for any  $x \in \mathbb{R}^n$  and  $j \in \mathbb{Z}$ . Therefore, Proposition 1 follows from Corollary 2 with  $(v, P_0, P, Q, R, S)$  replaced by  $(G(u), 1, p, q, r, s)$ .  $\square$

*Proof of Corollary 1.* The identity

$$\begin{aligned} & Q_j F_\rho(u)(x) - Q_j F_\rho(v)(x) \\ &= \int (F_\rho(u)(x+y) - F_\rho(u)(x) - F_\rho(v)(x+y) + F_\rho(v)(x)) \psi_j(y) dy \end{aligned}$$

holds. The fundamental theorem of calculus implies that the identity

$$\begin{aligned} & F_\rho(u)(x+y) - F_\rho(u)(x) - F_\rho(v)(x+y) + F_\rho(v)(x) \\ &= ((u-v)(x+y) - (u-v)(x)) \int_0^1 F'_\rho(\theta u(x+y) + (1-\theta)u(x)) d\theta \\ &\quad + (v(x+y) - v(x)) \\ &\quad \times \int_0^1 (F'_\rho(\theta u(x+y) + (1-\theta)u(x)) - F'_\rho(\theta v(x+y) + (1-\theta)v(x))) d\theta \end{aligned}$$

holds. The identity above and (2) imply that the estimate

$$\begin{aligned} & |F_\rho(u)(x+y) - F_\rho(u)(y) - F_\rho(v)(x+y) + F_\rho(v)(x)| \\ &\lesssim (|u(x+y)|^{\rho-1} + |u(x)|^{\rho-1}) |(u-v)(x+y) - (u-v)(x)| \\ &\quad + (|u(x+y)| + |u(x)| + |v(x+y)| + |v(x)|)^{(\rho-2)+} \\ &\quad \times (|(u-v)(x+y)| + |(u-v)(x)|)^{\min\{\rho-1, 1\}} |v(x+y) - v(x)| \end{aligned}$$

holds. Therefore, Corollary 1 follows from Corollary 2 and the estimates above.  $\square$

**4. Proofs of the Chain rules when  $s \in (1, 2)$**

*Proof of Proposition 3.* We note that it is shown in [5, (3.23) and (3.26)] that the estimate

$$\begin{aligned} |(\tau_y - 2 + \tau_{-y})F_\rho(u)(x)| &\lesssim |u(x)|^{\rho-1} |u(x+y) - 2u(x) + u(x-y)| \\ &\quad + \max\{|u(x)|, |u(x+y)|, |u(x-y)|\}^{(\rho-2)+} \\ &\quad \times (|u(x+y) - u(x)|^{\min\{\rho, 2\}} + |u(x-y) - u(x)|^{\min\{\rho, 2\}}). \end{aligned} \tag{6}$$

holds. Moreover, the symmetry of  $\psi$  and  $\int \psi = 0$  imply that we have

$$\begin{aligned}
 Q_j F_\rho(u)(x) &= \int F_\rho(u)(x+y)\psi_j(y)dy \\
 &= \int F_\rho(u)(x-y)\psi_j(y)dy \\
 &= \frac{1}{2} \int (F_\rho(u)(x+y) + F_\rho(u)(x-y))\psi_j(y)dy \\
 &= \frac{1}{2} \int (\tau_y - 2 + \tau_{-y})F_\rho(u)(x)\psi_j(y)dy.
 \end{aligned} \tag{7}$$

From (6) and (7), the estimate

$$\begin{aligned}
 &|Q_j F_\rho(u)(x)| \\
 &\lesssim |u(x)|^{\rho-1} \int |u(x+y) - 2u(x) + u(x-y)| |\psi_j(y)| dy \\
 &\quad + \int \max\{|u(x)|, |u(x+y)|, |u(x-y)|\}^{(\rho-2)_+} |u(x+y) - u(x)|^{\min\{\rho, 2\}} |\psi_j(y)| dy
 \end{aligned} \tag{8}$$

follows. By Lemma 2, when  $k < j$ , we estimate

$$\begin{aligned}
 &\int |\tilde{Q}_k Q_k u(x+y) - 2\tilde{Q}_k Q_k u(x) + \tilde{Q}_k Q_k u(x-y)| |\psi_j(y)| dy \\
 &\lesssim M Q_k u(x) \int_{|y| < 2^{-k}} 2^{2(k-j)} |\psi_j(y)| dy + \int_{|y| > 2^{-k}} (M Q_k u(x+y) + M Q_k u(x)) |\psi_j(y)| dy \\
 &\lesssim 2^{2(k-j)} (M Q_k u(x) + M^2 Q_k u(x)),
 \end{aligned}$$

where for the last estimate, we have used the estimates

$$\int_{|y| > 2^{-k}} |\psi_j(y)| dy \lesssim \int_{|y| > 2^{j-k}} |\psi(y)| dy \lesssim \int_{|y| > 2^{j-k}} |y|^{-n-2} dy \lesssim 2^{2(k-j)}.$$

Similarly, when  $k \geq j$ , it is estimated by

$$\begin{aligned}
 &\int |\tilde{Q}_k Q_k u(x+y) - 2\tilde{Q}_k Q_k u(x) + \tilde{Q}_k Q_k u(x-y)| |\psi_j(y)| dy \\
 &\lesssim \int (M Q_k u(x+y) + M Q_k u(x)) |\psi_j(y)| dy \\
 &\lesssim (M Q_k u(x) + M^2 Q_k u(x)).
 \end{aligned}$$

Therefore, the estimates above and Lemma 3 imply that we have

$$\begin{aligned}
 &\|2^{js} |u|^{\rho-1} \int |u(\cdot+y) - 2u(\cdot) + u(\cdot-y)| |\psi_j(y)| dy\|_{L^p(\ell^2_j)} \\
 &\lesssim \|2^{ks} |u|^{\rho-1} (M Q_k u + M^2 Q_k u)\|_{L^p(\ell^2_k)} \\
 &\lesssim \|u\|_{L^q(\rho-1)}^{\rho-1} \|u\|_{\dot{H}^s_s}.
 \end{aligned} \tag{9}$$

Moreover, for  $\rho < 2$ , by the Lemma 4, the second term of the RHS of (8) satisfy the estimate

$$\begin{aligned}
 \|2^{js}\|u(\cdot+y) - u(\cdot)\|_{L^{\rho}_{\Psi_{j,y}}}^{\rho} \|_{L^p(\ell^2)} &= \|2^{js/\rho}\|u(\cdot+y) - u(\cdot)\|_{L^{\rho}_{\Psi_{j,y}}}^{\rho} \|_{L^{\rho p}(\ell^2)}^{\rho} \\
 &\lesssim \|2^{ks/\rho}M^{(\rho)}Q_k u\|_{L^{\rho p}(\ell^2)}^{\rho} \\
 &\lesssim \|u\|_{\dot{H}^{\rho/\rho}}^{\rho} \\
 &\lesssim \|u\|_{L^{q(\rho-1)}}^{\rho-1} \|u\|_{\dot{H}^{\rho}}, \tag{10}
 \end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality to obtain the last estimate. For the details of the Gagliardo-Nirenberg inequality, we refer the reader to [8, Corollary 2.4] and references therein. Combining (9) and (10), we obtain Proposition 3. The case where  $\rho > 2$  is shown similarly.  $\square$

*Proof of Proposition 4.* For simplicity, we denote  $d_{\rho,s}$  by  $d$ . We note that the estimate

$$\begin{aligned}
 &|(\tau_y - 2 + \tau_{-y})(F_{\rho}(u) - F_{\rho}(v))(x)| \\
 &\lesssim \max_{z \in \{x, x+y, x-y\}} |u(z)|^{\rho-1} |(\tau_y - 2 + \tau_{-y})(u - v)(x)| \\
 &\quad + \mu^{(\rho-2)+} \max_{z \in \{x, x+y, x-y\}} |(u - v)(z)|^{\min\{\rho-1, 1\}} |(\tau_y - 2 + \tau_{-y})v(x)| \\
 &\quad + \mu^{(\rho-2)+} (|(\tau_y - 1)u(x)| + |(\tau_{-y} - 1)u(x)|)^{\min\{\rho-1, 1\}} |(\tau_y - 1)(u - v)(x)| \\
 &\quad + \mu^{(\rho-2)+} |(\tau_{-y} - 1)u(x)| \min\left\{\sigma, \max_{z \in \{x, x+y, x-y\}} |(u - v)(z)|^{\min\{\rho-1, 1\}}\right\} \tag{11}
 \end{aligned}$$

holds, where

$$\begin{aligned}
 \mu &= \max\{|u(x+y)|, |u(x)|, |u(x-y)|, |v(x+y)|, |v(x)|, |v(x-y)|\}, \\
 \sigma &= |(\tau_y - 1)u(x)| + |(\tau_{-y} - 1)u(x)| + |(\tau_y - 1)v(x)| + |(\tau_{-y} - 1)v(x)|.
 \end{aligned}$$

For the details of the estimate above, see [3, Lemma:A.1]. Therefore, for the proof of proposition 4, it is sufficient to show

$$\begin{aligned}
 &\|2^{js}\|B(u, v)(\cdot, y)\|_{L^1_{\Psi_{j,y}}} \|_{L^p(\ell^2)} \\
 &\lesssim (\|u\|_{\dot{H}^{\rho}} + \|v\|_{\dot{H}^{\rho}})(\|u\|_{L^{q(\rho-1)}} + \|v\|_{L^{q(\rho-1)}})^{\rho-1-d} \|u - v\|_{L^{q(\rho-1)}}^d, \tag{12}
 \end{aligned}$$

where  $\rho < 2$  and

$$B(u, v)(x, y) = |u(x+y) - u(x)||v(x+y) - v(x)|^{\rho-1-d} |u(x+y) - v(x+y)|^d.$$

(12) is used to control the last term on the RHS of (11). The other terms on the RHS of (11) may be treated like the argument above so we omit the detail. Put

$$q_0 = q \frac{\rho - 1}{d} \quad \text{and} \quad r_0 = \frac{pq_0}{q_0 - p}.$$

Then the Hölder inequality and Lemma 1 imply that we have

$$\begin{aligned} & \|B(u, v)(x, y)\|_{L^1_{\Psi_j, y}} \\ & \leq \| |(u - v)(x + y)|^d \|_{L^{q_0/p}_{\Psi_j, y}} \| |u(x + y) - u(x)| |v(x + y) - v(x)|^{\rho-1-d} \|_{L^{r_0/p}_{\Psi_j, y}} \\ & \lesssim (M^{(q(\rho-1)/p)} |u - v|(x))^d \\ & \quad \times \| |u(x + y) - u(x)| \|_{L^{(\rho-d)r_0/p}_{\Psi_j, y}} \| |v(x + y) - v(x)| \|_{L^{(\rho-d)r_0/p}_{\Psi_j, y}}^{\rho-d-1}. \end{aligned}$$

Then Lemma 1 and Corollary 2 imply that the estimates

$$\begin{aligned} & \|2^{js} B(u, v)(\cdot, y)\|_{L^1_{\Psi_j, y}} \|_{L^p(\ell^2)} \\ & \lesssim \|M^{(q(\rho-1)/p)} |u - v|\|_{L^{dq_0}}^d \\ & \quad \times \|2^{js/(\rho-d)} |u(\cdot + y) - u(\cdot)|\|_{L^{(\rho-d)r_0/p}_{\Psi_j, y}} \|_{L^{r_0(\rho-d)}(\ell^2)} \\ & \quad \times \|2^{js/(\rho-d)} |v(\cdot + y) - v(\cdot)|\|_{L^{(\rho-d)r_0/p}_{\Psi_j, y}} \|_{L^{r_0(\rho-d)}(\ell^2)}^{\rho-d-1} \\ & \lesssim \| |u - v|\|_{L^{q(\rho-1)}}^d \|u\|_{\dot{H}^{s/(\rho-d)}_{r_0(\rho-d)}} \|v\|_{\dot{H}^{\rho-d-1}_{r_0(\rho-d)}} \end{aligned}$$

hold. Then (12) follows from the estimates above and the Gagliardo-Nirenberg inequality.  $\square$

REMARK 1. In the proof above, one cannot take  $d = \rho - s$  because

$$\|2^j |u(\cdot + y) - u(\cdot)|\|_{L^{(\rho-d)r_0/p}_{\Psi_j, y}} \|_{L^{r_0(\rho-d)}(\ell^2)}$$

is not controlled by Lemma 4.

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